Fourier Analysis and its Applications

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11 The Poisson kernel

We still have to show that the solution obtained in the last slide does attain the value $f(\theta)$ on the boundary. Here are some exercises

Exercises:

1. Show that the Poisson kernel is non-negative and
$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\Pi_r(\theta-t)dt = 1. \tag{2.19}$$

2. Show that
$$\lim_{r\to 1-}|u(re^{i\theta})-f(\theta)| = 0. \tag{2.20}$$
Hint: Write $f(\theta)$ as the integral w.r.t over $[-\pi,\pi]$ of $f(\theta)\Pi_r(\theta-t)/(2\pi)$. Now let $\epsilon>0$ and I be an interval of length ϵ centered at θ. The integral over I is small for one reason and the integral over $[-\pi,\pi]-I$ is small for a different reason. We shall return to these shortly.

Poisson formula for a ball There is a corresponding result for the ball in $\mathbb{R}^3$ but to derive that we need to spend a little time with associated Legendre equations.

Theorem: Suppose given a continuous function $f(\mathbf{x})$ on the unit ball B centered at the origin in $\mathbb{R}^3$ then the solution of the boundary value problem
$$\Delta u = 0 \text{ on } B, \quad u\Big|_{\partial B} = f$$
is given by
$$u(\mathbf{y}) = \int_{\partial B}\frac{(1-\|\mathbf{x}\|^2)f(\mathbf{x})dS(\mathbf{x})}{(1+\|\mathbf{x}\|^2-2\|\mathbf{x}\|\cos\alpha)^{3/2}}, \tag{2.21}$$
where α is the angle between $\mathbf{x}$ and $\mathbf{y}$.

For the proof see page 180 of *G. B. Folland, Fourier Analysis and its applications, Amer. Math. Soc, Indian Reprint by Universities Press, New Delhi, 2012.*

Solving the heat equation We discuss this as series of solved exercises:

(i) Let $u(x,t)$ be a smooth solution of the heat equation in the upper half-plane $t\geq 0$ such that $u(x,t)=u(x+2\pi,t)$ for all $x\in\mathbb{R},\ t\geq 0$. Show that the energy $\int_{-\pi}^{\pi}(u(x,t))^2dx$ is a monotone decreasing function of time. Prove the same result if the integral is over any interval of length 2π. Deduce that a smooth $2\pi-$periodic solution of the heat equation with a given initial condition, if it exists, is unique.

Solution: Call the integral $E(t)$ which really represents the *Energy* and the method itself is called the *Energy Method* .
$$E'(t) = 2\int_{-\pi}^{\pi}u(x,t)u_t(x,t)dx = 2\int_{-\pi}^{\pi}u(x,t)u_{xx}(x,t)dx$$
Integrate the last equation by parts and we get $E'(t) = -2\int_{-\pi}^{\pi}(u_x(x,t))^2dx \leq 0$.

So we get

$$0 \leq E(t) \leq E(0).$$

Now if the intial value $u(x, 0) = 0$ then $E(0) = 0$ and so $E(t) = 0$ for all $t \geq 0$ and we infer that the solution is identically zero. From this the uniqueness follows.

We now turn to existence.

(ii) Suppose that $f(x)$ is a $2\pi-$ continuous periodic function. Further conditions will be imposed as we go along. The solution of $u_t = u_{xx}$ is sought in the form

$$u(x,t) = a_0(t) + \sum_{n=1}^{\infty} (a_n(t) \cos nx + b_n(t) \sin nx) \tag{2.22}$$

Let us substitute (2.22) into the heat equation $u_t = u_{xx}$ and we see that

$$a_0'(t) = 0, \quad a_n'(t) = -n^2 a_n(t), \quad b_n'(t) = -n^2 b_n(t) \tag{2.23}$$

$$a_0(t) = \alpha_0, \quad a_n(t) = \alpha_n e^{-n^2 t}, \quad b_n(t) = \beta_n e^{-n^2 t}.$$

Let us put $t = 0$ in equation (2.22) and we see that

$$f(x) = \alpha_0 + \sum_{n=1}^{\infty} (\alpha_n \cos nx + \beta_n \sin nx) \tag{2.24}$$

which is the Fourier expansion of the function $f(x)$ and can be determined easily since $f(x)$ is already prescribed. So finally the solution $u(x, t)$ assumes the form:

$$u(x,t) = \alpha_0 + \sum_{n=1}^{\infty} \exp(-n^2 t)(\alpha_n \cos nx + \beta_n \sin nx) \tag{2.25}$$

We need to check that as $t \to 0+$ the series displayed in (2.25) does converges to (2.24). The problem amounts to a passage across the summation sign of a limiting operation namely

$$\lim_{t \to 0+} \sum_{n=1}^{\infty} (\dots) = \sum_{n=1}^{\infty} \lim_{t \to 0+} (\dots)$$

We shall not carry out this program in detail but observe that at least when $f(x)$ is continuous and *piecewise smooth,* the Fourier series converges uniformly and so passage to limit across the summation sign is valid. See page 203 of the book of *Bachman-Narici* already cited earlier.

So we have established the existence of the solution for continuous piecewise smooth intial conditon.

Exercise: Determine the solution of the heat equation with intial condition $f(x) = \pi^2 - x^2$ on $|x| \leq \pi$ extended as a $2\pi-$periodic function on the entire real line.

Abel summability: Let (a_n) be a sequence of complex numbers. Recall Abel's limit theorem from elementary analysis:

Theorem: If $\sum_{n=0}^{\infty} a_n$ converges then the function $f(t)$ given by

$$f(t) = \sum_{n=0}^{\infty} a_n t^n$$

which is holomorphic on $|t| < 1$ (why?) satisfies

$$\lim_{t \to 1-} f(t) = \sum_{n=0}^{\infty} a_n \tag{2.26}$$

where, the limit is taken as t varies over the real axis.

The theorem is non trivial because in the equation (2.26)

$$\text{LHS} = \lim_{t \to 1-} \left(\sum_{n=0}^{\infty} a_n t^n \right) = \lim_{t \to 1-} \lim_{N \to \infty} \left(\sum_{n=0}^{N} a_n t^n \right)$$

whereas

$$\text{RHS} = \lim_{N \to \infty} \sum_{n=0}^{N} a_n = \lim_{N \to \infty} \lim_{t \to 1-} \left(\sum_{n=0}^{N} a_n t^n \right)$$

So the result is about the validity of an exchange of limits ! We say that the series

$$\sum_{n=0}^{\infty} a_n \tag{2.27}$$

is Abel summable if the associated power series

$$\sum_{n=0}^{\infty} a_n t^n \tag{2.28}$$

has radius of convergence 1 and further, if $f(t)$ is the sum function the following limit

$$\lim_{t \to 1-} f(t)$$

exists. This limit is then defined to be the Abel sum of the series $\sum a_n$. It is important that the limit is taken as t varies along the real axis. This condition can be relaxed somewhat (*Stolz region*)

Abel summability and Fourier series Let us now consider a *continuous* $2\pi-$periodic function on the real line with Fourier series

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \tag{2.29}$$

We know that the Fourier series (2.29) may not converge pointwise but a nice analogue of Abel-summability holds. Let us consider the associated series

$$a_0 + \sum_{n=1}^{\infty} r^n (a_n \cos nt + b_n \sin nt) \tag{2.30}$$

The series (2.30) converges for $0 \le r < 1$ and denoting by $u(re^{it})$ its sum, it is of interest to know whether for a fixed $\theta \in \mathbb{R}$,

$$\lim_{(r,t)\longrightarrow(1,\theta)} u(re^{it}) = f(\theta). \tag{2.31}$$

Recall the Poisson kernel:

$$\Pi_r(s) = \frac{(1-r^2)}{1+r^2-2r\cos(s)}$$

We already have the formula for $u(re^{it})$ namely,

$$u(re^{it}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Pi_r(s-t)f(s)ds = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t-s)\Pi_r(s)ds \tag{2.18}$$

Also,

$$f(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\Pi_r(s)ds.$$

Subtracting from (2.18), we get

$$u(re^{it}) - f(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi}(f(t-s)-f(\theta))\Pi_r(s)ds$$

We exploit the fact that the Poisson kernel is a *positive kernel*:

$$|u(re^{it}) - f(\theta)| \le \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(t-s)-f(\theta)|\Pi_r(s)ds \tag{2.32}$$

Let $\epsilon > 0$ be arbitrary. Uniform continuity of f gives a δ so that

$$|f(x)-f(y)| < \frac{\epsilon}{2}, \qquad \text{for } |x-y| < 2\delta.$$

Break the integral (2.32) in two pieces:

(a) Integral over $[-\delta, \delta]$. On this interval we have for $|t-\theta| < \delta$,

$$\frac{1}{2\pi}\int_{-\delta}^{\delta}|f(t-s)-f(\theta)|\Pi_r(s)ds < \frac{\epsilon}{2}\int_{-\pi}^{\pi}\frac{\Pi_r(s)}{2\pi}ds = \frac{\epsilon}{2}. \tag{2.32$'$}$$

We have used the fact that the integral of $\Pi_r(s)$ over $[\pi, \pi]$ is 2π.

(b) For the integral over $|s| > \delta$ we argue as follows. Let M be an upper-bound for $|f(s)|$.

$$\frac{1}{2\pi}\int_{|s|>\delta}|f(\theta-s)-f(\theta)|\Pi_r(s)ds < \frac{M}{\pi}\int_{|s|>\delta}\Pi_r(s)dt \tag{2.32$''$}$$

Now we observe that on $|s| > \delta$,

$$0 \le \Pi_r(s) \le \frac{1-r^2}{1+r^2-2r\cos\delta} \longrightarrow 0, \qquad \text{as } r \to 1.$$

So for a suitable $\eta > 0$ and $1-\eta < r < 1$, we have

$$\Pi_r(s) < \frac{\epsilon}{4M}$$

and the integral $(2.32)''$ over $|s| \geq \delta$ is again less than $\epsilon/2$. Consolidating $(2.32)'$ and $(2.32)''$, we get for $|t - \theta| < \delta$ and $1 - \eta < r < 1$, the estimate

$$|u(re^{it}) - f(\theta)| < \epsilon.$$

and (2.31) follows namely,

$$\lim_{(r,t) \longrightarrow (1,\theta)} u(re^{it}) = f(\theta). \tag{2.31}$$

Summarizing,

Theorem: Suppose $f(t)$ is a continuous $2\pi-$periodic function on the real line with Fourier series (2.29) then, for fixed $\theta \in \mathbb{R}$, the function $u(re^{it})$ which is the sum of (2.30) converges to $f(e^{i\theta})$ as $(r, t) \longrightarrow (1-, \theta)$.