An Introduction to Point-Set-Topology– Part II
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Lecture 08
Local Compactness-continued

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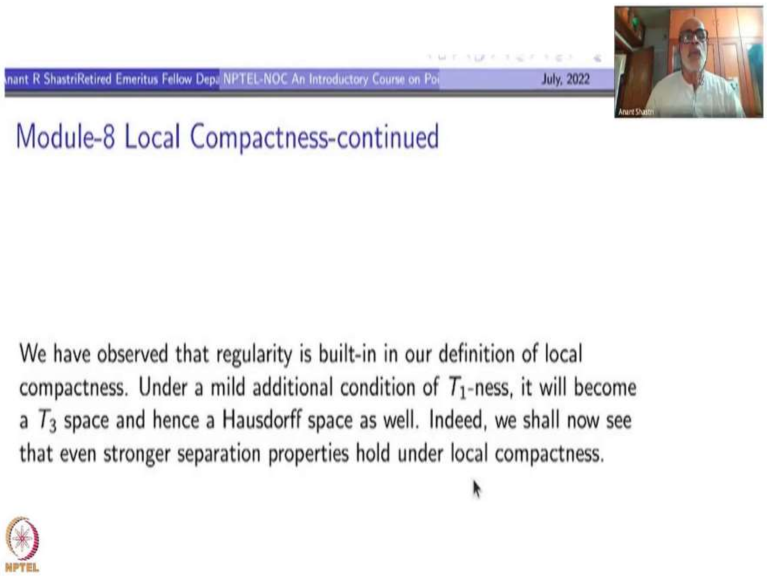

Welcome to module 8 of NPTEL NOC course on Point Set topology part-II. So, we shall continue the study of local compactness. We have already observed that in our definition of local compactness, regularity is built in. Under certain mild additional conditions like T_1-ness, it will be a T_3 space and then it will be Housdorff space also.

So, what we would like to see is that under this local compactness, many stronger separation properties are possible. So, here is an illustration. Let us begin with that.

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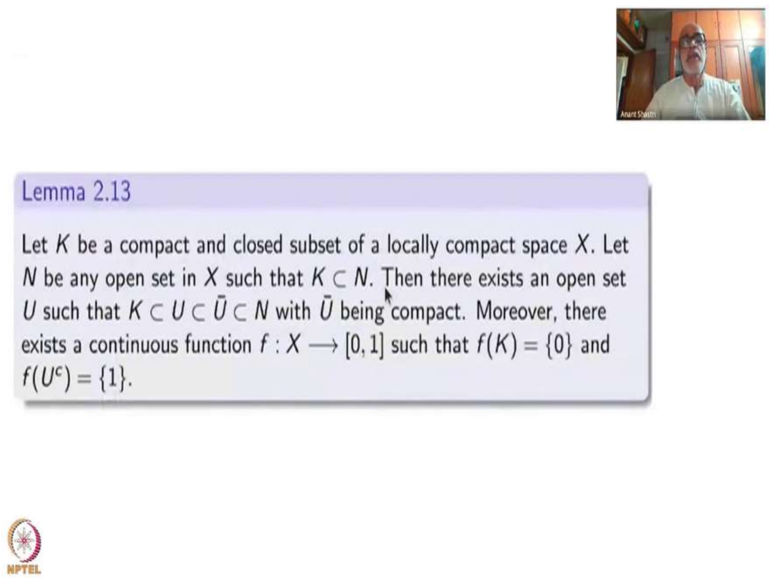

The first this is a lemma here: Take a compact and close subset K of a locally compact space X. Let N be an open subset in X which contains K. Then there exists an open set U such that K is contained in U and $\bar{U}$ is contained inside N and $\bar{U}$ is compact.

Moreover, it even stronger than this here viz., the separation is actually obtained from a function, namely there is a continuous function f from X to $[0,1]$ such that $f(K)$ is $\{0\}$ and $f(U^c)$ is $\{1\}$. You started with an open neighbourhood N of a compact and closed set, so, we could have stated that $f(N^c)$ is contained in $\{1\}$. What we have stated is correct and actually follows easily from this.

So, how do you perceive this? We have seen earlier that local compactness ensures that points and closed subsets can be separated by open sets, which means regularity. But now from a point, we are extending the same conclusion to closed compact sets. That is the first thing. Not only that, you can actually get functions of the above type to do the job. So, proof is not at all difficult.

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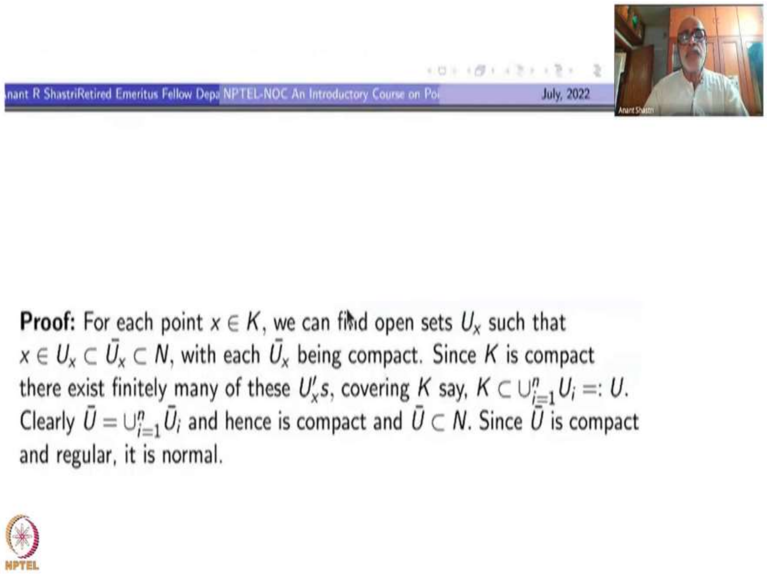

Let us go through the proof. For each point $x \in K$, by local compactness, you can find open sets U_x such that $x \in U_x$, is in $\bar{U_x}$ and that is in N. This is pointwise conclusion is from local compactness.

Since K is compact, if you take all the U_x as x varies in K, that will be form an open cover for K. Therefore there is a finite subcover. That means our K is contained inside some finite union of these members, so I will call them $U_1, U_2, \ldots, U_n$. I will take U to be their union. Each $\bar{U_i}$ contained inside N. Therefore, $\bar{U}$ which is the union of $\bar{U_i}$ (because it is a finite union) that is also contained inside N. Moreover it being a finite union of compact sets, $\bar{U}$ is compact also.

So, what I have? K contained inside U contained inside $\bar{U}$ contained inside N, with $\bar{U}$ compact. So, first part is over.

Now see $\bar{U}$ is compact. We are working inside a locally compact space X and $\bar{U}$ is a closed subset. So, it is locally compact also and hence it is regular also. Therefore, it is normal also.

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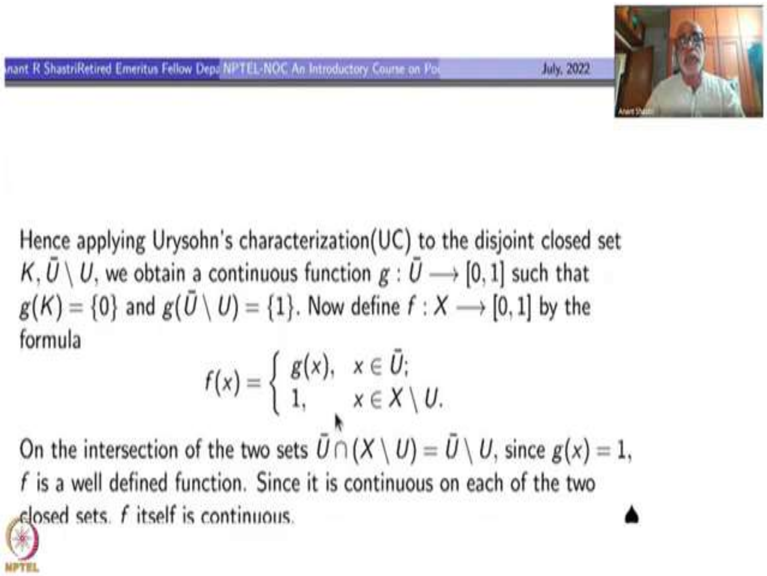

Therefore, here you can apply the Urysohn's characterization of normal spaces to this K and $\bar{U} \setminus U$. U is an open subset $\bar{U}$. Therefore K and $\bar{U} \setminus U$ are disjoint closed subset of $\bar{U}$. You get a continuous function g from $\bar{U}$ to $[0,1]$ such that $g(K)$ is $\{0\}$ and $g(\bar{U} \setminus U)$ is $\{1\}$. So, this is the standard Urysohn's characterization which you have seen in the part I, of any normal spaces. Any pair of disjoint closed subsets of a normal space can be separated by a continuous function itself like this. That is what we have used here.

Now, you define f from X to $[0,1]$, by the formula $f(x) = g(x)$ inside $\bar{U}$ and outside $\bar{U}$, I will extend it by identically the constant 1. Infact, on entire of $X \setminus U$, I can put it equal to 1. Therefore, it will agree on the intersection of these two closed subsets. And on $\bar{U}$ it is g and hence is continuous and on this $X \setminus U$, it is constant 1, that is also continuous. Therefore, f is continuous from X to $[0,1]$.

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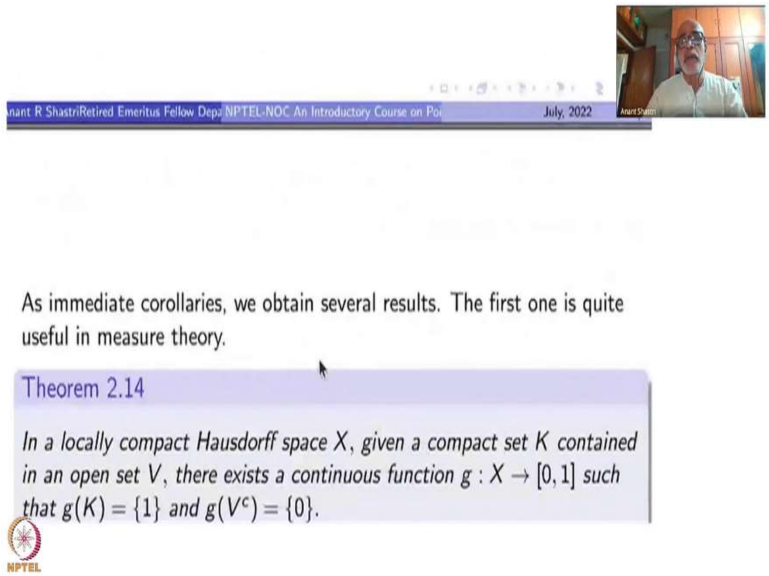

As an immediate corollary, we obtain several interesting results. The first one is quite useful in measure theory. In a locally compact Hausdorff space, given a compact set K contained inside an open subset V, there exist a continuous function g from X to $[0,1]$ such that $g(H)$ is $\{1\}$, $g(V^c)$ is $\{0\}$.

By interchanging g to $1-g$, you can easily interchange the conclusion as well. That is always possible. That should not confuse you.

So, what it says is in a locally compact Hausdorff space, given an open set V containing a compact set, we have a continuous function g such that $g(K) = \{1\}$ and $g(V^c) = \{0\}$.

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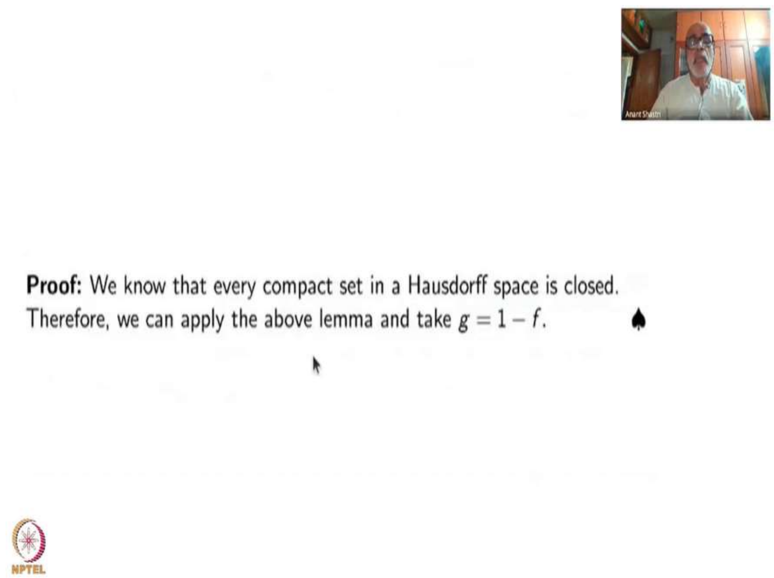

All that I do is to use the fact that every compact set in a Hausdroff space is a closed set. So, therefore, we can apply this lemma, that is all. So, this is just a restatement of that part of lemma in a special case that is all. So, this is what we get.

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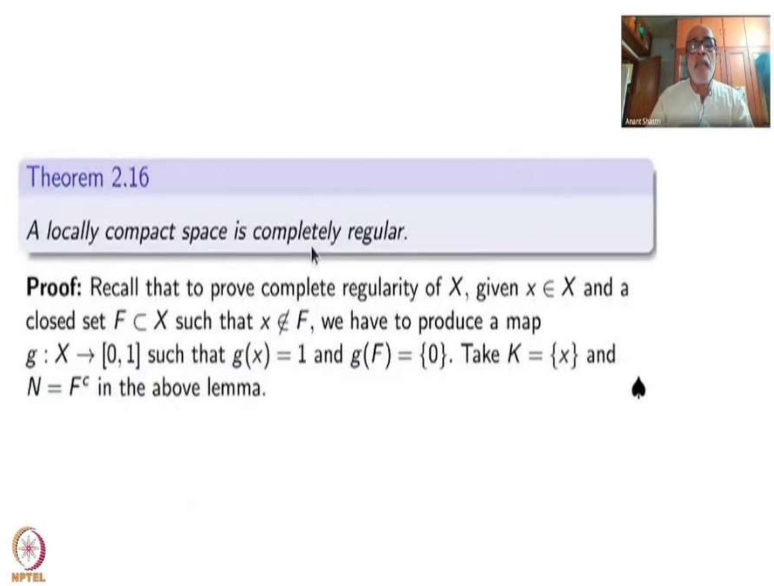

By the way, we will later on prove the same thing same result in a different way also, different proof also we will give. Also, it should be noted that with a little bit of effort, the hypothesis that K is closed can be removed.

Next, a locally compact space is completely regular. This is another corollary. Why I am calling this is a corollary because they are easy consequences of our lemma. A locally compact space is completely regular. Regularity we have already seen, it is built-in in our definition. Now, complete regularity means what, given any point in x and the closed subset F outside, x must be outside F, $\{x\}$ is disjoint from F, we have to produce a continuous function g from X to $[0, 1]$ such that $g(x)$ is 1, $g(F)$ is $\{0\}$. So, apply the previous lemma taking K equal to $\{x\}$ and N equal to complement of F. Starting with x and F as above, get an open set U_x such that x is in U_x and U_x is compact and is contained in F^c. Now apply the lemma with $K = \bar{U_x}$ and N equal to F^c. Over. So, locally compact spaces are completely regular.

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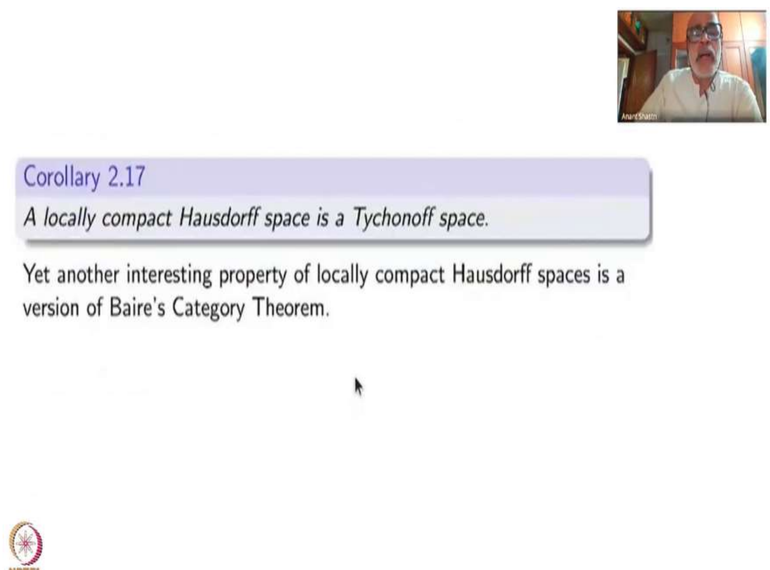

So, this is what we meant by saying that there is more separation in locally compactness than which is obvious from the definition, namely the built-in property was regularity, here now we have got complete regularity of very nature free compact set an open subset of that one can be separated and so on.

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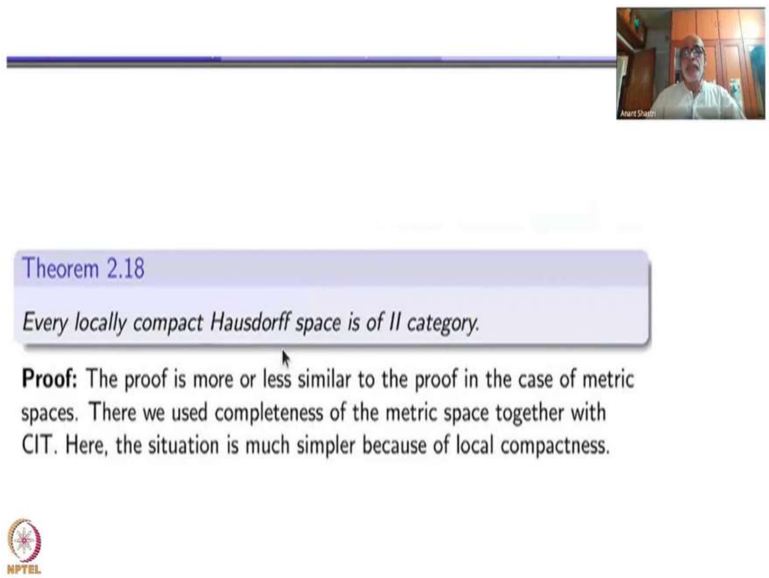

Let us go ahead with this one. A locally compact Hausdroff space, if you add Hausdorffness or just put T_1ness, it becomes a Tychonoff space. Why? Because we already we have seen that it is completely regular. Plus T_1 is, by definition Tychnoff space. That is all.

Now, we shall go to a different kind of result for locally compact spaces. So, local compactness it sits inside in the juncture of so many different kinds of concepts in topology, so, this is what we will try to see. So, next thing is the Baire’s Category type result for locally compactness. In other words, the Baire’s theorem is true for locally compact spaces.

So, every locally compact space is second category or you can say that it is a Baire’s space or you can say it is not first category and so on. So let me tell you that first of all Baire's property was proved for complete metric spaces in part I. The proof here is more or less similar, in fact, it is even simpler. The simplicity comes because of the local compactness you will see that. So, for proving this one in the case of metric spaces we have to do quite a bit of preparation and all that. Here, everything is done already, so that we can immediately prove this one. So, what is the proof?

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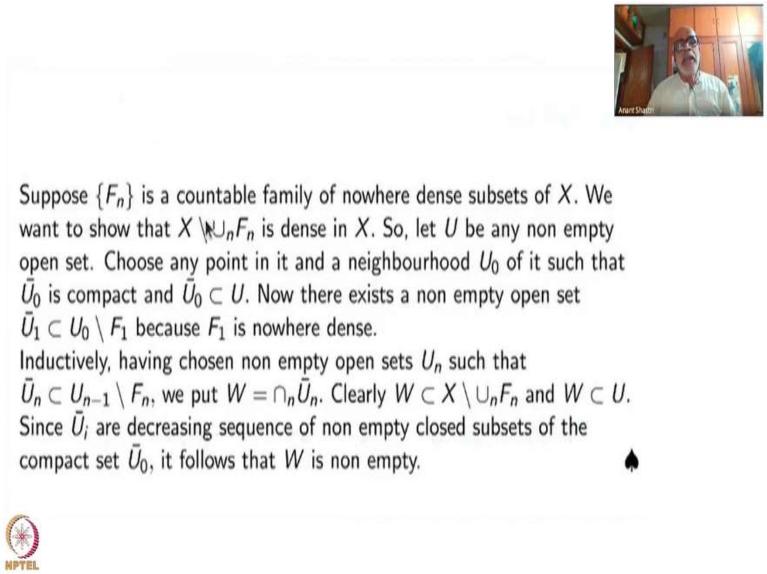

Start with any countable family $\{F_n\}$ of nowhere dense subsets of X. We must prove that the union of all these F_n's will not cover X. Actually, we should prove a stronger thing, namely, the complement of union of all these F_n's is itself dense in X. That is what we will prove and that is equivalent to saying that the space X is a Baire's space or it is second category. So, let U be any non-empty open set in X. We will show that none of this F_n's will cover U.

So, choose any point inside this open set, (in fact, taking a point etc is not necessary, we will see that later,) take a neighborhood U_0 such that $\bar{U_0}$ is compact and $\bar{U_0}$ is contained inside U. So, what have used here, local compactness, only for that purpose, I want to start with a point inside this non-empty open set U.

Now, $\bar{U_0}$ is compact. Now, there exists a non-empty open set U_1 with its closure contained inside $U_0 \setminus F_1$. Why? Because F_1 is nowhere dense, i.e., $\bar{F_1}$ does not contain any non-empty open set, that is the meaning of nowhere dense set.

So, if closure does not contain, then F_1 also does not contain any non-empty open set. That just means that $U_0 \setminus F_1$ is non-empty. So, there is a non-empty open subset U_1, exactly same as we did by taking a point and applying local compactness as before such that $\overline{U_1}$ is compact and contained in $U_0 \setminus F_1$. So, I am not writing all that here.

So, I am taking an open subset U_1 of U_0 such that $\overline{U_1}$ is contained inside U_0 and this $\overline{U_1}$ does not intersect F_1. Repeat this process now, to get U_2 such that $\overline{U_2}$ is contained inside $U_1 \setminus F_2$, $\overline{U_3}$ contained inside $U_2 \setminus F_3$ and so on. So, keep doing that what you have got is a sequence of open sets U_n such that $\overline{U_n}$ contained inside $U_{n-1} \setminus F_n$ for each n. That means $\overline{U_n}$ does not intersect F_n

So, what we have, we have a decreasing sequence of open sets U_n, closure of each contained inside the previous one and each of these is compact. Take W equal to the intersection. This W clearly is in the complement of all the F_n's. Why, because none of the F_n's intersects W. So, W does not intersect the union of F_n's.

And of course, W is contained inside U. $\overline{U_i}$ are decreasing sequence of non-empty closed subsets of the original $\overline{U_0}$ and $\overline{U_0}$ is compact. That is all I need. (Once $\overline{U_0}$ is compact all $\overline{U_n}$ are automatically compact anyway.) So, a decreasing sequence of non-empty closed subsets in a compact set, their intersection is non-empty. That follows by the De Morgan's law applied to compactness. If finite union does not cover the whole space, then the entire union also will not cover the whole space that is the property of compactness.

So, we are producing non-empty open subset inside this U_0 and outside all of the F_n's. So that is all what we wanted to prove. So, you see we got it in just three or four lines here. I elaborated it fully, we have a complete proof of Baire's theorem for locally compact spaces.

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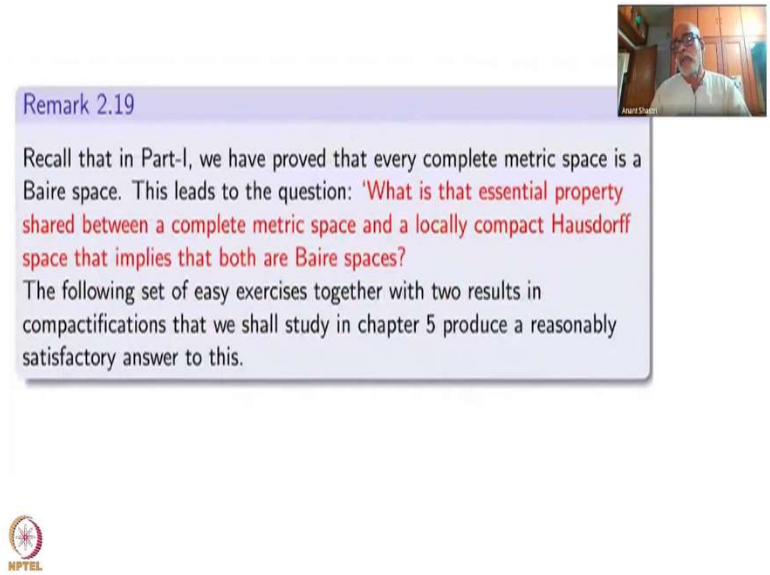

Now, I am going to discuss something much deeper. But here I want full participation from you people. If for some chance you feel that it is too much for you, you can take a break, you can sit aside and think a bit, but make a sincere attempt to try this one, this is the way one becomes a mathematician or one starts doing research work.

See, you have observed two phenomena; (i) a complete metric space is a Baire's space, (ii) a locally compact space is a Baire's space. So, immediately you should ask what is common between them, what makes both of them a Baire's space? Where is the hypothesis coming from? Apparently, if you just look at local compact spaces and complete metric spaces, they have nothing to do with each other.

Indeed, there are lots of complete metric spaces which are not locally compact and there are a lot of locally compact spaces which are not metrizable either. So, what is it that makes both of them Baire's spaces? So, I have tried to explain this one in a set of exercises, which are all doable. In any case, you can try them. Then your TAs will help you to solve them if you have difficulties, only after you try. Otherwise, you can take it easy. For the exam purpose and so on we will not bother you with this one that is what I meant by saying that you can take it easy.

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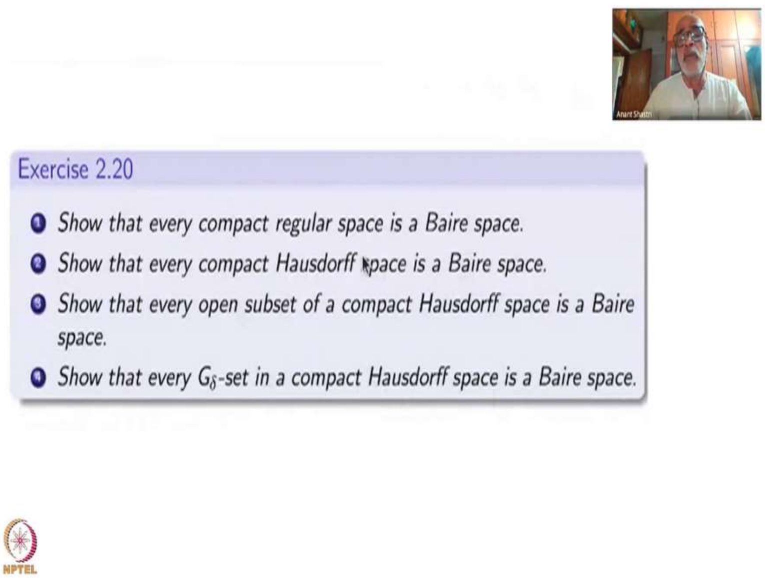

So, let us go through this exercises. just I am not going to tell you what is it but they are all easily done. The first thing is every compact regular space is a Baire's space. See, what we have done? Locally compact space is Baire's space. But when I say compact, a compact space, in our definition, may not be locally compact, remember that. So, I have to put regularity here in addition.

But the proof? If you have understood the proof of locally compact space is Baire's space just done above, you can prove this one also.

Next every compact Hausdorff spaces is a Baire's space. This is the consequences of one of the earlier exercises, if you think properly.

The third thing is every open subset of a compact Hausdorff space is Baire's space. If you prove (iii), (ii) will also get proved as a special case. But why I have given it like this first you prove this (i) and (ii) and then reduce (iii) to (ii) Directly proving (iii) that every open subset of a compact space is Baire's space that will take you to more effort.

Now, all these three exercises tell you that there are more Baire's spaces than just locally compact spaces or complete metric spaces. Only for that reason I have stated them here.

Now, comes the crucial thing. Every G_δ set in a compact Hausdroff space is a Baire's space. So, G_δ concept comes here. Remember G_δ just means intersection of a countable family of open sets. Open set in a compact Hausdorff space is a Baire's space is exercise (iii). From open set you have come to G_δ sets here. So this is the set of first 4 exercises. Once you have done them, we will turn our attention to complete metric spaces.

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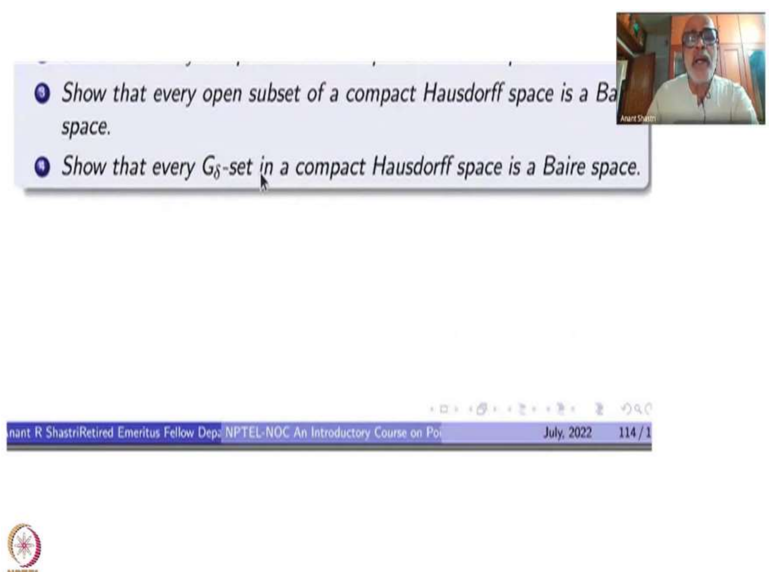

Indeed, metric space hypothesis is not fully necessary, you can just work all of these things in a pseudo metric space. A pseudo metric space is just a metric space except the property 1 of definiteness need not hold: $d(x,y)=0$ need not imply $x=y$, that is all. I am just recalling that. So, take a pseudo metric space. It will also give you a topology. Show that every closed subset in it is a G_δ.

So, now, we have what one single property common to both these families, pseudo metric spaces, every closed subset is a G_δ. And here every G_δ set in a compact Hausdroff is a Baire space. So, that must be something. That is the key for these two things to have Baire's property.

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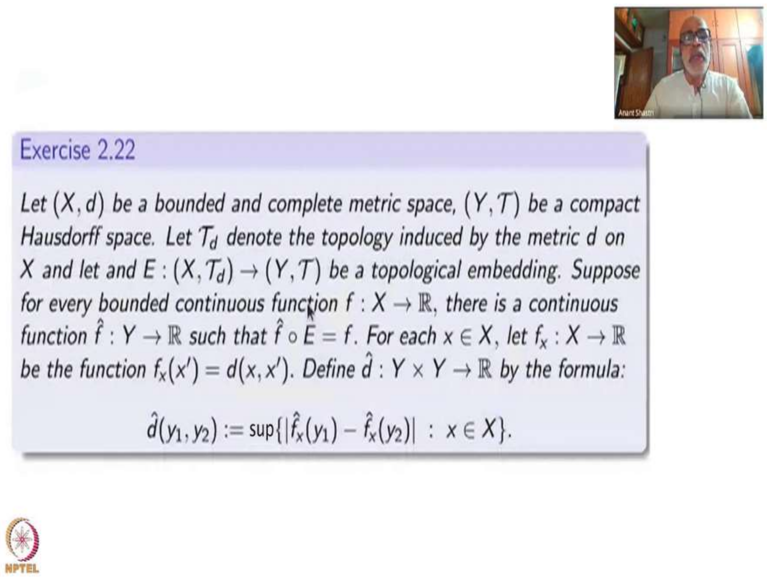

So, let (X,d) be a bounded and complete metric space. The boundedness can be always achieved, because every metric is equivalent to a bounded metric--- by just taking $d/(1+d)$; that will be a bounded metric always and it will give you the same topology. So, this is a mild restriction but it is needed in this exercise. Start with a bounded and complete metric space.

Now, let $(Y,\mathcal{T})$ be a compact Hausdroff space? Let $\mathcal{T}_d$ denote the topology induced by this metric d on X. Let E from $(X,\mathcal{T}_d)$ to $(Y,\mathcal{T})$ be a topological embedding. See, here you have metric space, here you do not have any metric, but both of them have topologies. In the topological sense, let E be an embedding.

Embedding means what? A continuous mapping which is a homeomorphism onto its image.

Suppose for every bounded continuous function f from X to $\mathbb{R}$, you have an extension of that to the whole of Y, a continuous extension. Now, how can I can talk about continuous extensions? X is embedded here, so you can think of X as a subspace of Y.

More precisely all that I want is a continuous function $\hat{f}$, so that $\hat{f} \circ E$ is f. So, suppose all the above conditions are true. Now, for each x belonging to X, let f_x from X to $\mathbb{R}$ denote the function, viz., the distance from x, $f_x(x')$ is equal to distance between x and x'. This distance d is the metric coming from X here.

With this function, I want to define a map $\hat{d}$ from $Y \times Y$ to $\mathbb{R}$ by the formula $\hat{d}(y_1, y_2)$ is the infimum of $\hat{f}(x)(y_1) - \hat{f}(x)(y_2)$, take the modulus. So these are all non-negative real numbers, take their infimum. So, there you will get a non-negative function on $Y \times Y$.

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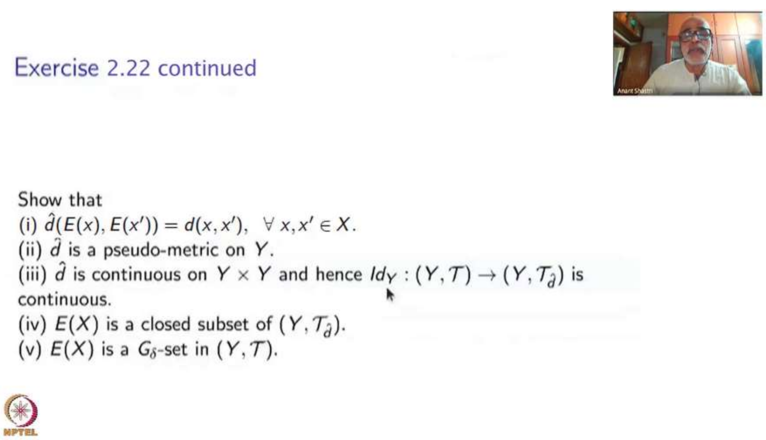

What is this? This has all the characters of a metric except it is not exactly a metric; it is a pseudo metric. On $E(X)$, i.e., X thought of as a subspace of $(Y, \hat{d})$ is the same as d, i.e. $\hat{d}(E(x), E(x'))$ is equal to $d(x, x')$, the original distance. So, what has happened is that $\hat{d}$ is an extension of d on X to the space Y. So, d hat is not a metric perhaps, it may fail to be a metric but it is a pseudo metric.

Also, $\hat{d}$ is continuous on $Y \times Y$, remember Y has already a topology on it. $\hat{d}$ will give another topology on Y. We have to wait for that yet. So, but this $\hat{d}$ is continuous on $Y \times Y$. And hence, the identity map from $(Y, \mathcal{T})$ to $(Y, \mathcal{T}_{\hat{d}})$ is continuous.

The fourth property is that the image of X under E is a closed subset of $(Y, \mathcal{T}_d)$, in the new topology, pseudo metric topology it is a closed subset. That is why I have done all this. I have put an arbitrary metric space inside X inside $(Y, \mathcal{T}_d)$ which is a compact metric space. Now, $E(X)$ becomes a G_δ set in $(Y, \mathcal{T})$. So, this will really complete the picture why complete metric spaces and locally compact spaces have this beautiful property namely, being Baire's space, they share this property.

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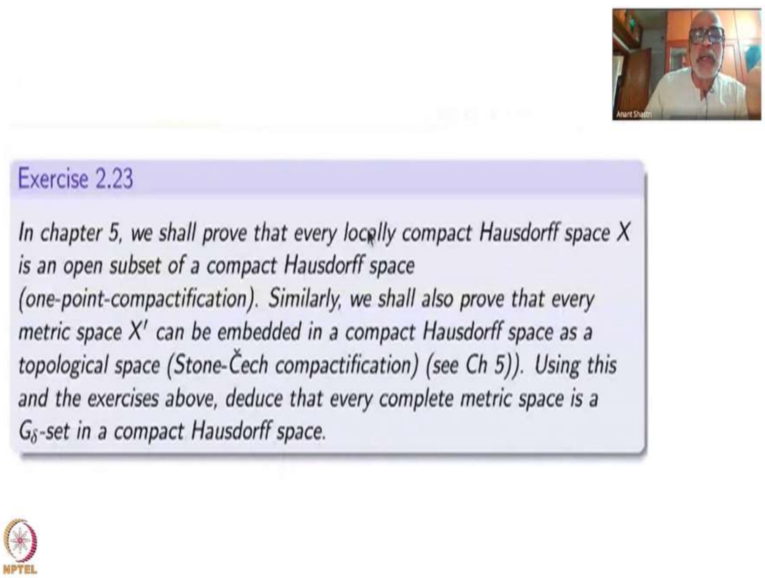

So, that is the conclusion of this last exercise which you can read by itself, but you will have to wait to see the full thing, you will have to wait till chapter 5, when we will do compactifications of various things. Namely, a locally compact Hausdroff space can be compactified what is called as one-point compactification.

Whereas, a pseudo metric space can be compactified, what is called the Stone-Cech compactification and so on. That will fully complete the picture. But that requires some preparations. Till then you can take these things granted.

After we finish chapter 5, you can again come back to these exercises, part of this exercise you might have solved by that time and then we will be able to see the full picture. So, that is enough for today. So, thank you. So, we meet again next time.