Fourier Analysis and its Applications

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13 Fejer's theorem (Contd).

Applications of Fejer's theorem An immediate corollary of Fejer's theorem is the following:

corollary The set of all trigonometric polynomials is dense in the space of $2\pi-$periodic continuous functions in the following sense. Given a $2\pi-$periodic continuous function f, for every $\epsilon > 0$ there is a trigonometric polynomial $P_n(x)$ such that

$$\sup_{|x|\leq\pi} |f(x) - P_n(x)| < \epsilon/\sqrt{2\pi}.$$

Exercise: Show that $\|f(x) - P_n(x)\| < \epsilon$. From continuous functions we can easily pass over to functions in $L^2[-\pi, \pi]$.

Theorem (Trigonometric polynomials are dense in $L^2[-\pi,\pi]$): If $f \in L^2[-\pi, \pi]$ then given any $\epsilon > 0$, there is a trigonometric polynomial $P_N(x)$ such that $\|f(x) - P_N(x)\| < \epsilon$.

Proof: Proceeds in four steps. Let $\epsilon > 0$ be arbitrary.

Step - I : By *Luzin's theorem*, there is a continuous function $g(x)$ on $[-\pi, \pi]$ such that

$$\int_{-\pi}^{\pi} |f(x) - g(x)|^2 dx < \epsilon^2/8.$$

Step - II : Let M be the supremum of $|g(x)|$. There is a $\delta > 0$ such that

$$2\int_{|x|\geq\pi-\delta} |f(x)|^2 dx + 2\int_{|x|\geq\pi-\delta} M^2 dx < \epsilon^2/8.$$

Step - III : Now we choose a continuous function $G(x)$ such that $G(x) = g(x)$ on $|x| \leq \pi - \delta$ and $G(\pm\pi) = 0$. Further $|G(x)|$ also has upper bound M. This is possible by *Tietze's extension theorem*. Then

$$\begin{aligned}
\int_{-\pi}^{\pi} |f(x) - G(x)|^2 dx &= \int_{|x|\leq\pi-\delta} |f(x) - g(x)|^2 dx + 2\int_{|x|\geq\pi-\delta} (|f(x)|^2 + |G(x)|^2) dx \\
&\leq \frac{\epsilon^2}{3} + 2\int_{|x|\geq\pi-\delta} (|f(x)|^2 + M^2) dx < \epsilon^2/4 \\
\therefore \|f - G\| &< \epsilon/2.
\end{aligned}$$

Step - IV : Extend G as a $2\pi-$periodic continuous function which in view of $G(\pm\pi) = 0$ is continuous. Now by Fejer's theorem we select a Trig. Poly $P(x)$ such that $\|G - P\| < \epsilon/2$. So that

$$\|f - P\| \leq \|f - G\| + \|G - P\| < \epsilon$$

Proof of Parseval formula via Fejer's theorem: Parseval formula states that if $f \in L^2[-\pi,\pi]$ then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2dx = |a_0|^2 + \frac{1}{2}\sum_{j=1}^{\infty}(|a_j|^2+|b_j|^2) \tag{3.4}$$

(i) It is useful to recall at this point that if $P_N(x)$ is a trigonometric polynomial of degree N then its $n-th$ partial sum $S_n(P_N,x)$ *agrees* with $P_N(x)$ *for* all $n \geq N$.

(ii) Let us also recall that if $f \in L^2[-\pi,\pi]$ then by Pythagorous's theorem,

$$\|f - S_n(f,x)\|^2 + \|S_n(f,x)\|^2 = \|f\|^2 \tag{3.5}$$

We see that (3.4) will follow from (3.5) if we show that

$$\|f - S_n(f,x)\|^2 \longrightarrow 0, \quad n \to \infty.$$

Let us apply (3.5) to $f - P_N$ where P_N is a trigonometric polynomial with $n \geq N$:

$$\|f - P_N - S_n(f-P_N,x)\|^2 + \|S_n(f-P_N,x)\|^2 = \|f-P_N\|^2$$

Clearly $S_n(f-P_N,x) = S_n(f,x) - S_n(P_N,x) = S_n(f,x) - P_N$ so that

$$\|f - S_n(f,x)\|^2 + \|S_n(f-P_N,x)\|^2 = \|f-P_N\|^2$$

whereby we conclude that

$$\|f - S_n(f,x)\| \leq \|f - P_N\|, \quad n > N.$$

Now we have seen that given any $\epsilon > 0$ there is a trigonometric polynomial $P_N(x)$ such that $\|f(x) - P_N(x)\| < \epsilon$ and hence for $n > N$ we have that $\|f - S_n(f,x)\| < \epsilon$.

Kronecker's theorem and Weyl's equidistribution theorem: Let us recall a classical result due to *Leopold Kronecker* sometimes also known as *Dirichlet's theorem*. Suppose α is an irrational number. Consider the sequence of numbers

$$\{\alpha\}, \{2\alpha\}, \{3\alpha\}, \dots \tag{3.6}$$

where $\{\theta\}$ denotes the fractional part of θ namely $\{\theta\} = \theta - [\theta]$. The numbers in the list are all in $[0,1]$ and they are all distinct. Suppose

$$\{k\alpha\} = \{l\alpha\}, \quad k < l.$$

Then $(k-l)\alpha = [k\alpha] - [l\alpha]$ which would mean α is a ratio of two integers. Contradiction since α was assumed to be irrational. Thus the sequence (3.6) must have limit points in $[0,1]$. Kronecker's theorem asserts that the sequence (3.6) is dense in $[0,1]$. Let us provide a simple proof of this fact.