Fourier Analysis and its Applications

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III - Fejer's theorem and its applications

We shall now turn to the third type of convergence known as the *Cesaro-summability* of the partial sums of the Fourier series. Let us recall a simple result in real analysis - Cauchy's First limit theorem:

Theorem (Cauchy's first limit theorem): Suppose (b_n) is a sequence of real or complex numbers converging to l. Then the sequence of arithmetic means

$$\frac{1}{n}(b_1 + b_2 + \cdots + b_n)$$

also converges to l.

Proof: First let us do some preliminary algebra:

$$\Big|\frac{1}{n}(b_1 + b_2 + \cdots + b_n) - l\Big| = \frac{1}{n}\Big|(b_1 - l) + (b_2 - l) + \cdots + (b_n - l)\Big|$$

Let us now split the RHS into two pieces:

$$\Big|\frac{1}{n}(b_1 + b_2 + \cdots + b_n) - l\Big| \leq \frac{1}{n}\Big(|b_1 - l| + \cdots + |b_{n_1} - l|\Big) + \frac{1}{n}\Big(|b_{n_1+1} - l| + \cdots + |b_n - l|\Big)$$

This is true for any n_1 and $n > n_1$ but we shall specify the n_1 presently. However before that observe that since a convergent sequence is bounded, there is an $M > 0$ such that $|b_n - l| \leq M$ for all n. Hence

$$\Big|\frac{1}{n}(b_1 + b_2 + \cdots + b_n) - l\Big| \leq \frac{n_1 M}{n} + \frac{1}{n}\Big(|b_{n_1+1} - l| + \cdots + |b_n - l|\Big)$$

We now bring in the epsilons. Let $\epsilon > 0$ be arbitray. There is an n_1 such that $|b_n - l| < \epsilon/2$ for all $n \geq n_1$ with this n_1 the last equality implies:

$$\Big|\frac{1}{n}(b_1 + b_2 + \cdots + b_n) - l\Big| \leq \frac{n_1 M}{n} + \Big(\frac{n - n_1}{n}\Big)\frac{\epsilon}{2} < \frac{n_1 M}{n} + \frac{\epsilon}{2}, \quad n > n_1.$$

Now if we select n_0 such that $n_0 > n_1$ and also $n_0 > 2n_1 M/\epsilon$ then we get

$$\Big|\frac{1}{n}(b_1 + b_2 + \cdots + b_n) - l\Big| < \frac{n_1 M}{n} + \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2}, \quad n > n_0.$$

completing the proof.

Remark: The converse is obviously false. So the original sequence may *fail* to converge but the sequence of arithmetic means *may* converge. Thus the convergence of the sequence of arithmetic means may be regarded as some kind of a *generalization* of the notion of ordinary convergence. The theorem proved above is significant since it says that if the sequence converges in the ordinary sense it converges in the generalized sense and to the *same* limit. Let us now give a name to this generalized notion of convergence.

Definition (Cesaro convergence): A sequence (b_n) is said to converge in the *Cesaro sense* to l if

$$\lim_{n\to\infty} \frac{1}{n}\Big(b_1 + b_2 + \cdots + b_n\Big) = l.$$

In particular Cesaro convergence implies convergence as we have shown above. We have seen that if $f(x)$ is a $2\pi-$periodic continuous function then the sequence $S_n(f,x)$ of partial sums of the Fourier series of f *need not converge pointwise.* The *Theorem of Fejer* asserts that it does converge in the sense of Cesaro. Moreover the arithmetic means converge *uniformly!* We recall

Definition (uniform convergence): A sequence of functions $F_n(x)$ all defined on a common domain E is said to converge uniformly to $F(x)$ if

$$\sup_{x\in E} |F_n(x) - F(x)| \longrightarrow 0, \qquad \text{as } n \to \infty$$

We have already used this notion earlier to prove the Riemann Lebsegue lemma.

Exercises on Cesaro convergence

(1) Suppose that (b_n) is a monotone increasing/decreasing sequence of real numbers then the sequence of arithmetic means

$$\frac{1}{n}\Big(b_1 + b_2 + \cdots + b_n\Big)$$

is also increasing/decreasing.

(2) Suppose (b_n) is a sequence of positive real numbers converging to l, then the sequence of geometric means

$$g_n = \Big(b_1 \cdot b_2 \ldots b_n\Big)^{1/n}$$

also converges to l. Further, if $l > 0$ then the sequence of harmonic means

$$h_n = n\Big(b_1^{-1} + b_2^{-1} + \cdots + b_n^{-1}\Big)^{-1}$$

also converges to l.

(3) Deduce that if (b_n) is a sequence of *positive* real numbers such that b_{n+1}/b_n converges to l then $b_n^{1/n}$ also converges to l. This important result is known as *Cauchy's second limit theorem.*

(4) Deduce that $n^{-1}(n!)^{1/n}$ converges to $1/e$.

The last exercise can be regarded as a WEAK form of the celebrated *Stirling's approximation formula.* We now state the precise version:

Theorem (Stirling's approximation formula):

$$n! \sim n^n e^{-n}\sqrt{2\pi n}, \qquad \text{for } n >> 1.$$

Theorem (Fejer's theorem): Suppose $f(x)$ is a $2\pi-$periodic continuous function on the real line and $S_n(f, x)$ is the $n-$th partial sum of the Fourier series of $f(x)$ then

$$\lim_{n\to\infty} \frac{1}{n+1}(S_0(f,x) + S_1(f,x) + S_2(f,x) + \cdots + S_n(f,x)) = f(x) \tag{3.1}$$

uniformly.

Proof: We begin with the formula

$$S_j(f,x) = \int_{-\pi}^{\pi} D_j(x-t)f(t)dt \tag{3.2}$$

where $D_j(\theta)$ is the Dirichlet kernel: $D_j(\theta) = \frac{1}{2\pi}\sin(j\theta + \frac{\theta}{2})/\sin(\theta/2)$ We put $j = 0, 1, 2, \ldots, n$ and we find that

$$S_0(f,x) + \cdots + S_n(f,x) = \int_{-\pi}^{\pi} f(t)\tilde{K}_n(x-t)dt, \quad \text{where}$$
$$\tilde{K}_n(\theta) = \frac{1}{2\pi\sin(\theta/2)}\Big(\sin\frac{\theta}{2} + \sin\frac{3\theta}{2} + \cdots + \sin\frac{(2n+1)\theta}{2}\Big)$$

Multiply the numerator and denominator by $2\sin\frac{\theta}{2}$ we get

$$\begin{aligned}
\tilde{K}_n(\theta) &= \frac{1}{4\pi\sin^2(\theta/2)}\Big(2\sin^2\frac{\theta}{2} + 2\sin\frac{\theta}{2}\sin\frac{3\theta}{2} + \cdots + 2\sin\frac{\theta}{2}\sin\frac{(2n+1)\theta}{2}\Big) \\
&= \frac{1}{4\pi\sin^2(\theta/2)}\Big(1 - \cos\theta + \cos\theta - \cos 2\theta + \cdots + \cos n\theta - \cos(n+1)\theta\Big) \\
&= \frac{1-\cos(n+1)\theta}{4\pi\sin^2(\theta/2)}
\end{aligned}$$

The Fejer's kernel Let us not forget that we have to divide by $n+1$ since we are taking the average of $S_0, S_1, \ldots, S_n$: The function $K_n(\theta)$ given by

$$K_n(\theta) = \frac{1-\cos(n+1)\theta}{4(n+1)\pi\sin^2(\theta/2)} = \frac{\sin^2((n+1)\theta/2)}{2(n+1)\pi\sin^2(\theta/2)} \tag{3.3}$$

is called the *Fejer kernel*. It has the pleasant feature that it is *positive* and this will be significant in the proof of Fejer's theorem. It is clear that

$$\int_{-\pi}^{\pi} K_n(\theta)d\theta = 1 \tag{3.4}$$

since we have

$$\int_{-\pi}^{\pi} D_j(\theta)d\theta = 1, \qquad \text{for each } j = 0, 1, 2, \ldots, n.$$

Now we have proved

$$\begin{aligned}
\frac{1}{n+1}(S_0(f,x)+S_1(f,x)+\cdots+S_n(f,x)) &= \int_{-\pi}^{\pi} K_n(x-t)f(t)dt \\
&= \int_{-\pi}^{\pi} K_n(t)f(x-t)dt \\
\text{And,} \qquad f(x) &= \int_{-\pi}^{\pi} K_n(t)f(x)dt \\
\text{Subtracting, } \frac{(S_0(f,x)+\cdots+S_n(f,x))}{n+1} - f(x) &= \int_{-\pi}^{\pi} K_n(t)(f(x-t)-f(x))dt \\
\therefore \Delta_n(x) = \Big|\frac{(S_0(f,x)+\cdots+S_n(f,x))}{n+1} - f(x)\Big| &\leq \int_{-\pi}^{\pi} K_n(t)|f(x-t)-f(x)|dt
\end{aligned}$$

To show $\Delta_n(x) \to 0$, We must now bring in the epsilons and deltas in the argument ! Let $\epsilon > 0$ be arbitrary. By uniform continuity of $f(x)$ there is a $\delta > 0$ such that

$$|f(x-t)-f(x)| < \frac{\epsilon}{2}, \quad |t| < \delta.$$

Splitting the integral in the last slide we get

$$\begin{aligned}
\Delta_n(x) &= \int_{|t|<\delta} K_n(t)|f(x-t)-f(x)|dt + \int_{\delta\leq|t|\leq\pi} K_n(t)|f(x-t)-f(x)|dt \\
&\leq \frac{\epsilon}{2}\int_{|t|\leq\delta} K_n(t)dt + \int_{\delta\leq|t|\leq\pi} K_n(t)|f(x-t)-f(x)|dt \\
&\leq \frac{\epsilon}{2} + 2M\int_{\delta\leq|t|\leq\pi} K_n(t)dt
\end{aligned}$$

where M is the maximum of $|f|$ on $[-\pi,\pi]$. We now deal with the second integral. If $\delta \leq |t| \leq \pi$ then $\delta/2 < |t/2| \leq \pi/2$ and $\sin^2\delta/2 < \sin^2(t/2)$.

$$0 \leq K_n(t) = \frac{\sin^2((n+1)t/2)}{2(n+1)\pi\sin^2(t/2)} \leq \frac{1}{(n+1)\sin^2(\delta/2)}$$

So we get

$$\begin{aligned}
0 \leq \Delta_n(x) &\leq \frac{\epsilon}{2} + 2M\int_{\delta\leq|t|\leq\pi} K_n(t)dt \\
&\leq \frac{\epsilon}{2} + \frac{4M\pi}{(n+1)\sin^2(\delta/2)}
\end{aligned}$$

Now we select $n_0 > 8M\pi/(\epsilon\sin^2(\delta/2))$ then for all $n > n_0$, we have $0 \leq \Delta_n(x) < \epsilon$.

Applications of Fejer's theorem An immediate corollary of Fejer's theorem is the following:

corollary The set of all trigonometric polynomials is dense in the space of $2\pi-$periodic continuous functions in the following sense. Given a $2\pi-$periodic continuous function f, for every $\epsilon > 0$ there is a trigonometric polynomial $P_n(x)$ such that

$$\sup_{|x|\leq\pi} |f(x) - P_n(x)| < \epsilon/\sqrt{2\pi}.$$

Exercise: Show that $\|f(x) - P_n(x)\| < \epsilon$. From continuous functions we can easily pass over to functions in $L^2[-\pi, \pi]$.