An Introduction to Point-Set-Topology (Part 2)
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Lecture No. 62
Connected Sum

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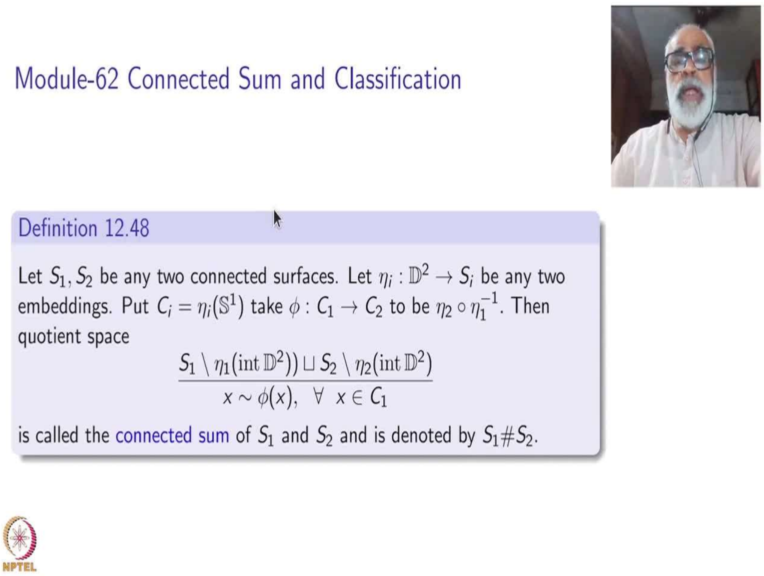

Hello, welcome to NPTEL NOC and introductory course on points set topology part II, module 62, connected sum and classification of surfaces.

Last time we have introduced the notion of rubber sheet models and stated a big theorem which is supposed to give you all compact connected surfaces. The same result we will try to explain in a more geometric way this time. So, I have defined one single notion here. Start with any two connected surfaces. This time let us not bother about the boundary. Preferably we shall work with boundary-less manifolds. Maybe boundaries are there, but keep it far away from the our cite of operations.

Start with any two connected surfaces S_1, and S_2. Take two embeddings of the disc $\mathbb{D}^2$, the closed disc $\mathbb{D}^2$, say, η_1 and η_2, one in S_1 and other in S_2. Put $C_i = \eta_i$ of the boundary circle. Take ϕ from C_1 to C_2 to be $\eta_2 \circ \eta_1^{-1}$, where I mean by η_1^{-1}. I am restricting η_1 to the boundary circle C_1, and taking the inverse. Then the quotient space obtained as follows, namely, you take S_1 throw away the interior of the image of η_1, similarly take S_2 and throw away the interior of of the image of η_2, and their disjoint union, (so, you started with two surfaces now you have made two holes one in each, there. Hole means what? Removing the interior of a disc, so boundary of the disc is there, so those things are C_1 and C_2 here), and now identify x

with $\phi(x)$ for each $x \in C_1$, where ϕ is $\eta_2 \circ \eta_1^{-1}$. So, that quotient space is called the connected sum of S_1 and S_2 and is denoted by this notation S_1 connected-sum S_2. That is the way I read this one.

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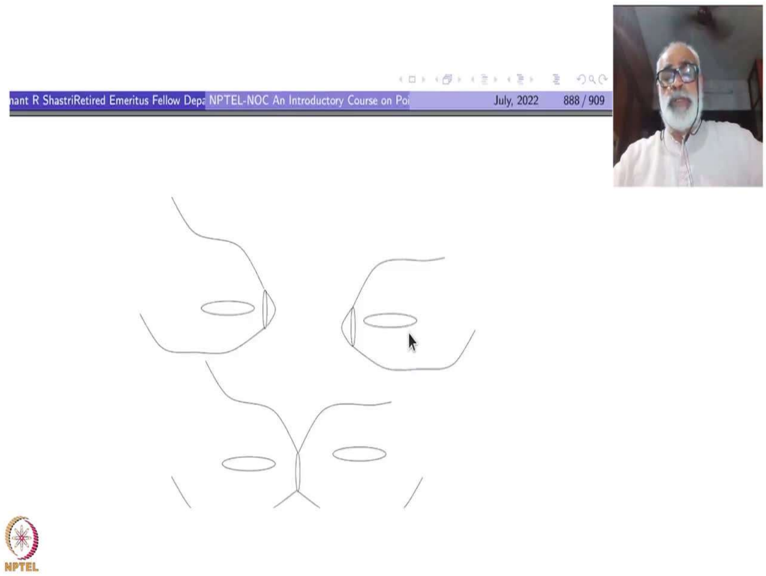

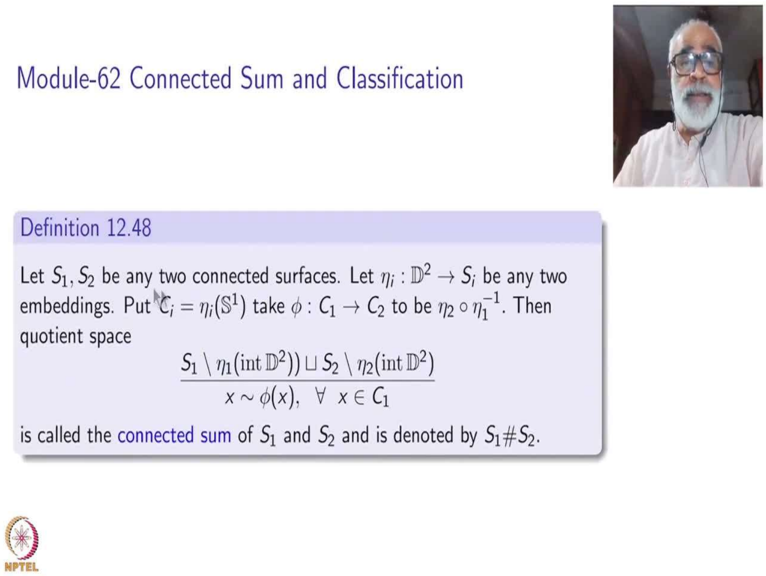

So, here is a picture. The picture may be confusing, I do not know. This is one surface there is another surface, there is a disc here there is a disc here, these are embedded discs. So, you remove the interior of this remove the interior of that there is a circle here left out namely image of a circle. So, this looks like an ellipse here. So, identify them like this. So, bring them together like this identify the two circles by a homeomorphism that told homeomorphism is not an arbitrary homeomorphism. It is from this circle see this one is η_1, so $\eta_1^{-1} \circ \eta_2$, the other way round. so there is a homeomorphism from here to here you identify these two circles.

For example, if you take a disc, remove the interior of a smaller disc from that you will get an annulus. Do the same thing with another copy of the disc also. Now you have got two interior circles in both the annuali, you identify them, what do you get? You will get back again a cylinder. Actually annulus and annulus they are both cylinders. So, when you identify two of the circles one circle from here and other circle from another one, you get one cylinder.

So the word connected sum precisely refers to this one. you start with connected surfaces here the end result is also a connected surface. To begin with there are two of them here the end one single is connected surface. So, two of them you start but each of them must be connected then you make them connected by this performance. If you just join them at one point it will fail to be a manifold, though that would have been the simplest way of doing a connected sum. Indeed, in general topology we will do like that only and call it one point union. But while dealing with manifolds, it will fail to yield a manifold. So you have to do this trick, namely remove a small disc here, remove a small disc there and look a the resulting boundary circles and identify them.

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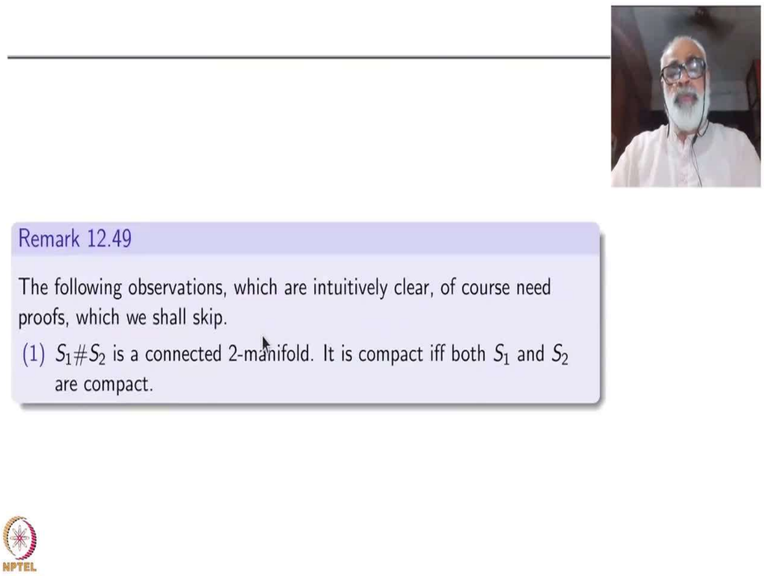

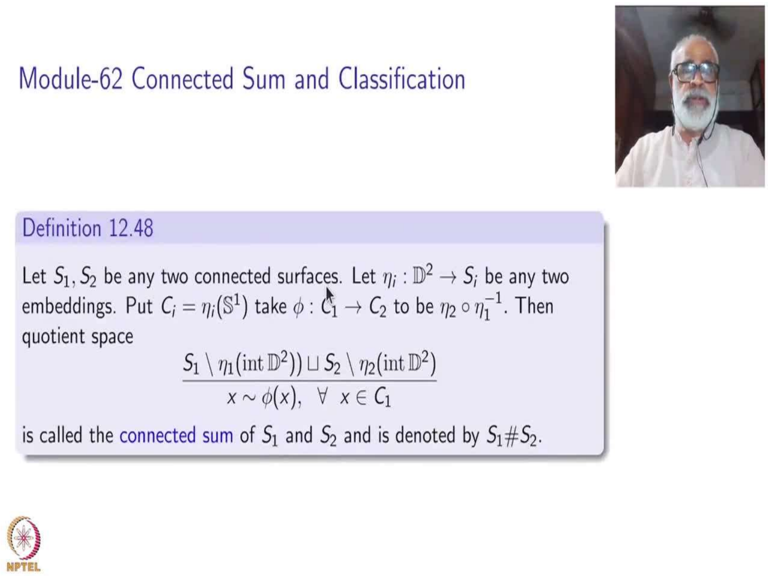

The following observations which are intuitively clear of course, need proofs which we shall skip. So, what are these observations?

S_1 connected sum S_2 is a connected 2-manifold. It is compact if and only if, both S_1 and S_2 are compact.

See, there was no statement about this being compact in the definition of connected sum. So I could have taken any two connected surface that is all. So, far whatever I have stated they are not at all difficult to verify. So, only because we have time limits here we cannot go on explaining everything.

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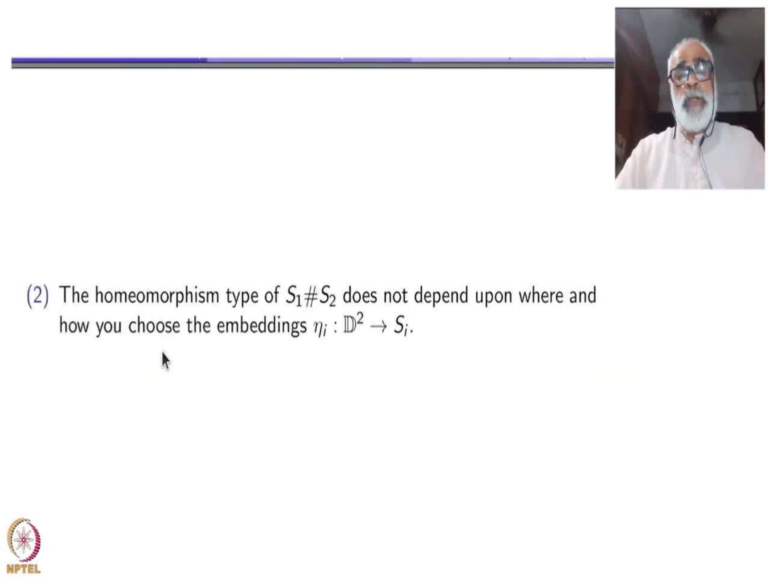

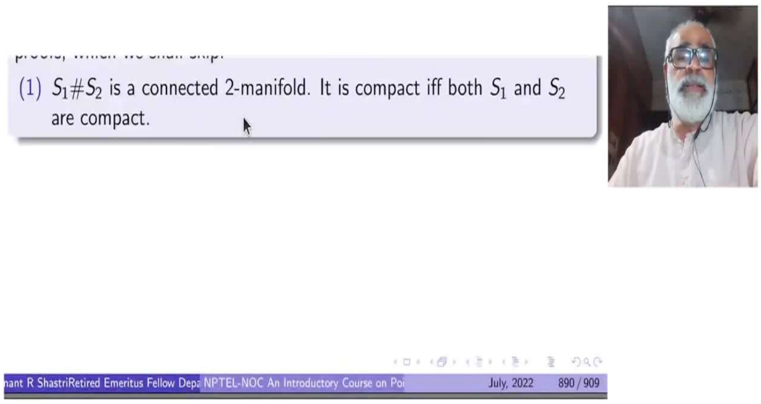

The homeomorphism type of S_1 connected S_2 does not depend upon where and how you have chosen these embeddings η_1 from $\mathbb{D}^2$ to S_1, and η_2 from $\mathbb{D}^2$ to S_2, up to homeomorphism, the connected sum is the same. This is a deeper statement, and requires some proof.

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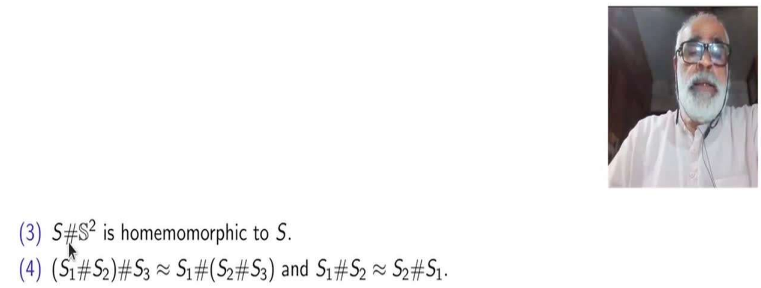

If you take S connected sum $\mathbb{S}^2$, where S is any surface and $\mathbb{S}^2$ is the standard 2-sphere, what do you get? You will get back S itself, Why? Because you have to remove a 2-disc from here and disc from here. When you remove disc from $\mathbb{S}^2$ what do you get? You have another disc, the complement. Now you are filling that disc back in the gap that we have produced here in S. That is all. So, all that amounts doing nothing so that nothing is up to a homeomorphism.

So, taking connected sum with the 2-sphere produces no effect. That means it is a two sided identity for the binary operation of taking connected sum. So, that is why I told you that $\mathbb{S}^2$ is like the zero element. This connected sum more or less, can be thought of as an additive operation. You can see that it is associative, and commutative. The only thing is there is no inverse here, it is like addition on the set of natural numbers. You see there is a beautiful algebra here out of these what are called Cobordism theory. But that I cannot go into the detail. In Cobordism theory this connected sum operation is the sacrosanct there. So, this can be defined in all dimensions also, but now we are learning it in dimension two.

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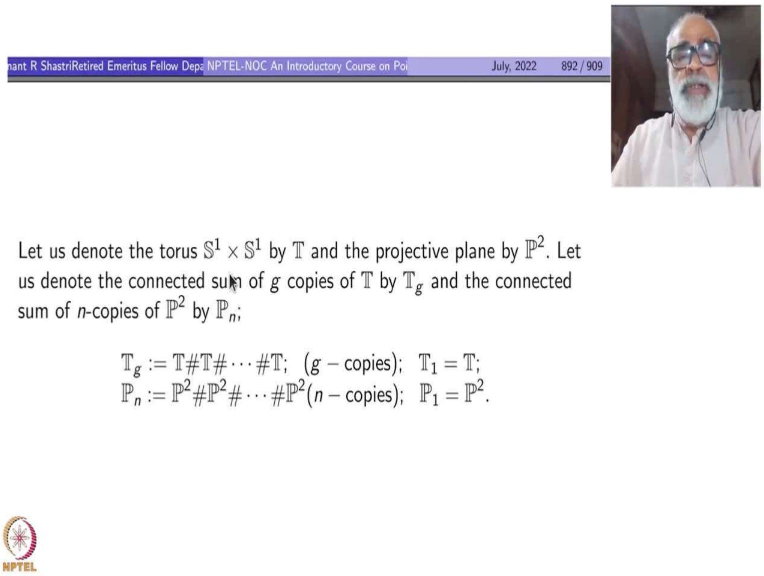

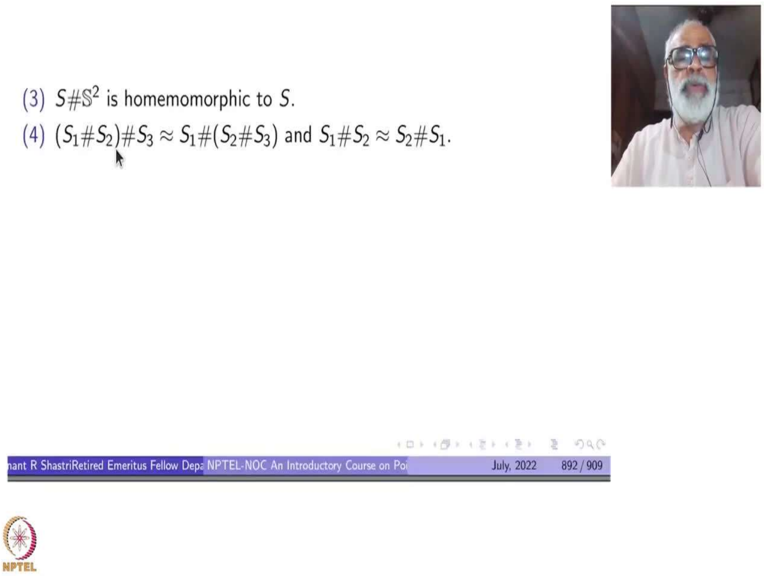

Let us denote the torus $\mathbb{S}^1 \times \mathbb{S}^1$ by this simple symbol this $\mathbb{T}$. I will just read it $\mathbb{T}$ and the projective plane by $\mathbb{P}^2$, this we have already done. Let us denote the connected sum of g copies of $\mathbb{T}$ by $\mathbb{T}_g$. Similarly, let the connected sum of n copies of the projective space $\mathbb{P}^2$ be denoted by just $\mathbb{P}_n$.

So, again this g and n are coming here. just like in the last lecture. When you perform the connected sum operation more than once, there is no need to put brackets here, because associativity. So the notation $\mathbb{T}_g$ is unambigious. Similarly, $\mathbb{P}^n$ is $\mathbb{P}$ connected sum $\mathbb{P}$ connected sum ..., n copies. What is $\mathbb{T}_1$? It is nothing but $\mathbb{T}$ itself; $\mathbb{P}_1$ is just $\mathbb{P}^2$ Because only one copy is there. You have performed no operation. So, this are the short notation for this long thing.

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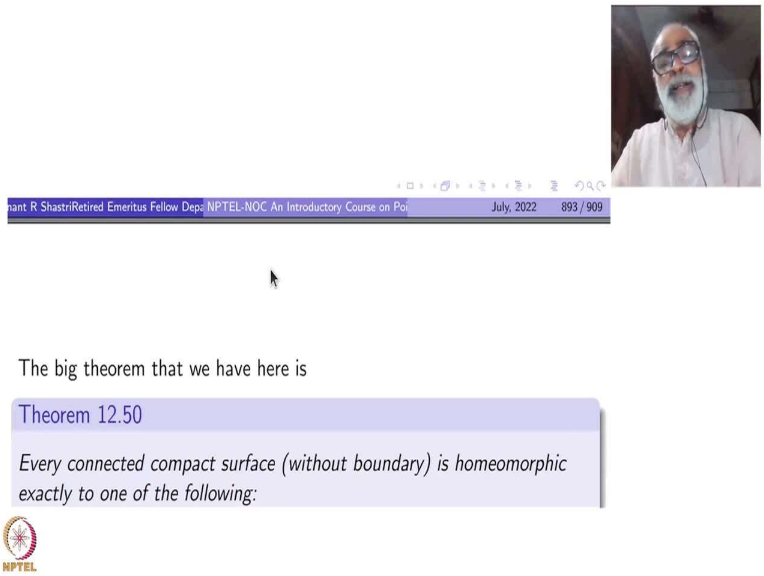

Now, the big theorem here is like we did after long, the last time, but this time we are having easy time. It is the following:

Every connected compact surface without boundary, (I want to emphasize that here), is homeomorphic to exactly one of the three things.

$(a)\mathbb{S}^2;\ (b)\mathbb{T}_g, g \in \mathbb{N};\ (c)\mathbb{P}_n, n\sqsupset nN.$

Actually there are three lists here, this is a single element, but this an infinite sequence this is also infinite sequence, indexed by natural numbers.

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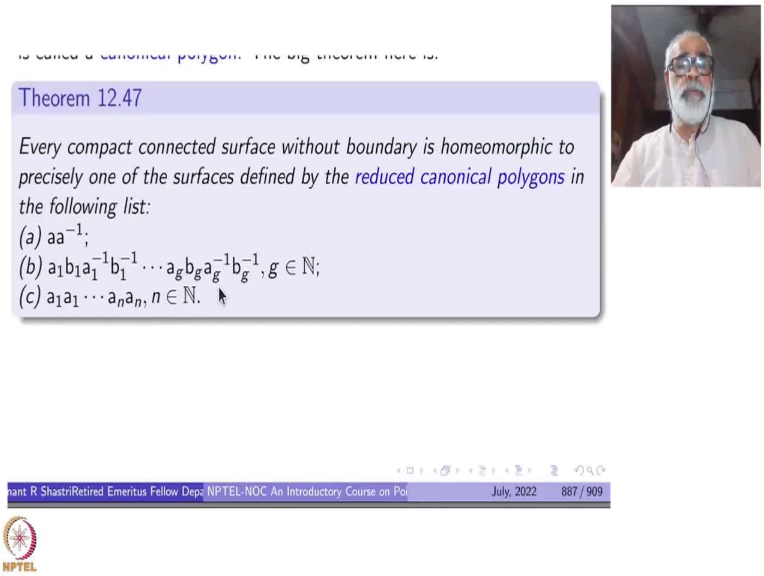

Exactly similar to the previous theorem that we had I will just show you that theorem which we did last time, this is the theorem. The list (a), we have told you it represents $\mathbb{S}^2$. These are going to be in second list $\mathbb{T}_g$'s and these are going to be $\mathbb{P}_n$'s. Every compact connected surface without boundary is homeomorphic to precisely one of the surfaces defined by these canonical polygons. So, this is some kind of technique to obtain such a result.

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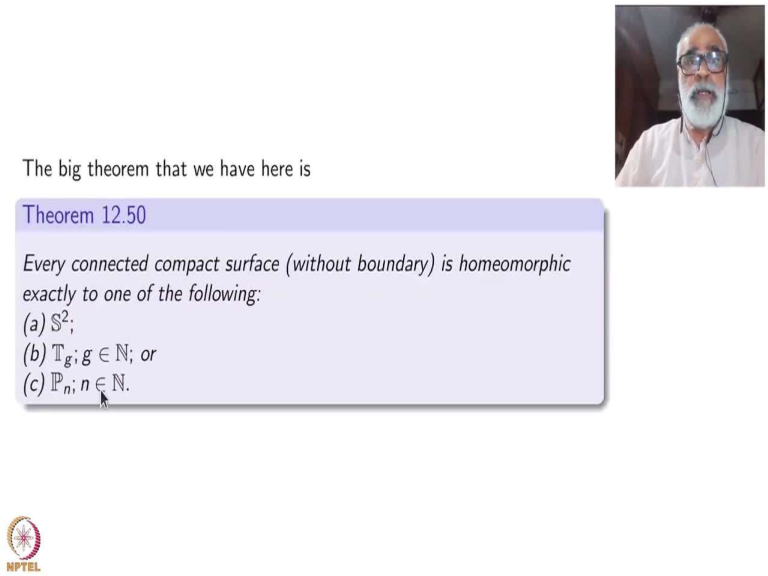

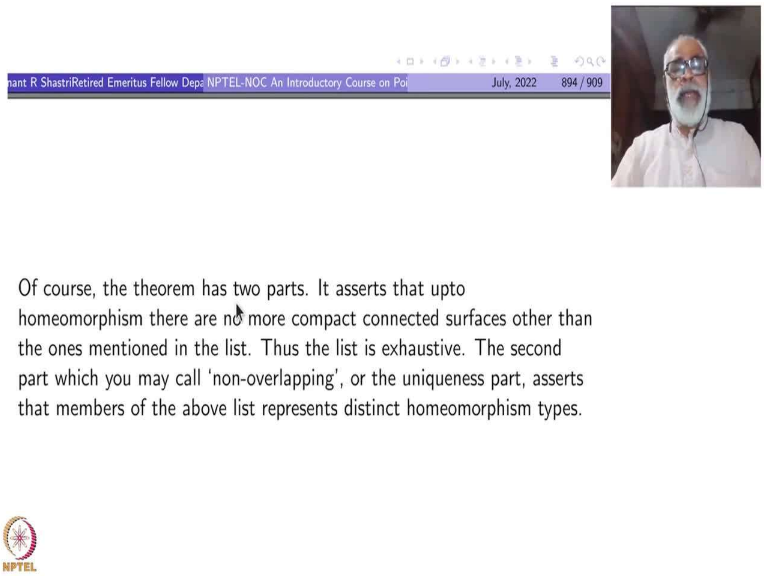

Finally, we would like to have this description. So, why these two are the same, I will try to explain that one. Not a proof, but just some explanation. The theorem has two parts, you see every compact connected surface is exactly one of them. First of all take any compact connected surface you must be able to find them here, means what? Up to homeomorphism. Moreover, every element here in the list is a different homeomorphism type. So that is the second part here, there are two parts of the statement of the classification.

It asserts that up to homeomorphism there are no more compact connected surfaces other than the ones mentioned in the list, the list is exhaustive. The second part is that which you may call is non-overlapping property of the list. namely the uniqueness part, asserts that members of the above list represent distinct homeomorphism types.

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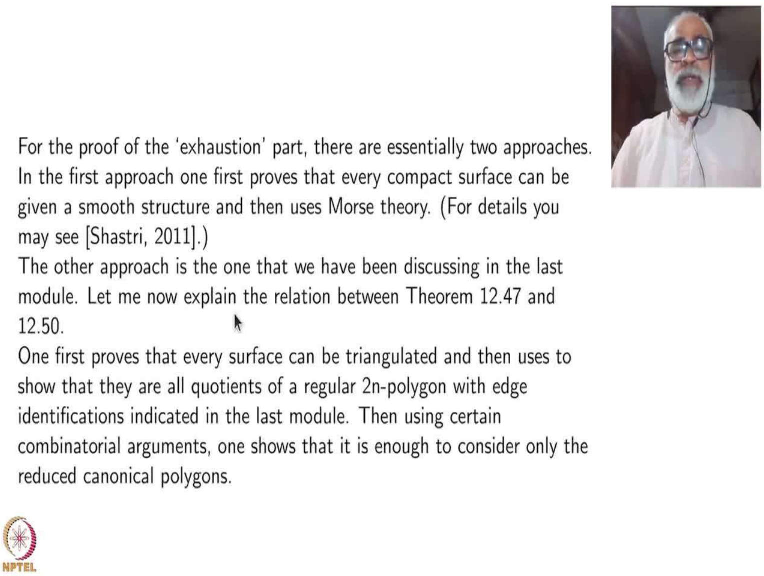

The proof of the exhaustion part: there are essentially two different approaches. In first approach one proves that every compact surface can be given a smooth structure and then uses what is called Morse theory. You can it from read many books. But you can also read it from my book itself.

The other approach is the one that we have been discussing in the last module, namely this rubber sheet geometry, the canonical polygons or whatever you want to call them.
So, let me explain this a little more because we have done some work there. Let me explain the relation between the previous theorem and this present theorem, namely the second approach. One first proves that every surface can be triangulated, what is the meaning of triangulation? Just the ability to cut a surface into finitely many pieces each of them homeomorphic to a triangle, homeomorphic to a triangle is simply a homeomorphic to a disc, but you like to think of them as a triangular pieces with three edges, so this is the idea. The proof of this is not at all easy. The proof of that every surface can be given a smooth structure is even much harder. So, you have to choose.

So, right now you have to assume that your surfaces are all triangulated. From the triangulation, what you can do is? You can get a regular $2n$-polygon with edge identifications as indicated in the last module. You cut the surface into all these triangles, while cutting down what you do, you keep a label wherever you perform the cut. you are introducing two different edges by each cut. So, label both the edges with the same letter and exponent. Best method is to put an arrow to indicate the direction. So, this will help you in recovering the same surface back aafter you do some patch works. That is the whole idea.

So, look at all these triangles. Now lay them out on the table one by one side by side, as if you are solving a zig-saw puzzle. When you are doing jigsaw puzzle nobody tells which piece you should take first and so on right? So you pick up any one of the pieces first and try to arrange the next one and so on. How do you do that? There is only one rule, namely, wherever an edge indicated by a letter you look for that letter in one of the triangles it will be there, so take that particular piece and lay it down next the previous one so as the two edges indicated by the letter match up. In this Zig-saw puzzle, you are allowed to stretch the triangles whichever way you like so that every time you have a convex polygon on the table.

Readjust whatever shape you have got into a convex polygon. That is the extra freedom here. That is all.

So, when you finished all the pieces, you have a union of finite many triangles and that is a convex polygon. Now what will happen? There will be all these boundary edges of this convex polygon. Remember for each of these edges is paired with another edge. Where are these twins. They are also there in the same polygon but on the boundary now, because everything in the interior they have got their pair, they have been paired already. So, these are the last ones which are left out. So, you will have exactly an even number of sides in your polygon, a $2n$-sided polygon. That will give you a paper scheme. You perform identifications as indicated by the paper scheme, you will get back the surface you started with. That is how the exhaustion part of the theorem is proved.

So, every surface can be got by a rubber sheet scheme. Now, there is another part, now we have to say that the list contains all the representatives, representatives means what? Up to homeomorphism. So, that is the harder part. that we have done in algebraic topology course part II, Also you can read it from my book in algebraic topology.

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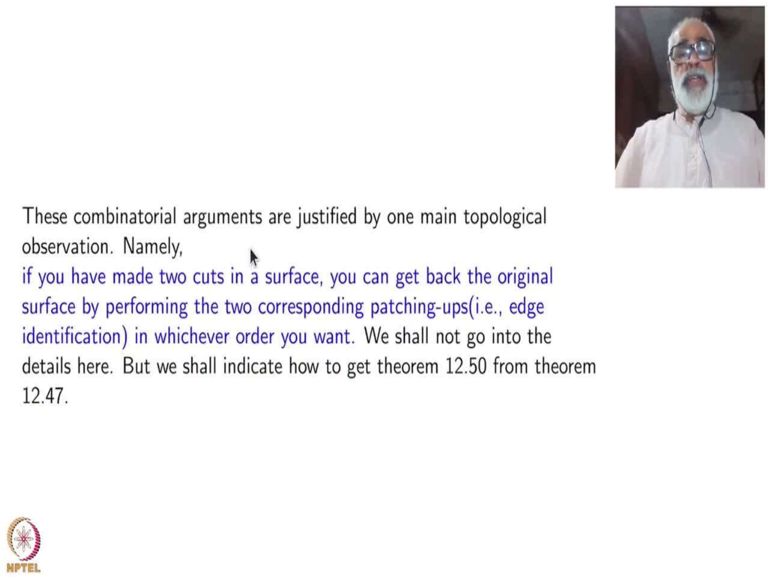

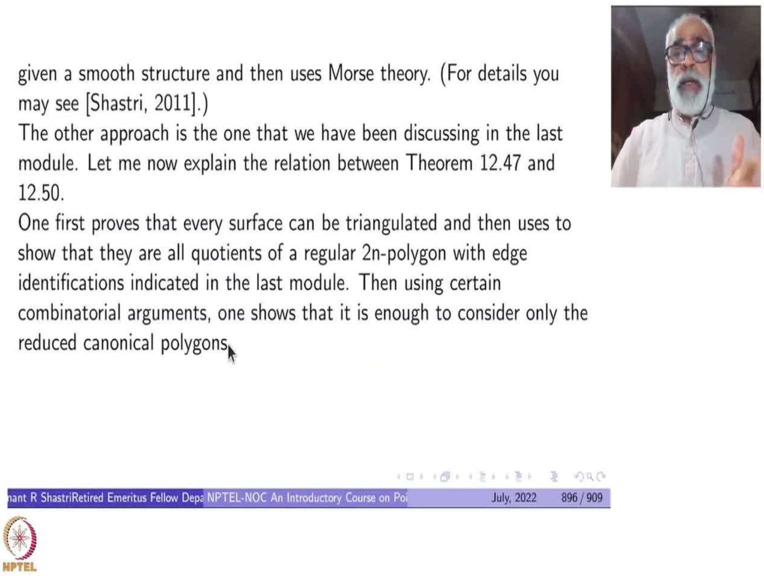

So, this is kind of combinatorial argument. This to cutting down a surface and re-assemble obtain canonical polygons as listed last time. They are justified by one main topological observation. What is that? If you have made two different cuts in a surface you can get back the original surface by performing the two corresponding patch-up operations in whichever order you want. You may first stitch back the second cut and then stitch the first one or the other way round. It is no problem. The resulting surface that you get will be the same provided you have performed all the stitching back wherever you have cut that is all. So, this is the beauty. You can cut as many times as you want, every time you have cut, sometime or the other, that cutting must be stitched back, must be identified that is all. We shall not go into details here, but we shall indicate how to get 12.50 from 12.47 now. So, that is the main thing, there are two different descriptions, I want to relate them for now.

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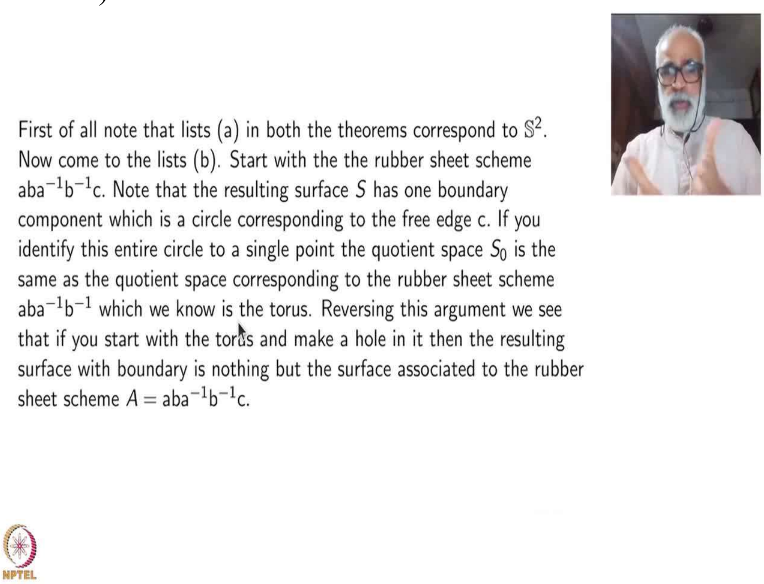

First of all, note that list (a) in both cases corresponds to $\mathbb{S}^2$. Now come to the list (b). Start with the rubber sheets scheme $\{aba^{-1}b^{-1}\}$ and some c, does not matter. Note that the resulting surface S has one boundary component c, because this is not identified with any other edge. And that c will become a circle because both the endpoints are getting identified. If you identify this entire circle to a single point, make that circle smaller and smaller and bring it to a single point, that quotient space S_0 is the same as the quotient space corresponding to the rubber sheet scheme $\{aba^{-1}b^{-1}\}$, as if that edge has been shortened to a single point, before you identify other edges, so no free edge left out that is all.

We now that gives you a torus. Therefore, this scheme represents a torus with a hole, do you understand this? I will repeat it. So, what I start, with? I do not start with a torus with a hole. I take the scheme $aba^{-1}b^{-1}$, I know that gives a torus. But now I have a different scheme here, namely, there is a pentagon and it is marked with $aba^{-1}b^{-1}c$, the fifth side is just named c. When you carry out the identifications, c will become a circle in the quotient S_0. Whatever this quotient is, I do not know what it is, does not matter, but there is a hole in it a boundary component.

Now, you move that hole to a smaller, make it smaller circle, and make it a single point. So, that surface is the same thing as if you have got it from the scheme $\{aba^{-1}b^{-1}\}$. Therefore, the surface S_0 must be what? The torus with a hole. That is the whole idea. So, reversing this argument, we see that if you start with a torus and make a hole in it, then the resulting surface with boundary is nothing but the surface obtained by this $aba^{-1}b^{-1}c$. So, this is the scheme for that.

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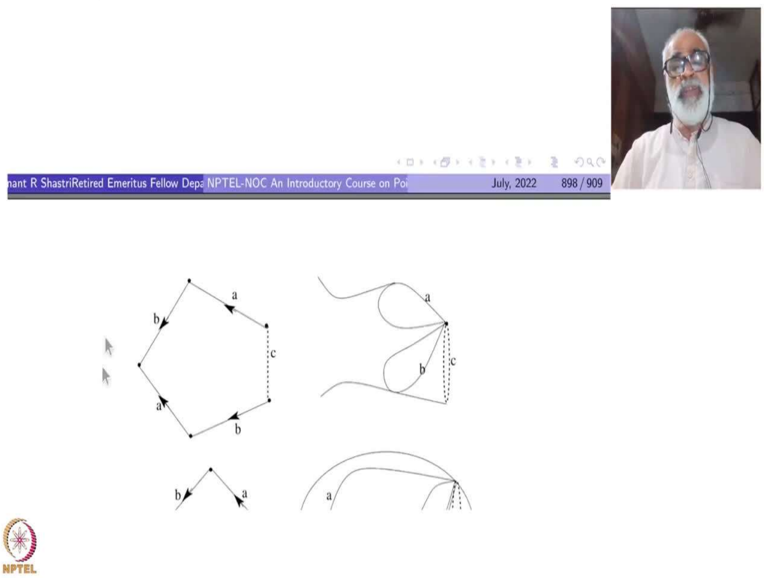

So, this is a picture here. So, $aba^{-1}b^{-1}c$. So, that c is there, which we do not want. So, once you perform the identifications, in the quotient space, a becomes a loop, a^{-1} also becomes a loop but identified with the same loop. Similarly b and b^{-1} become another loop, they are in the interior of the surface. The edge c also becomes another single but in the boundary of the surface. So, make it smaller and smaller is the same thing as putting a disc here over this circle. Removing a disc here you get back this one. This picture is as if you have got it from $aba^{-1}b^{-1}$. So, this is a picture. So, making the circle to a single point is same thing as putting a disc here. Removing the disc from the torus gives you the surface same as given by the scheme, $aba^{-1}b^{-1}$. Let us call this scheme A.

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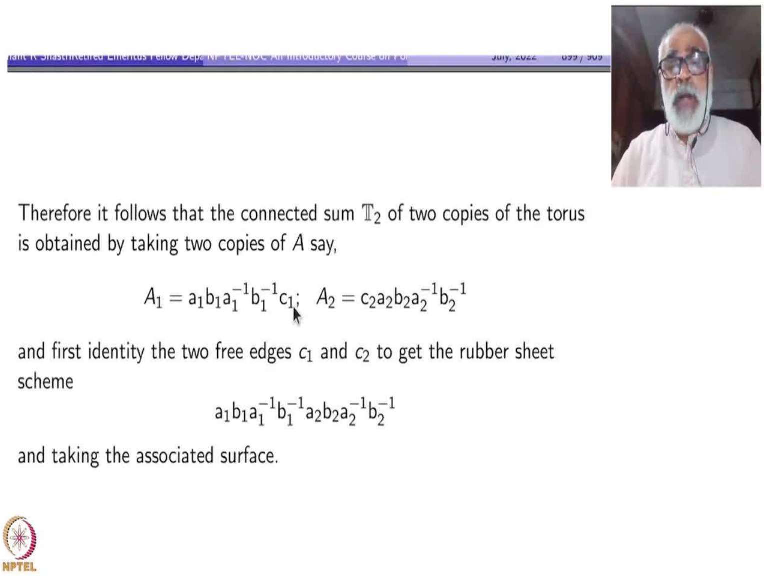

Therefore, it follows that the connected sum T_2, of two copies of the torus $T = T_1$ is obtained by taking two copies of the scheme A namely A_1 and A_2 say. So, A_1 is what? $a_1b_1a_1^{-1}b_1^{-1}c_1$, and A_2 is what? $a_2b_2a_2^{-1}c_2$. I have changed the order cyclically and written it as $c_2a_2b_2a_2^{-1}b_2^{-1}$, because finally I want to bring c_1 here and c_2 there together and merge them.

First identify the two free edge c_1 and c_2, (you can identify them whichever order you want because the two edges are in two different schemes, that is one of the main guiding principle here. You get a scheme what is it? You identify c_1 and c_2, two boundary edges of two discs Therefore the union is again a disc with the common edge $c_1 = c_2$ disappearing in the interior of the new disc. On the boundsary you have the scheme $a_1b_1a_1^{-1}b_1^{-1}$ followed by $a_2b_2a_2^{-1}b_2^{-1}$

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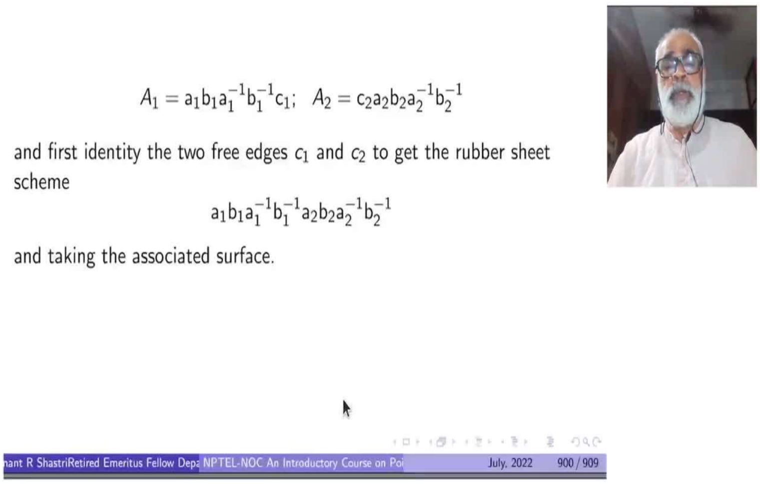

This is a picture. So, c_1 and c_2 they are brought together here in this picture then realigned the whole thing, this dotted thing here is c_1c_2, one identified with other, rest of them is $a_1b_1aba^{-1}b^{-1}a_2b_2a_2^{-1}b_2^{-1}$. In the old theorem this is another item in (b) with $g = 2$ here.

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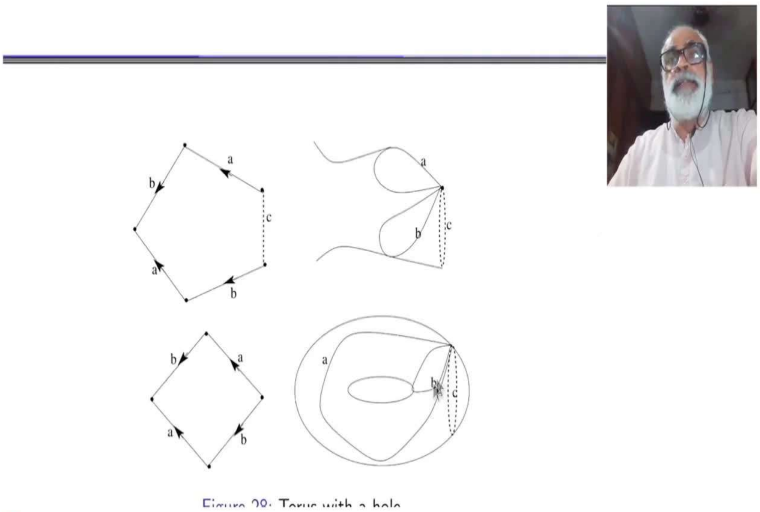

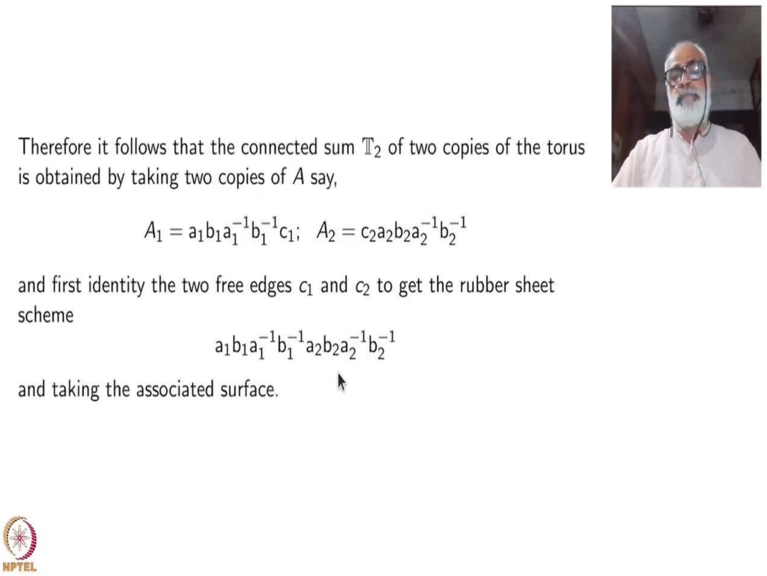

On the other hand, if you perform the scheme identifications on each of the schemes first, what you get? Two copies of S_0, each representing the torus minus a disc, with the boundaries represented by the two edges c_1 and c_2 respectively. Join c_1 and c_2 this time at the last, what do you get? You get the connected sum of the torus with itself. Therefore represented the connected sum of the torus with itself is represented by the scheme number 2 in the list (b).Now an ordinary induction tells you that if you take g of them like $a_1b_1a_1^{-1}b_1^{-1}\ldots a_gb_g$ that is nothing but the connected sum of g copies of the torus. It follows that the two lists (b) in the two theorems give you the same class of surfaces.

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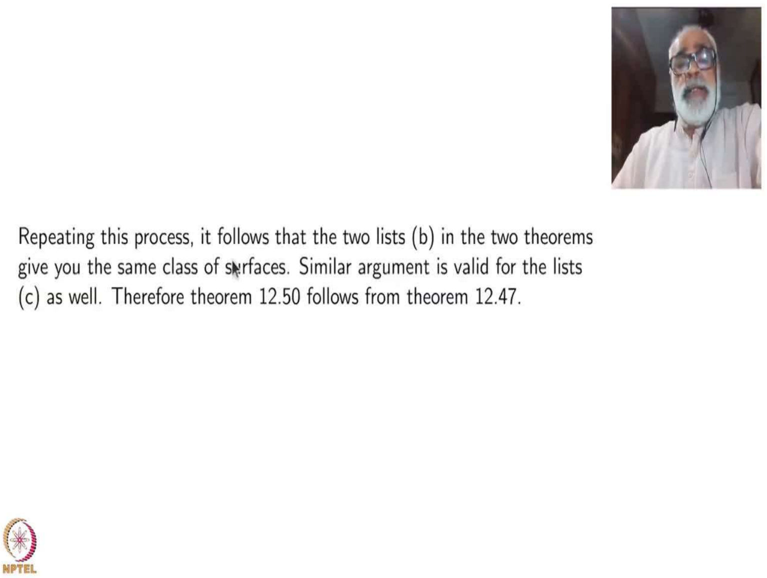

Exactly similar argument is valid for the list (c) as well. Therefore, 12.50 follows from 12.47.

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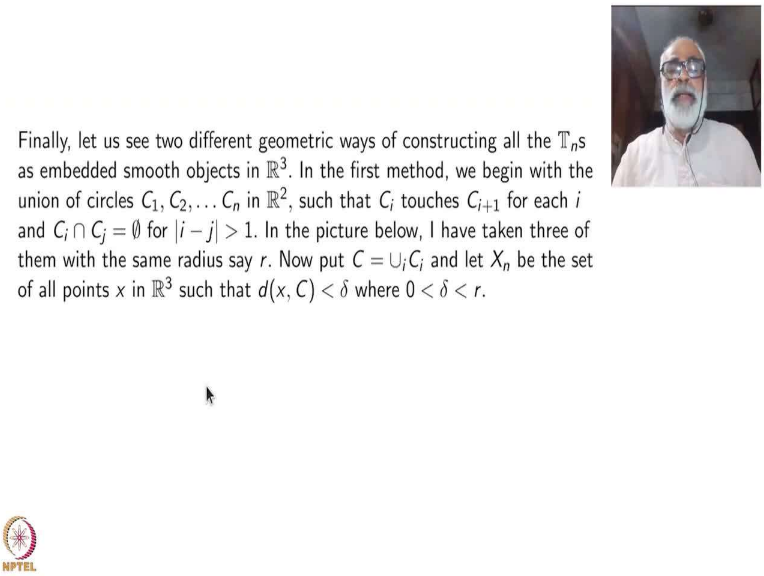

Finally, let us see two different geometric ways of constructing all the T_n's as embedded smooth objects in $\mathbb{R}^3$. I want to tell you two different methods, because they are both interesting and useful. There may be many. In the first method, we begin with the union of n circles C_i in $\mathbb{R}^2$, such that C_1 touches C_2, C_2 touches C_3, and so on like a chain, so they are not all disjoint. What are the disjoint? C_1 and C_3 are disjoint of course. C_2 and C_4 are disjoint, and so on. So, whenever $i - j > 1$, they are disjoint when they are consecutive, they have a single point in common. They are touching each other. In the picture below I have taken three of them because picture has to be limited, you cannot take arbitrary n anyway in a picture.

So, three of them with the same radius r that is just an extra thing so that you get a nicer picture that is all. Now put C equal to union of these C_i's finite union, and let X_n be the set of all points x inside $\mathbb{R}^3$ such that the distance between x and these circles union of circles is less than δ, wherever where $0 < \delta < r$, this an open subset, take less than or equal to δ, you get a closed and bounded subset and so compact. Its boundary in $\mathbb{R}^3$ is given by the equation $d(x, C) = \delta$. δ should be chosen correctly namely less than r. Then this boundary is a surface. So that is the whole idea.

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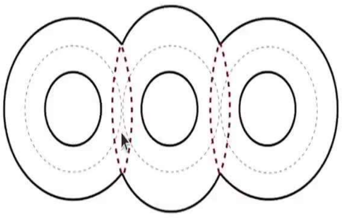

Figure 30: Connected sum of three tori

So, this is a picture here. The dotted circles here, these are the beginning circles C_i, they are touching at one point one after another they are inside $\mathbb{R}^2$. The entire solid, the set of all points which are at a distance less than or equal to some δ, (δ must be chosen positive and less than r). So these are circles of radius r. So, this δ must be less than r, strictly less than r. For the sake of this picture, I have taken $\delta = r/2$. Take all the points which are at a distance less than $r/2$ like that. And then look at the boundary that boundary is the connected sum of three tori in the picture. If you have taken n of them, it is a connected from of n tori. If you take only one circle, first thing you have to see is that it is actually the torus. So, that part is just some elementary three dimensional calculus, you should know some differential calculus to see that this is actually a nice smooth surface.

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Of course, it takes some 3-variable calculus to show that the boundary of X_n is diffeomorphic to T_n, whereas it is intuitively quite clear.
In the second approach, I merely give you the equation:

$$[x^2 + (y - n - 1)^2 - (n + 1)^2] \prod_{k=1}^{n} [x^2 + (y - k)^2 - 1/9] = -z^2. \quad (39)$$

This time I take Y_n to be the set of all points $(x, y, z) \in \mathbb{R}^3$ which satisfy the equation (39).

Of course, it takes some three variable calculus to show that the boundary of X_n is diffeomorphic to $\mathbb{T}_n$, whereas it is intuitively quite clear. You take a circle, you thicken it, like a wire. The boundary of that is a torus. So that is what we are doing more generally with this union of circles.

One approach is over, so you have got a model of the surface. The boundary surface, which is $\mathbb{T}_n$, is inside $\mathbb{R}^3$ over. So, actually, I should write $\mathbb{T}_g$ here, in my notation, I should write X_g and $\mathbb{T}_g$ to be consistent.

In second approach, I merely start with an equation. So, this should be liked by algebraic geometers for example. So, you can take equation here, of course, all with real coefficients in three variables and hence should represent some subset of $\mathbb{R}^3$. You can look at what is this funny equation. See the left hand side is just some polynomial in x and y, the right hand side is just $-z^2$. Also, the variable z is separated.

So, look at this LHS. There is one special factor here and then there are similar factors here k ranging from 1 to n. What are they? The first factor here is $x^2 + y^2 - (n+1)^2 - (n+1)^2$. If you put this equal to 0, that will be the equation of the circle with the radius $n+1$ and the center equal to $(0, n+1)$. Similarly, here these factors represent circles with center with $(0, k)$ and the radius $1/3$. I have taken the product of these polynomials and putting them equal to minus z^2. This time I take y_n to be the set of all points x, y, z inside $\mathbb{R}^3$ which satisfy this equation (41) and claim that this represents $\mathbb{T}_n$, homeomorphic to $\mathbb{T}_n$.

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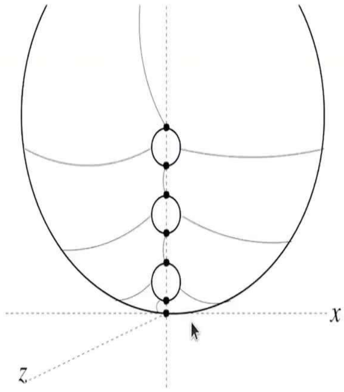
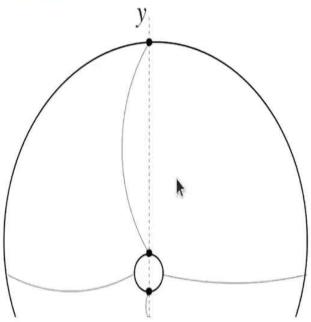


Figure 31: A representation of A surface of genus n



Of course, it takes some 3-variable calculus to show that the boundary of X_n is diffeomorphic to T_n, whereas it is intuitively quite clear.
In the second approach, I merely give you the equation:

$$[x^2 + (y - n - 1)^2 - (n+1)^2] \prod_{k=1}^{n} [x^2 + (y-k)^2 - 1/9] = -z^2. \quad (39)$$

This time I take Y_n to be the set of all points $(x, y, z) \in \mathbb{R}^3$ which satisfy the equation (39).

Look at this picture. So, how many circles are there? $1, 2, 3, \ldots$ and one big circle. What is this big circle? This is the circle represented by the first fact on the LHS. So, where are all these circles? Inside $\mathbb{R}^2 \times \{0\}$, the xy-plane. The z-axis is shown this way. So, I will tell you a little more about this. I do not want to prove anything here. Namely, take this polynomial on the left hand side just put equal to 0. What is the meaning of that? The third coordinate equals 0 means intersection of Y_n with with $\mathbb{R}^2 \times \{0\}$. What is that? If some product is 0 that means at least one of the factor must be 0, one by one we get all these circles, all of them have their centers on the y-axis.

The radius of the first circle is $n+1$, the radius of all other circle is $1/3$. And the centers are $(0,1), (0,2), \ldots, (0,n)$. The largest circle with radius $n+1$ actually encloses all the other circles, where as these smaller circles themselves do not encircle any of the other. That kind of picture is very important. So, they are mutually external to each other. The disc bounded by any one of them does not contain any other circle. That picture is important.

Now, what happens? Look at this area here bounded by the big circle and lying outside of the all the small circles. So, how do you get that area, how do you express any point in this? I am not interested in the areas as such, but points in this domain, so how do you get that one. They are inside this one means, the value of the first factor must be less than 0. Equal to 0 would have given the big circle. Less than 0 will give the inside region of the circle. Bigger than 0 will be outside. Same thing here I want it to be outside all small circles, so the value of each of the other factors is positive at all these points. Therefore the value of the product, the entire LHS is negative.

Indeed, conversely, whenever the value of the entire product is negative it represents a point inside this region. Take the negative of that negative number, that will be a positive number. Take the square root of that, that is your z. So, that is the meaning of this equal to $-z^2$. Then there are two square roots, accordingly you will get two values for z and hence two points in $\mathbb{R}^3$, one above the xy-plane and another below the xy-plane. So, take that one, you take the positive z, you will get the graph of that function precisely this part and then you will get another copy below this with $-z$. those 2 graphs is precisely this one. So, z equal to something is the graph of that function of two variables, namely square root of the minus of the LHS. You get two graphs corresponding to two square roots. The union of these two graphs this is surface.

If you take a point (x, y) not belonging to this region, then the value of the LHS is equal positive and so minus of that does not have a real square root. That just means that the set a point (x, y, z) statisfies (41) only of (x, y) belongs to this region, where the value of LHS is less than or equal to zero. That is a trick here.

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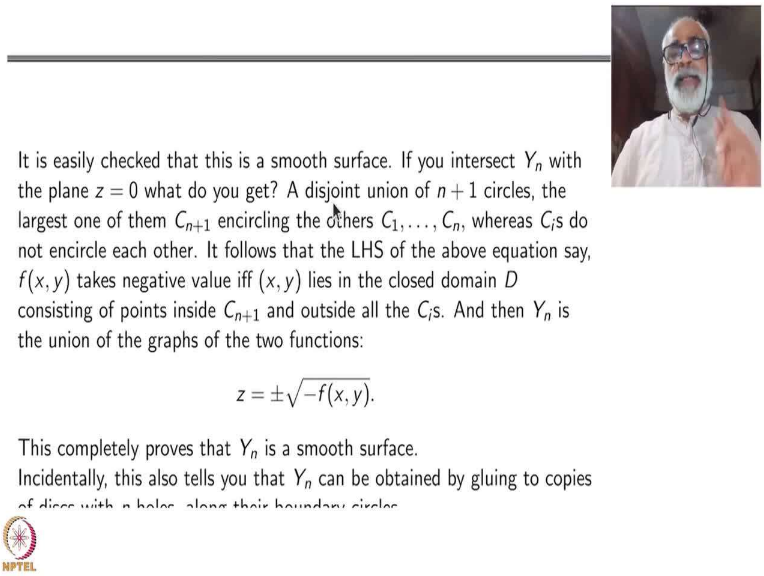

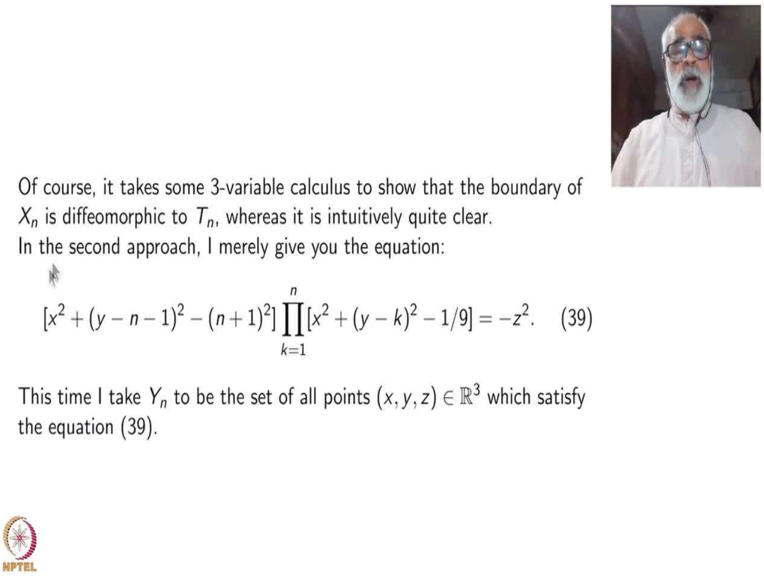

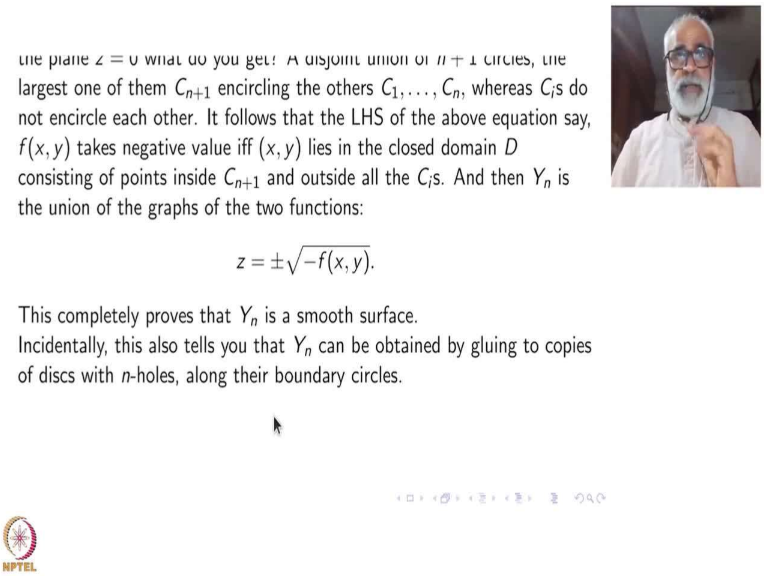

So, it is easily checked that Y_n is a smooth surface. If you intersect Y_n with plane $z = 0$, what you get? A disjoint union of $n + 1$ circle, all that I have written down here. Finally, Y_n is nothing but union of two graphs z equals plus minus square root of minus $f(x, y)$ where $f(x, y)$ denotes the polynomial function on the LHS of (41). That is enough justification to claim that Y_n is a smooth surface.

So, in two different ways I was shown that there are embedded surfaces inside $\mathbb{R}^3$. Up to homeomorphism, they represent members of list (b). The lone member in (a), obviously is there already, we know that. Members in list (c) cannot be represented by embedded surfaces in $\mathbb{R}^3$. Look at $\mathbb{P}^2$, you cannot embed it in $\mathbb{R}^3$. Look at the Mobius band, Mobius band is also non orientable which has a boundary. But that can be embeded in $\mathbb{R}^3$. The Mobius band is the same thing as making a hole inside $\mathbb{P}^2$, as soon as you make a hole you have a boundary circle. Take any surface in the list (c). Namely, P_n connected sum of n copies of $\mathbb{P}^2$. Make a hole in that. That surface can be represented by an embedded object inside $\mathbb{R}^3$. Not very difficult to prove, but I will not prove that here.

Alternatively, you do not want to make a hole then what best you can do? You can get an `embedding' except there will be one single crossing around a circle that is called a self-intersection of the immersed surface, such things are called immersions not embeddings. So, there will be two circles in the original surface and a map which will be an embedding except the map takes these two circles, to the same circle, they will be identified to a single circle in the image, so that is the meaning of that. So, that is the best thing you can do for all the non orientable surfaces namely those which occur in the list (c).

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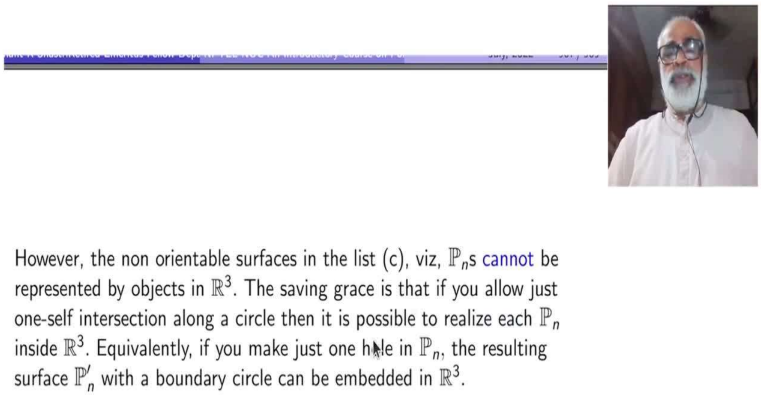

So, P_n's cannot be represented by objects in $\mathbb{R}^3$. The saving grace is that if you allow just one self-intersection along the circle then it is possible or equivalently suppose you grant us to remove a disc, then you can have n embedding, namely, take the immersion as above and wherever there is an intersection you cut that circle inside that there will be a disc and remove that disc. So you remove that disc which was extra then you do not have to cross it out. So, that is how you can get this embeddings of the surface after removing a disc.

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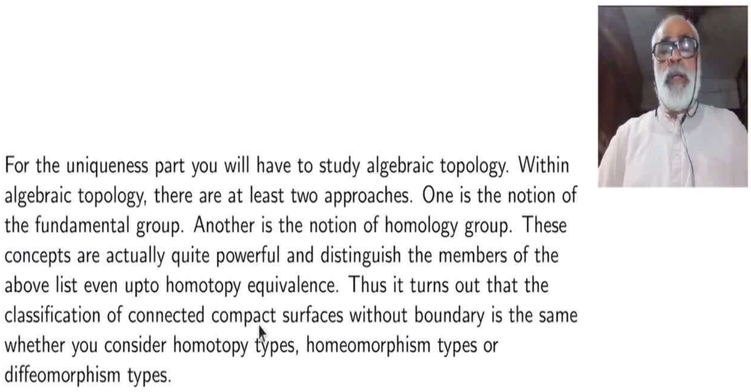

Now comes the uniqueness part, I will tell you a few things. The uniqueness part cannot be proved by pure points set topology. It is not like classifying the one-dimensional manifolds which could be done which we have done just using point set topology. You have to bring some tools from algebraic topology, you have to bring some invariants, you may use some heavy machinery like complex analysis, not elementary complex analysis etc. So, like Riemann did, in the beginning. So, you have to have some high machinery here.

So, I will tell you the simplest thing is that you can use the fundamental group. Just use fundamental group. You compare the fundamental groups of all these objects, they will be all distinct groups, distinct means what? Non isomorphic. That is not difficult to prove once you know what is fundamental group and how to compute these things for surfaces. Now if you know that if two spaces are homeomorphic to each other, then they will have isomorphic fundamental groups. Not only that if they are homotopy equivalent to each other, then also they have isomorphic funcdamental groups.

Since the list will give you different fundamental groups for different members, they must be a non-homeomorphic. So, that is the way the proof is completed. Instead of fundamental group you can use what is called homology groups. These homology groups may take a little more time, but they are much easier to understand, easier to compute than fundamental groups actually. They will also give you that these lists have different homology groups, just you have to look at H_1 and H_2 both of them you have to consider. H_2 will be good enough to distinguish between the list (b) and (c), list (a) is easy to differentiate. But within the list (b) or (c) you will have to use H_1 to distinguish two members. Homology groups come in a sequence H_0, H_1, H_3 and so on.

So finally, the big thing here is which is very special for surfaces and 1-dimensional manifolds. It turns out that the classification of connected compact surfaces without boundary is the same whether you consider homotopy types, homeomorphism types, or diffeomorphism types, this is a very deep result. We have not done the full thing here, even for 1-dimensional case.

The classification that we have done would have been just by homotopy type, likely you have to careful with boundaries, do not use boundaries, connected manifolds without boundary. Boundary you can later on include, once you have complete classification without boundary, Similarly here, up to homotopy you can determine the whole thing. You will have to assume that they are without boundary, otherwise, they will have some other problems which can be sorted out.

So, we have come to an end of the chapter as well as the end of the course here now. So, I would like to thank all of you first of all, if you are really come all the way to the end of this course, many people drop out in between, does not matter. So, thank you all, I thank my team of TA's, who have been especially useful for even getting the material in order and all that and they will be also helping you all the time in understanding this one, they have been helping you for in the tutorials or what you call assignments and so on, also in the discussion forum. So, big thanks for my team.

Not only that, the team from NPTEL they have been very-very encouraging and very helpful. So, I thank them all. See you some other time. Thank you.