Fourier Analysis and its Applications

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01 Genesis and a little history

Dieses Buch widme ich meiner Frau in Dankbarkeit

I - Trigonometric series and Fourier series

The subject is popular in many undergraduate programs and no doubt you have seen some elementary presentations of Fourier series in the context of solving *boundary Value Problems* for the Laplace, heat and wave equations for instance. The main issue is to

Expand a "fairly general function f(x)" as a series of sines and cosines !

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \tag{1.1}$$

§1.1 Nascent stages of development of the theory A very important historical source is the book of *Ivor Gratten-Guinness, Convolutions in French Mathematics. 1800-1840, Volume 2, The Turns, Birkhauser Verlag,, Basel, 1990.* Let us look into a few aspects during the nascent stages of the development of this theory. Page references refer here to this book.

1. Attempts at expressing a "fairly general function $f(x)$ as a series of the form (1.1) predate Fourier by at least fifty years in the works of Daniel Bernoulli and Euler (p. 598).

2. Perplexing issue: how can a series of $2\pi-$periodic functions produce a general "apriori non-periodic" function? The problem seems to have *languished in this state until the time of Fourier* .

3. Fourier's researches were persuasive in advocating the method of separation of variables (so commonly found in modern undergraduate courses!) in PDEs

4. The method hitherto was unpopular inasmuch as the solution presents itself in a form in which the boundary conditions remain masked in the coefficients of the expansion (1.1)

5. In contrast the functional form was preferred such as Poission's integral for the solutions of the Laplace's equation of D'Alembert's solution of the wave equation in one space dimension. (pp. 595, 600).

6. Fourier obtained interesting series expansions such as (p. 596)

$$\frac{\pi}{4} = \sum_{N=0}^{\infty} \frac{(-1)^n \cos((2N+1)x)}{2N+1}, \quad 0 < x < \pi, \tag{1.2}$$

7. Convergence issues: Fourier was aware of these but established rigorously the convergence of (1.2) and likewise issues pertaining to completeness also cropped up (p. 596).

8. Besides trigonometric functions, Fourier also considers summands with Bessel functions (p. 605). This point was acknowledged by Lommel and Heine proposed the name Fourier-Bessel expansions for these.

9. **L. Dirichlet, C. Jordan and Paul du Bois Reymond** The first significant step towards a general convergence theorem is due to L. Dirichlet in 1829 (p.1131). Pointwise convergence at points of continuity for functions that are piecewise continuous and monotone. A precise formulation of this result will appear later. Dirichlet was of the view that the failure of pointwise convergence would stem from some lack of integrability of the funtion whose expansion is being sought.

10. Jordan generalized the results of Dirichlet to include functions of bounded variation. We shall define this class precisely later.

11. In the year 1873 Paul du Bois Reymond produced an example of a continuous function whose Fourier series failed to converge (See Kahane p.33).

Some twentieth century milestones: In 1926 A. N. Kolmogorov found a function in $L^1[-\pi, \pi]$ whose Fourier series diverges EVERYWHERE and N. N. Luzin conjectured that the Fourier series of a function in $L^2[-\pi, \pi]$ must converge pointwise almost everywhere. The conjecture of Luzin was finally settled in 1966 by Lennert Carleson, later generalized by Hunt for functions in $L^p[-\pi, \pi]$ with $p > 1$.

Work of B. Riemann: After Dirichlet the study of Fourier series was significantly furthered by B. Riemann in his *Habilitationschrift* (1854). In 1859 his epoch making memoir on the distribution of primes and the zeta funtion appeared where he gives a proof of the famous functional equation that bears his name using Fourier analysis. The theta function identity can be established by solving the heat equation in two ways (using Fourier series and Fourier transforms and comparing the results). The theta function identity is closely related to the functional equation for the Riemann zeta function.

Pervasive nature of Fourier analysis: The theory of Fourier series and Fourier transforms has been profoundly generalized and its use extends far beyond the purpose for which it was originally invented namely, solving partial differential equations.

Fourier analysis is today an indispensible tool in several diverse areas of mathematics such as Probability theory, Number theory and Geometry to name a few. Besides the book of I. Grattan-Guinness, the authoritative account of J.-P. Kahane and P.-G. Lemarié Rieusset ought to be consulted for any serious understanding of the historical developments of the subject. Chapter 3 of this work is devoted to Riemann's work.

J.-P. Kahane and P.-Gilles Lemarié-Rieusset, Fourier Analysis and Wavelets, Gordon and Breach Publishers, Luxemburg, 1995.

Impact of Fourier analysis in the late 19th and early 20th centuries: Fourier Analysis provided much impetus for the development of point-set topology as well as the development of measure theory. We cite here two important references that provide details on this

1. *A. Kechris, set theory and the uniqueness of trigonometric series, Lecture notes, Pasadena.* Reading this would be an excellent project for a student at the Master's level.

2. *R. Cooke, Uniqueness of trigonometric series and descriptive set theory,*1870-1985, Archive of the history of exact sciences, **45** (1993) 281-234.

I - The basic notions. Plan of the chapter

In this first chapter we focus on the following:

1. We look at the class of $2\pi-$periodic functions on the real line that are *Lipschitz* continuous and *Hölder* continuous of exponent α.
2. We prove the fundamental *Riemann-Lebesgue lemma* for functions in $L^1[-\pi, \pi]$.
3. We derive the *Dirichlet kernel* and examine the singularity of the kernel and understand the obstacle to proving pointwise convergence. When the function is merely continuous the argument breaks down whereas a little added regularity such as Hölder continuity enables us to establish a basic pointwise convergence theorem.
4. A simple example of a Lipschitz continuous function on $[-\pi, \pi]$ is $\cos ax$ and since it is an even function its $2\pi-$periodic extension is Hölder continuous on the real line. The basic convergence theorem we establish would give us several beautiful identities already!

We get the following

$$\cos ax = \frac{\sin \pi a}{\pi a} + \frac{2a \sin a\pi}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n \cos nx}{a^2 - n^2}$$

in particular putting $x = 0$ we get the partial fraction expansion or *Mittag-Leffler expansion* for $\operatorname{cosec}(\pi a)$ namely,

$$\operatorname{cosec}(\pi a) = \frac{1}{\pi a} + \frac{2a}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{a^2 - n^2}$$

and similar expansions for $\cot(\pi a)$ and other such functions. We shall discuss a special case of the *Poisson summation formula* for the Gaussian and derive from it the *theta function identity* of Jacobi. As mentioned earlier this is equivalent to Riemann's functional equation for the zeta function but we shall instead, if time permits, sketch a proof due to G. H. Hardy a proof of the functional equation directly from a Fourier series. Some details here will have to be omitted.

This will be followed by a discussion on *Bernoulli numbers* and special values of zeta functions.

We close the chapter with a discussion of the generating functions for *Bessel functions* of the first kind $J_n(x)$ and derive the *Schlömilch formula* and the integral representation for $J_n(x)$. These will be used in a later to give an application to a problem in celestial mechanics.

Chapter 1: The basic convergence theorem

To set the stage, let us begin with a 2π periodic function $f : \mathbb{R} \longrightarrow \mathbb{R}$ which at present is quite arbitrary except that its integral over $[-\pi, \pi]$ exists. For technical reasons it is important to work with *Lebesgue integrals* and so we assume that $f \in L^1[-\pi, \pi]$.

The objective is to express f as an infinite series:

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \tag{1.1}$$

A series of the form (1.1) is called a *trigonometric series.* The fundamental question is regarding the meaning of equality in (1.1). To list some of the alternatives, let $S_n(f, x)$ denote the $n-$th partial sum of the series (1.1).

Modes of convergence of a series

(i) We say the series (1.1) converges to $f(x)$ *pointwise almost everywhere* if $S_n(f, x) \longrightarrow f(x)$ as $n \to \infty$ for all x except on a set of Lebesgue measure zero.

(ii) We say the series (1.1) converges to $f(x)$ *in mean* if $\|S_n(f, x) - f(x)\| \longrightarrow 0$ as $n \to \infty$ where,

$$\|g(x)\| = \Big(\int_{-\pi}^{\pi} |g(x)|^2 dx\Big)^{1/2}$$

(iii) We say that the series (1.1) *Cesaro converges* to $f(x)$ is the sequence of arithmentic means

$$\frac{1}{n}\Big(S_1(f, x) + S_2(f, x) + \cdots + S_n(f, x)\Big) \longrightarrow f(x)$$

as $n \to \infty$.