Fourier Analysis and its Applications

Prof. G. K. Srinivasan
Department of Mathematics
Indian Institute of Technology Bombay

10 Dirichlet problem for a disc

Tying up some loose ends In case you haven't already figured it out !

(a) First the area formula (2.9). Let C be a smooth simple closed curve traced counter-clockwise. Let us apply Green's theorem:

$$\oint Pdx + Qdy = \iint \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)dxdy$$

Taking $Q = x$ and $P = 0$ we get the area formula (2.9).

(b) Recall that if $\gamma : [a,b] \longrightarrow \mathbb{R}^2$ is a regular smooth curve, its arc-length $s(t)$ is given by

$$\frac{ds}{dt} = |\dot{\gamma}(t)|$$

and hence

$$\Big(\frac{dx}{dt}\Big)^2 + \Big(\frac{dy}{dt}\Big)^2 = \Big(\frac{ds}{dt}\Big)^2$$

Now use the relation between s and t in the above proof of isoperimetric theorem.

(c) Now, if $f(t)$ and $g(t)$ are both in $L^2[-\pi,\pi]$ taking real values, with Fourier coefficients $\alpha_0, \alpha_n, \beta_n$ and $\gamma_0, \gamma_n, \delta_n$ $(n = 1, 2, 3, \dots)$ then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t) \pm g(t)|^2 dt = (\alpha_0 \pm \gamma_0)^2 + \frac{1}{2}\sum_{n=1}^{\infty}\Big((\alpha_n \pm \gamma_n)^2 + (\beta_n \pm \delta_n)^2\Big)$$

Subtracting the two expressions now gives

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)g(t)dt = \alpha_0\gamma_0 + \frac{1}{2}\sum_{n=1}^{\infty}\Big(\alpha_n\gamma_n + \beta_n\delta_n\Big)$$

Now take $f(t) = x(t)$ and $g(t) = y'(t)$ and noting that for a smooth function, the Fourier series can be differentiated term by term to yield the Fourier series for the derived function.

Maximum Modulus Theorem in complex analysis As a second application of Parseval formula, we prove the important *maximum modulus theorem* in complex analysis.

Theorem (MAx. Mod. Theorem): Suppose f is non-constant holomorphic function on a connected domain Ω in the complex plane and $|f|$ is bounded in Ω then the supremum of $|f|$ cannot be attained at any point of Ω. *Assume that the maximum modulus is attained at a point in* Ω which we may assume without loss of generality to be the origin. Let us consider the closed disc D of radius R centered at the origin and contained in Ω. The power series for f converges absolutely and uniformly in D:

$$\begin{aligned} f(z) &= a_0 + a_1 z + a_2 z^2 + \dots, \quad |z| \leq R. \\ &= a_0 + a_1 r e^{i\theta} + a_2 r^2 e^{2i\theta} + \dots, \quad 0 \leq r \leq R,\ 0 \leq \theta \leq 2\pi. \end{aligned}$$

We recognize here a Fourier series of the smooth 2π periodic function $\theta \mapsto f(re^{i\theta})$. For a fixed r with $0 < r \leq R$ we compute:

$$\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \sum_{m,n=0}^{\infty} a_m \overline{a_n} \int_0^{2\pi} e^{i(n-m)\theta} d\theta = 2\pi \sum_{n=0}^{\infty} |a_n|^2$$

Question: How to justify the exchange of summation and integration?

What we see in this displayed equation is exactly the Parseval formula for the function $\theta \mapsto f(re^{i\theta})$. Now by our assumption,

$$|f(re^{i\theta})| \leq |f(0)| = |a_0|, \quad 0 \leq \theta \leq 2\pi,\ 0 < r \leq R.$$

We infer,

$$2\pi \sum_{n=0}^{\infty} |a_n|^2 = \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta \leq 2\pi |a_0|^2.$$

forcing $a_1 = a_2 = a_3 = \cdots = 0$. This implies f is constant on Ω (how?). Contradiction.

Application to PDEs: Laplace equation on a disc: We shall now apply the theory of Fourier series for solving some classical PDEs. We take up the Dirichlet problem for the Laplace equation on the unit disc $D = \{(x, y) \ :\ x^2 + y^2 \leq 1\}$. The problem seeks a twice continuously differentiable function u such that

$$\Delta u = 0, \quad \text{on } D, \quad u(\cos\theta, \sin\theta) = f(\theta), \tag{2.11}$$

where we assume that f is Lipschitz and 2π periodic on $\mathbb{R}$. First we write the equation in polar coordinates. Recall that

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}. \tag{2.12}$$

So the PDE becomes:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0. \tag{2.13}$$

Seeking special solutions in the form $u(x, y) = v(r)g(\theta)$ where $g(\theta)$ is 2π periodic,

$$(r^2 v'' + rv')/v = -g''/g$$

Since g is a function of θ alone and v is a function of r alone, either side must be a constant say k^2 and we get the pair of ODEs

$$r^2 v''(r) + rv'(r) - k^2 v(r) = 0, \quad g''(\theta) + k^2 g(\theta) = 0.$$

These have solutions

$$v(r) = Ar^k + Br^{-k}, \quad g(\theta) = C\cos k\theta + D\sin k\theta.$$

Now since $g(\theta)$ is 2π-periodic, we must have $k \in \mathbb{Z}$ (how?). Also since the solution is continuous at the origin, $k \geq 0$. Thus we get the solution in the form

$$u(r\cos\theta, r\sin\theta) = a_0 + \sum_{n=1}^{\infty} r^n (a_n \cos n\theta + b_n \sin n\theta) \tag{2.14}$$

To determine the coefficients of this Fourier expansion we must use the boundary condition. Setting $r = 1$ we get

$$f(\theta) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) \tag{2.15}$$

from which we deduce the values of $a_0, a_1, \dots$ and $b_1, b_2, \dots$.

Exercises:

1. Determine the solution of the Laplace's equation in the unit disc with the prescribed boundary value $|\sin\theta|$.

2. Show that if u is a harmonic function (that is $\Delta u = 0$) then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} u(r\cos\theta, r\sin\theta)d\theta = u(0,0). \tag{2.16}$$

   This is called the *mean value theorem* for harmonic functions.

The Poisson kernel: Let us continue with the formula obtained in the last slide:

$$u(re^{i\theta}) = a_0 + \sum_{n=1}^{\infty} r^n(a_n\cos n\theta + b_n\sin n\theta) \tag{2.14}$$

But we know the formula for these Fourier coefficients. Inserting these we get

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt + \frac{1}{\pi}\sum_{n=1}^{\infty} r^n\Big(\cos n\theta\int_{-\pi}^{\pi} f(t)\cos nt dt + \sin n\theta\int_{-\pi}^{\pi} f(t)\sin nt dt\Big)$$

Since the integrals decay to zero by Riemann Lebesgue lemma, it is easy to justify exchange of sum and integral (with $0 \le r < 1$):

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\Big(1 + 2\sum_{n=1}^{\infty} r^n(\cos n\theta\cos nt + \sin n\theta\sin nt)\Big)f(t)dt \tag{2.17}$$

Exercise:

$$1 + \sum_{n=1}^{\infty} 2r^n\cos ns = \frac{1-r^2}{1+r^2-2r\cos s}$$

and we get the result

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{(1-r^2)f(s)ds}{1+r^2-2r\cos(\theta-s)} \tag{2.18}$$

The expression $\Pi_r(\theta-t) = (1-r^2)/(1+r^2-2r\cos(\theta-t))$ is called the *Poisson Kernel.*