An Introduction to Point-Set-Topology (Part II)
Professor Anant R Shastri
Department of Mathematics
Indian Institute of Technology Bombay
Lecture No. 61
Surfaces

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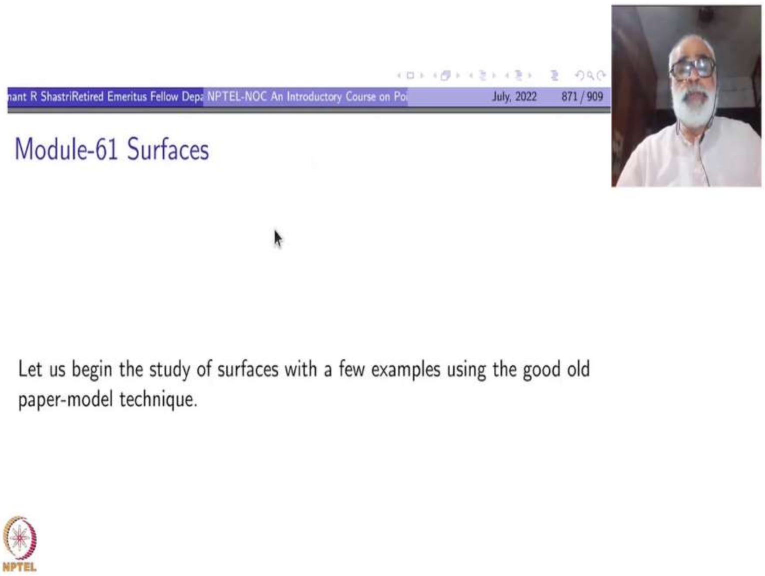

Hello, welcome to Module 61 of NPTEL NOC an introductory course on point set topology part II. So, in this chapter we will now study 2-dimensional manifolds. For short we call them surfaces, later on you may restrict the word `surface' for a suitable class of 2-manifolds. So, this chapter is going to be not a rigorous one, but intended to be expository, and introductory for things to come, like, motivating you people to study algebraic topology, and differential topology, and so on, manifolds in general.

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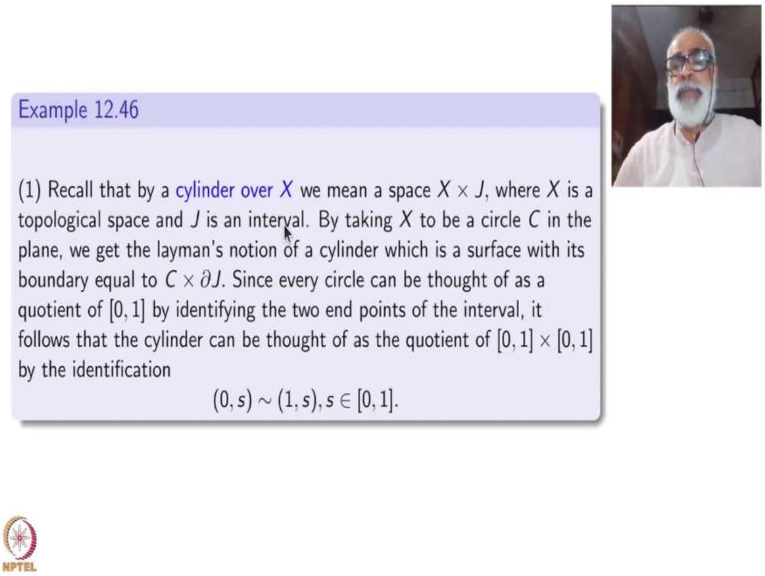

So let us begin the study of surfaces with the good old technique of paper models, which everybody uses. So, I begin with an example here which you are familiar with namely, in general, in mathematics we call a cylinder to be any space which looks like X cross an interval. Modelled on the most familiar object namely when X is equal to a circle. When X is the circle, circle cross an interval is a typical model for the cylinder that you have studied even in your 10 standard and so on. So, the surface that we have here is not general $X \times J$ but some curve cross interval, so that is the most general cylinder as such. But if you want real meaning of cylinder let us stick to the classical notion when X itself is a circle.

When I say circle, I mean only up to a homeomorphism. Therefore, you can take an ellipse also no problem. So, when you take circle cross J, J is an interval, its boundary will be circle cross boundary of J. At two different edges. For example, if J is just $[0, 1]$ the closed interval, then $C \times 0$ and $C \times 1$ are the boundaries of $C \times J$. So all these things can be verified. The point is I want to use a paper model for this one namely, I start with a rectangular piece of paper which is representative of (as a topological space) $[0, 1] \times [0, 1]$. But we were looking at modeling so you have to take a piece of paper which represents $[0, 1] \times [0, 1]$ as a topological space, the subspace of $\mathbb{R}^2$.

And then what we are going to do? To produce a circle from one of the factors, and keep the other fact the same, we identify, $(0, s)$ with $(1, s)$ fro all $s \in [0, 1]$. So, the first on the factor, the endpoints are identified. That I have to do for every s inside the second factor namely $(0, 1)$ here, so that is the identification on $J \times J$. So, very straightforward identification which gives you a cylinder, this much we already know.

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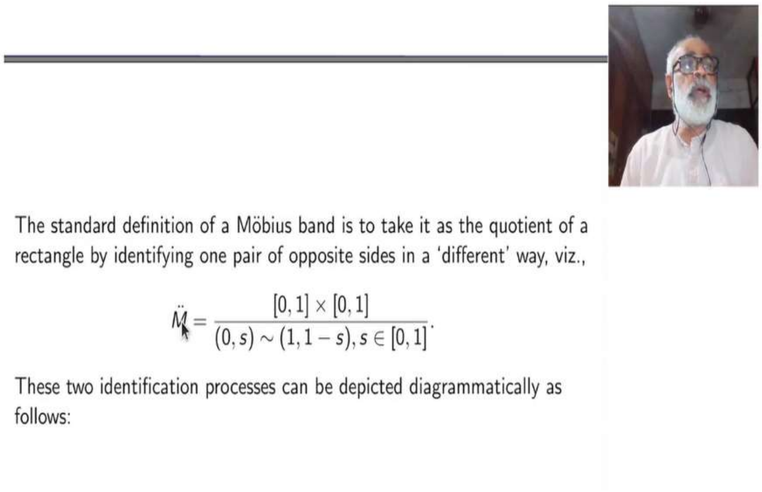

On the other hand, while identifying, if I just turn around the two sides namely, here I have this edge $\{0\} \times [0,1]$ and the second one is $\{1\} \times [0,1]$. Instead of s going to s, I turn it around, namely s going to $1-s$. So, $(0,s)$ is identified with $(1,1-s)$. Then what I get is the so called Mobius band. You must have studied this before. If you have not, you should do that now, namely, actually work out these two operations, and see that they give quite different objects. At whatever level you consider, these two are quite different objects. Even in the layman's language or in the language of a topologist. So, what are the different topological aspects of them, that itself is an interesting study. So, this you must have done. If you have not done it before, we can do it at some other time, there is no problem.

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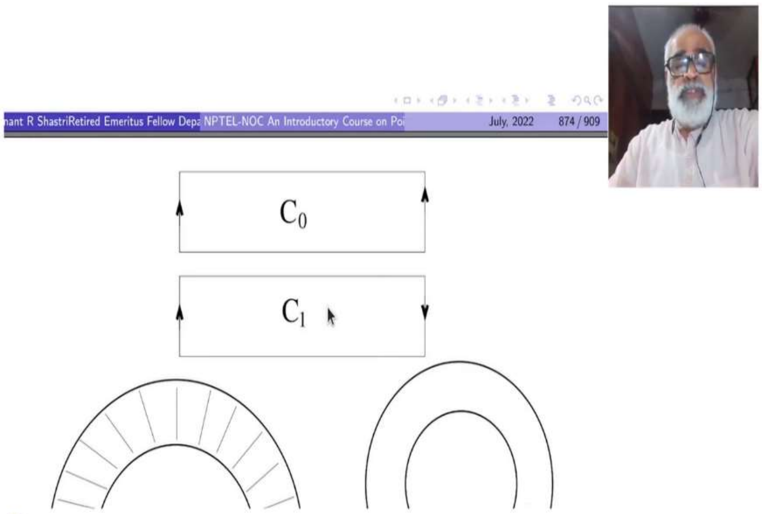

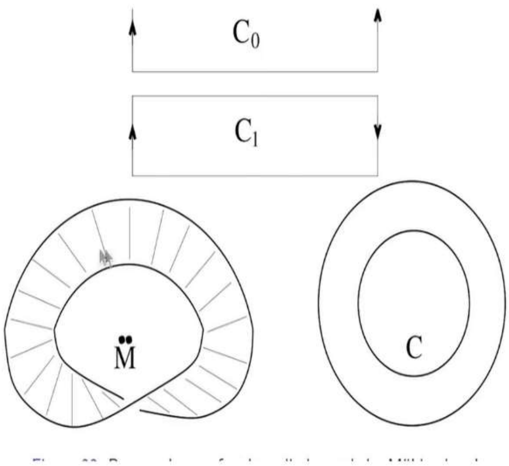

So here is the way the cylinder and the Mobius band are represented diagrammatically. Take this model here namely this is $[0,1] \times [0,1]$, there is no price for keeping the length the same there is no need. And then you identify this edge with this edge as the arrow indicates, the arrow is indicating how you are going from one to the other. Here the arrows are reversed you see, this is going to give you the Mobius band. Of course, when you actually identify it, this one will give you the annulus, which is equivalent to a cylinder and that will give you some surface like this where it will be a twist here, and the resulting object is not a part of the plane.

Here I have deliberately shown it as a subspace of the plane itself which is possible for a cylinder. However, this cannot be done for the Mobius band, there will be some crossing over. So, this will actually hang in the third dimension. So, that is a Mobius band. So, these are the paper schemes. So, just indicate this way, you have to understand that these two edges are identified, so that is the meaning of a paper scheme. So, with a piece of paper you can actually perform these identifications also, so that is all whole idea.

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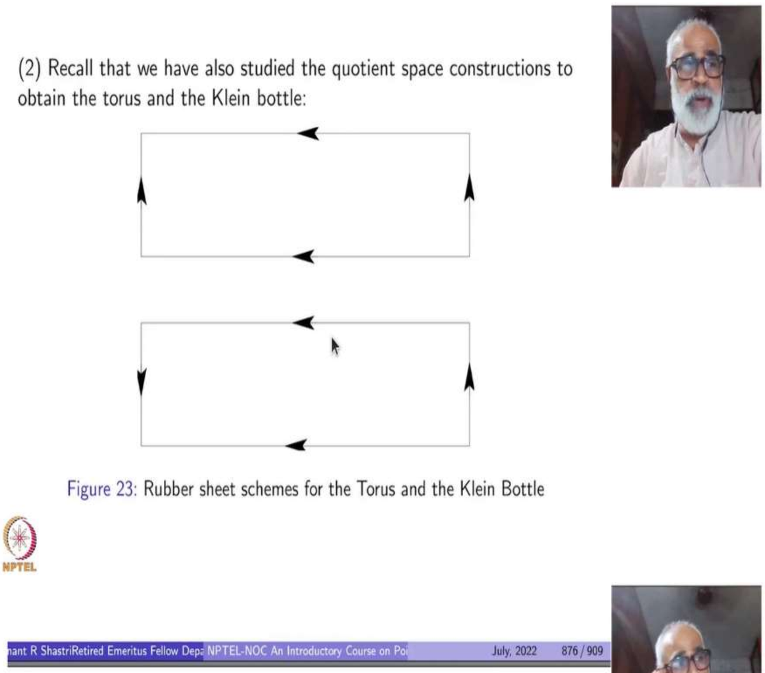

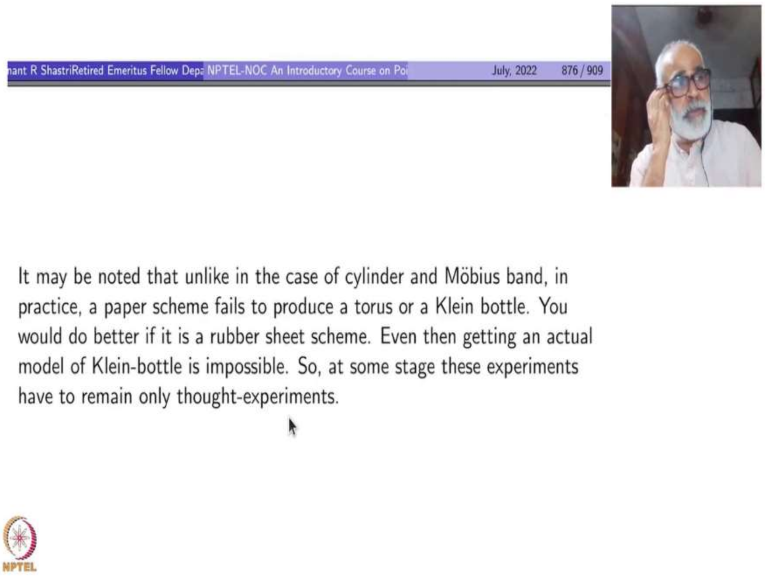

But there is other thing that you have studied in your part I, namely the Torus and the Klein Bottle. The Torus is got by in the first scheme here, namely again the same rectangular piece of paper, the opposite sides here are identified, these and these the same orientation like that, here also same orientation. Whereas, in the Klein bottle one pair is identified with the same orientation and the other one there is a twist, the orientation is reversed here, so this gives you the Klein Bottle. This will give you the Torus.

Now if you use a paper, you cannot actually perform these identifications to get even the Torus, forget about Klein Bottle. For Klein Bottle, even if you use a rubber sheet instead of a paper, you cannot perform the identifications inside $\mathbb{R}^3$. But for Torus, if you have a rubber sheet you can perform just like a like a cycle tube. But if you have a paper, of course 1 pair you can identify, after that the second pair cannot be identified, without crumpling the resulting cylinder. Once you have got a cylinder you cannot bend it to identify the two boundary circles there, because there is some rigidity with the paper also, it will crumble, it will not represent an embedded object in $\mathbb{R}^3$, as desired. So, for that we will have to use different tricks.

But, so finally, what I want to say is that the so called experiments that I am going describe here, they are partly experiments which we can actually perform; rest of them have to remain only thought experiments. Thanks to this word `thought experiment' the word introduced by Einstein, I think. So, this is what we can do finally.

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(3) Also recall that we have defined the projective plane $\mathbb{P}^2$ as the quotient of $\mathbb{S}^2$ by the antipodal action: $x \sim -x$. Let $q : \mathbb{S}^2 \to \mathbb{P}^2$ be the quotient map $q(x) = [x]$. Note that the open sets for $i = 1, 2, 3$,

$$U_i = \{(x_1, x_2, x_3) \in \mathbb{S}^2 \ : \ x_i > 0\}$$

are mapped homeomorphically onto open subsets of $\mathbb{P}^2$ and these open subsets cover the whole of $\mathbb{P}^2$. Therefore $\mathbb{P}^2$ is locally Euclidean. Because $\mathbb{S}^2$ is a Hausdorff space and quotient construction in this case is with respect to the finite group $\mathbb{Z}_2$-action via the antipodal map, it follows that $\mathbb{P}^2$ is Hausdorff also. For a similar reason, it follows that it is also II countable. Indeed it is also compact. Hence it is a 2-dimensional compact connected manifold.

(2) Recall that we have also studied the quotient space constructions to obtain the torus and the Klein bottle:

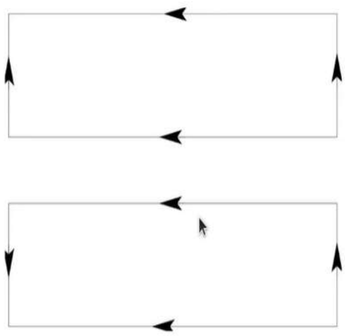
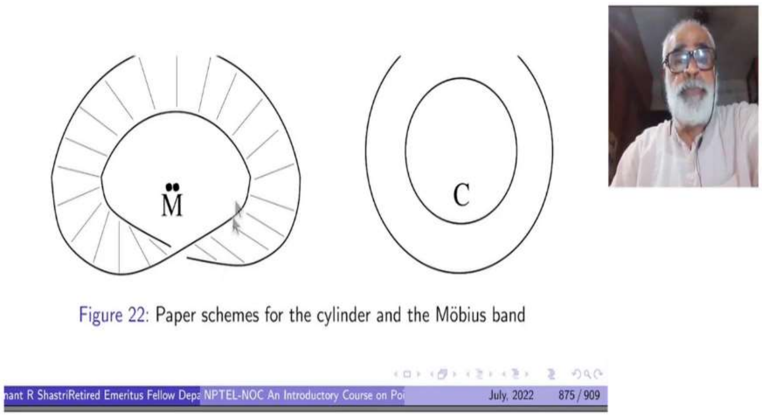
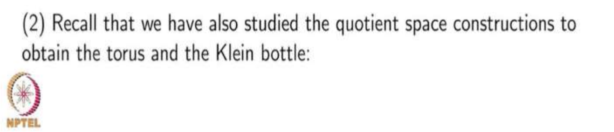
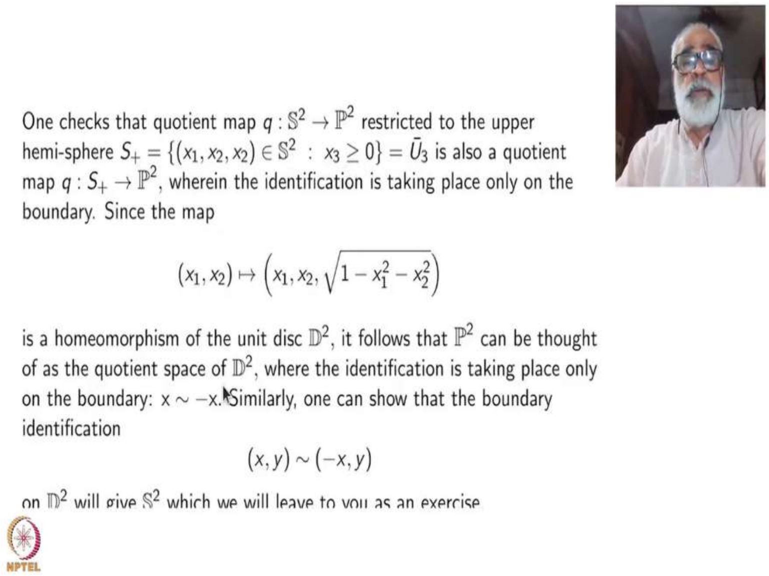


Figure 23: Rubber sheet schemes for the Torus and the Klein Bottle







Now let me recall that we have already verified that the standard 2-sphere is a surface. All $\mathbb{S}^n$ contained inside $\mathbb{R}^{n+1}$, they are n-dimensional manifolds that we have seen. But now I am going to give you I am going to recall that you have also defined the projective spaces. In particular, the projective space of dimension 2, $\mathbb{P}^2$, which can be obtained as a quotient of $\mathbb{S}^2$ by the antipodal action, namely x being identified with $-x$ for all x in $\mathbb{S}^2$. So, that is the projective space. We can use the notation q from $\mathbb{S}^2$ to $\mathbb{P}^2$ to denote the quotient map. Also we can use this square bracket $[x]$ to denote the equivalence class of x, where x is inside $\mathbb{S}^2$. The square bracket $[x]$ will be inside $\mathbb{P}^2$.

Note that if you take all points (x_1, x_2, x_3) inside $\mathbb{S}^2$, namely, satisfying summation x_i^2 equal to 1, suppose you put the condition, x_1 is positive or x_2 is positive or x_3 is positive and so on, those are coordinate neighborhoods in $\mathbb{S}^2$, each of them will be mapped homeomorphically onto some oper subsets of $\mathbb{P}^2$ because, the moment one of the coordinate is positive, when you take the action, it will become negative so, those two will be disjoint, so only one of them will be there, so q will be injective on each of them.

And all points of $\mathbb{P}^2$ are covered by these three open sets. Taking any $[x] = [(x_1, x_2, x_3)]$ in $\mathbb{P}^2$, you can change the sign to make one of the coordinates x_i to be positive signs and make it positive. So, the above three open sets cover, $\mathbb{P}^2$ entirely and they are each of them homeomorphic to an open subset of $\mathbb{R}^2$, because they are now homeomorphic to coordinate open subsets $\mathbb{S}^2$, via q, they are actually discs inside $\mathbb{S}^2$, they are homeomorphic to open some discs inside $\mathbb{R}^2$ also. Therefore, $\mathbb{P}^2$ is a locally a Euclidean space is of dimension 2. $\mathbb{S}^2$ is compact therefore $\mathbb{P}^2$ will be also compact because it is a quotient. Image of the compact set under continuous map is compact. So, compact and well you can immediately see that it is it is actually second countable also no problem, but what is important here is to see that $\mathbb{P}^2$ is a Hausdorff space. Then it will be a 2-manifold.

So to see Hausdorffness, there is a general thing you have studied, under group actions. namely, if you have a finite group action on any Hausdorff space, and the action is fixed point free then the quotient space is automatically a Hausdorff space. In particular, this action is just the $\mathbb{Z}_2$ action $x \mapsto -x$. You can immediately see that given any point x and $-x$, so you can separate them by this method same thing can be strengthened as follows. If you have two distinct points $[x], [y] \in \mathbb{P}^2$, they correspond to four distinct points in $\mathbb{S}^2$. Take the minimal distance d between any two of them and take open balls of radius smaller than $d/2$ it follows that their images will give didjoint neighbourhoods of $[x]$ and $[y]$ in $\mathbb{P}^2$.

So, $\mathbb{P}^2$ is a nice examples of a compact 2-manifold though a little non trivial one.

So remember this Klein Bottle is actually a little more complicated than the projective spaces. The Torus is simpler, you can see the image of it in $\mathbb{R}^3$ itself. All of them are manifolds without boundary, whereas, the cylinder and the Mobius band these are much simpler objects in some sense, they have boundary, one has two boundary components, and the other has only one boundary component, which is obviously a circle.

So, so far all these examples are all familiar objects to us. The whole idea is to get to the theory slowly through such examples. So, do not hesitate if you have any questions you can raise right now. So, I do not want to state much theorems here, but try to give you glimpses of what are the things happening here. So, I have given you explicitly how to get the Hausdorffness in one such case.

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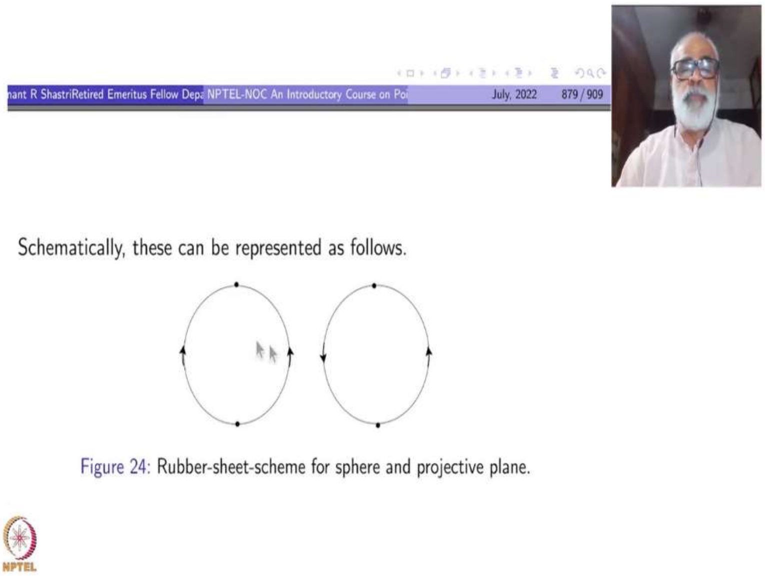

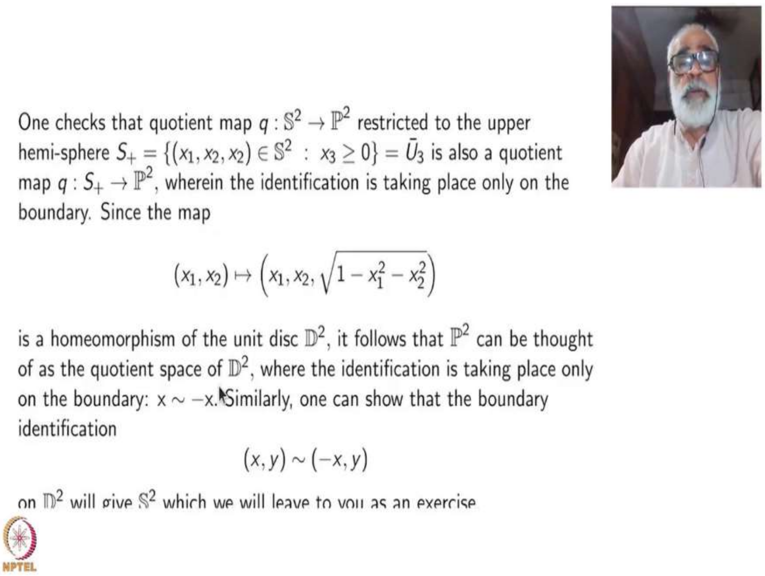

So schematically, even $\mathbb{S}^2$ and $\mathbb{P}^2$ can be represented by a paper scheme. Not with rectangles always. In general, you can just take some disc and identify some portions of the boundary. So, that is what I want to tell here.

Check that the quotient map q from $\mathbb{S}^2$ to $\mathbb{P}^2$. Restrict q to just the upper half sphere, viz., all (x_1, x_2, x_3) in $\mathbb{S}^2$, with $x_3 \geq 0$. So, let us call this $\overline{U_3}$. Then the entire thing below is covered because everything else is minus of that, we do not need that separate representations for minus of some point here in U_3. We do not need that. So, if you just take $\overline{U_3}$ and then restrict q the quotient map on it, that itself is a quotient map. Therefore, you can think of $\mathbb{P}^2$ as a quotient of the closed upper hemisphere which is homeomorphic to $\mathbb{D}^2$.

Now the identification is occurring only on the boundary of U_3, which is a circle contained inside $\mathbb{R}^2 \times \{0\}$. So, what is the identification? Identifications, again are x going to $-x$. But it is performed only on the boundary. So, you can think of $\mathbb{P}^2$ being obtained from the unit disc $\mathbb{D}^2$ in $\mathbb{R}^2$, by the identifications on the boundary circle, $x \sim -x$.

See, as the point x moves along the circle it as moves counterclockwise, $-x$ will also move counterclock wise. In the picture we have indicated with arrows. The boundary is divided into two parts, which are interchanged under the antipodal map. So, the identification can be just carried on as x moves along one of then arcs.

On the other hand, look at this picture here. Again, you have a 2-disc here and we are carrying out identifications along the boundary circle. This time we use the action $(x, y) \mapsto (x, -y)$. As the point moves like this its image will be moving like that, that is the homeomorphism. What is it? It is just z going to $\bar{z}$ you can say or just $(x, y) \mapsto (x, -y)$.

So, this is another paper model you can say for the 2-sphere. And this is for the projective space $\mathbb{P}^2$. Try to perform these two with a paper model, you may not succeed. However, there are models for the first one.

So this is just like a lady's purse, which is a snap-shut purse, just close it up like that. So, something like this you have seen perhaps, the purse like that. So, that will give you a 2-sphere representation.

However, logically (it is possible to show that) you cannot actually perform the identifications in the case of this projective space. There are deeper reasons the projective space cannot be embedded inside $\mathbb{R}^3$, so you cannot have a model representing projective space inside $\mathbb{R}^3$.

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Motivated by these experiments in the simple situations, we now consider the following general process for obtaining compact surfaces with or without boundary.
Begin with the 2-disc $\mathbb{D}^2$ (or any homeomorphic copy of it in $\mathbb{R}^2$) with a fixed orientation on the boundary circle, say, for definiteness, the counterclockwise one. Next, select a finite number of points (at least two) on the boundary circle which cut the boundary circle into finitely many arcs. Let us call these point vertices and the resulting arcs edges. Now label these arcs with letters and write them sequentially in a counter clockwise sense, to obtain a word which is well-defined upto cyclic permutation. That is because, you may start at any one of these arcs, while labeling. You are allowed to use the same letter at most twice. Next look at a letter x which occurs twice, you have a freedom to put the superscript -1, such as x^{-1}, when it occurs second time, or leave it as it is. For instance, the examples in the diagram correspond to the following

Schematically, these can be represented as follows.

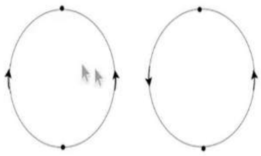


Figure 24: Rubber-sheet-scheme for sphere and projective plane.

(2) Recall that we have also studied the quotient space constructions to obtain the torus and the Klein bottle:

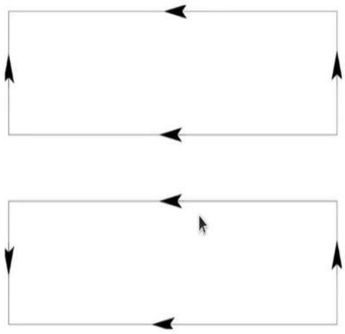
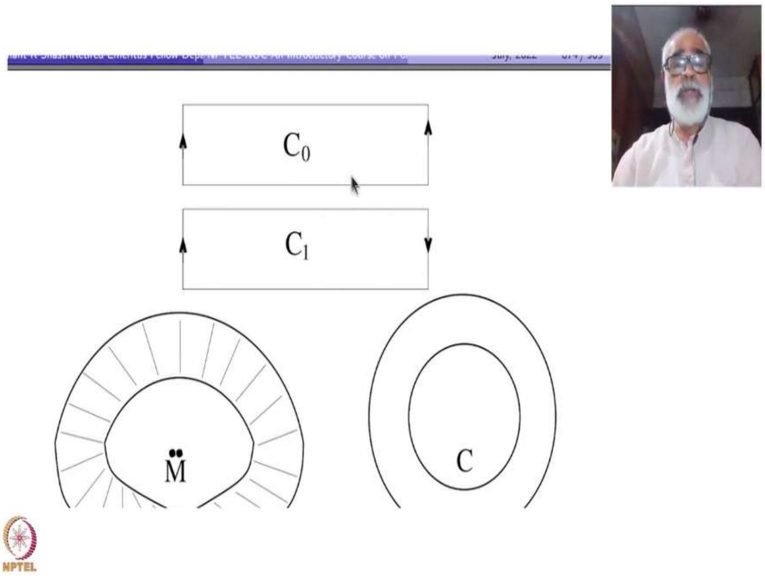


Figure 23: Rubber sheet schemes for the Torus and the Klein Bottle



So motivated by these experiments in simple situations, we now consider the following general process for obtaining compact surfaces with or without boundary. Anything which you want to perform it has to be a finite process, so compactness creeps in. So, you start with a piece of paper, so it is compact, so everything is compact now. Compact with or without boundary both of them we will consider. The Mobius band and the cylinder etc. are easy examples, let us not exclude them. Let them be there. So, begin with the 2-disc that itself is a 2-dimensional surface, it has a boundary, so boundary is non empty. So that is the starting point, very nice.

Now what do we want? Any homeomorphic copy of the disc will do the job. We will also call such as space also a disc. So, let us fix an orientation on the boundary once for all. The boundary is one single circle. So, for definiteness let us say the counterclockwise sense that is the standard orientation. Next, you select a finite number of points, at least two of them on the boundary so that the entire boundary is cut out into finitely many pieces each of them a copy of the closed interval $[0, 1]$. You do not actually perform the cuts but we just imagine that.

So like if you have an interval cut it into n parts means what? You are taking a partition $t_0 < t_1$ ... and so on. So, writing less than less than less than is not possible in the circle, but orientation makes sense. You can take points, k points of the circle and then you can label them in a counterclockwise sense $Z_1, Z_2, \ldots, Z_k$. So, these counterclockwise sense of labeling is unique up to a cyclic permutation, why? Because you can start from any one of the k points and then you have to end the labeling just before you hit again that point where you started. Do not repeat, that is all. So, that is the way you label the points, points which you have chosen on the circle.

Let us call them vertices and then the resulting arcs there, we will call them edges, this is just for convenience. Now label these arcs, (forget the labeling of vertices) with letters, again in a counterclockwise sense, just the way you did for vertices. That is what I am doing. Again, you may start at anyone of these edges while labeling, then you will get a different labeling, but that labeling will differ by a cyclic permutation that is all.

So, you are allowed while labeling the edges, to use the same letter at most twice, i.e., repetition is allowed but only once. That is all. So, a, b, c, d, and so on, but the next one may be again a next one may be, and after that you should never use a and b because you have used them already twice. So you have a finite sequence of letters.

So next, look at any one of the letter x which occurs twice in your sequence. So, look at it and you have freedom to put a superscript 1 or -1 on it. If you choose to put 1 that is as if you do not have any superscript there just the way you write the number 5 to mean 5^1. However, 5^{-1} is different. Similarly, a -1 superscript has a special significance which we will explain soon. The same thing you could do for all letters, but it serves no purpose to put a superscript on letter which occurs only once in the sequence.

So here is an example, one sequence consisting of two letters, a, a^{-1}, another one just letter two a, a. The third one is $a, c, a^{-1}d$, the 4th one here $acad$. So, here in the second one, I have a letter which is repeated but the exponent is not changed. Now the idea of a repeated letter is to indicate which edge is going to be identified with which one.
Let us look at the example $a, b, a^{-1}b^{-1}$. The edge is labeled a and the second one is b. That means we are not going to identify these to edges ever. But the next one is labeled a^{-1}. So, I am going to identify it with the first edge, but the way I do that is indicated by the exponent. Here it says, that I have to carry out the identification in the opposite direction. You see the standard orientation on each letter without any exponent is counterclockwise, but the exponent -1 indicates that you have to reverse it. In the diagram it easier to represent it just by putting an arrow in the current direction. similarly, here we have b^{-1} which means the

fourth edge will be identified with the second one in the opposite direction. So, this is the scheme for the Torus : $aba^{-1}b^{-1}$.

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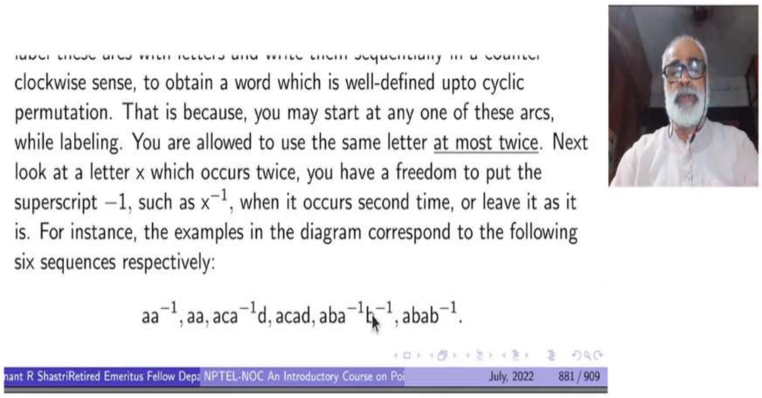

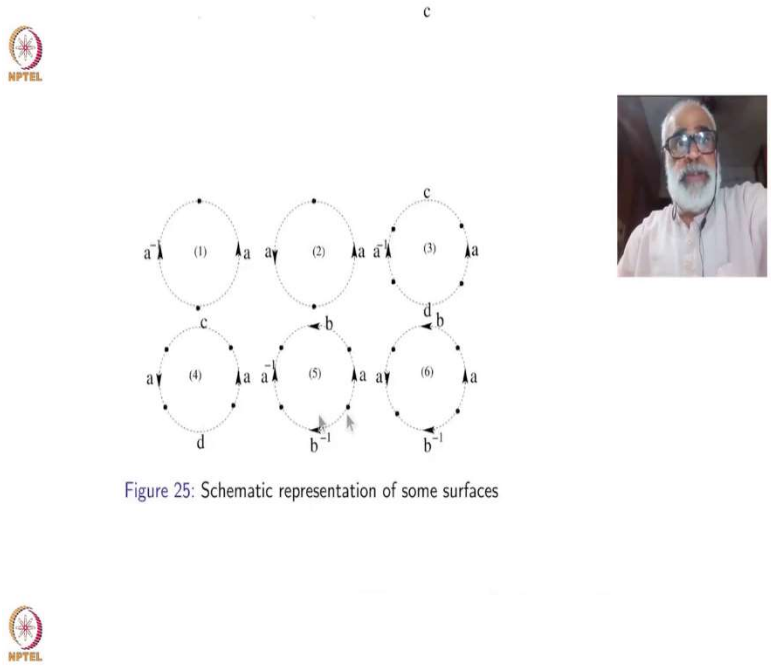

So, all these things are listed here, six different schemes and the corresponding diagrams.

So they are my starting examples, easy examples. I have redrawn them, I do not need now, any of these rectangle or convex polygons and so on, one single shape will do viz., the 2-disc itself. So, all of them are drawn on the disc now. So, the first one is $\{a, a^{-1}\}$, this will give you a 2-sphere. the second one $\{a, a\}$ represents $\mathbb{P}^2$. This $\{a, c, a^{-1}d\}$ will give you what? The pair $\{a, a^{-1}\}$ when we have identify this will give you the cylinder. The edges c and d will automatically become the two boundary circles. The fourth one will give you a Mobius band, $\{ac, ad\}$. So the edges c and d are left out, but after identification they will give you one single circle.

The fifth example is the Torus and the last one is the Klein Bottle. The schematic representation of the 6 of the surfaces that we can access easily.

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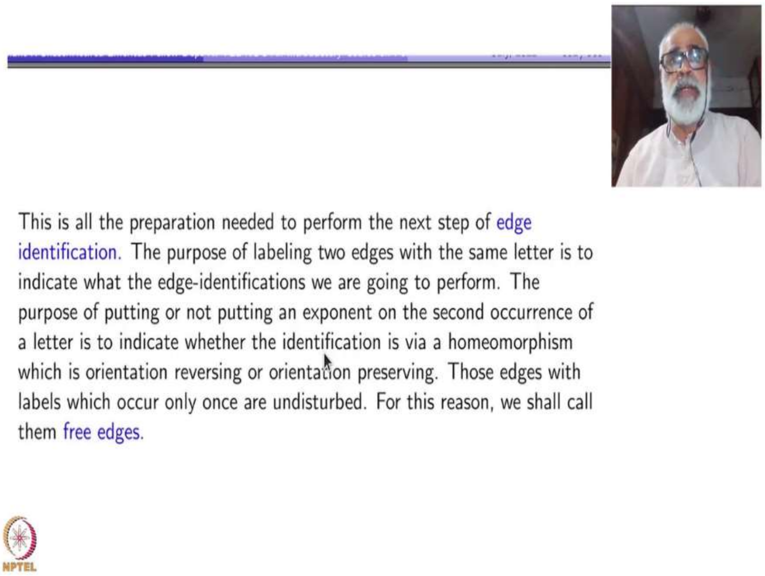

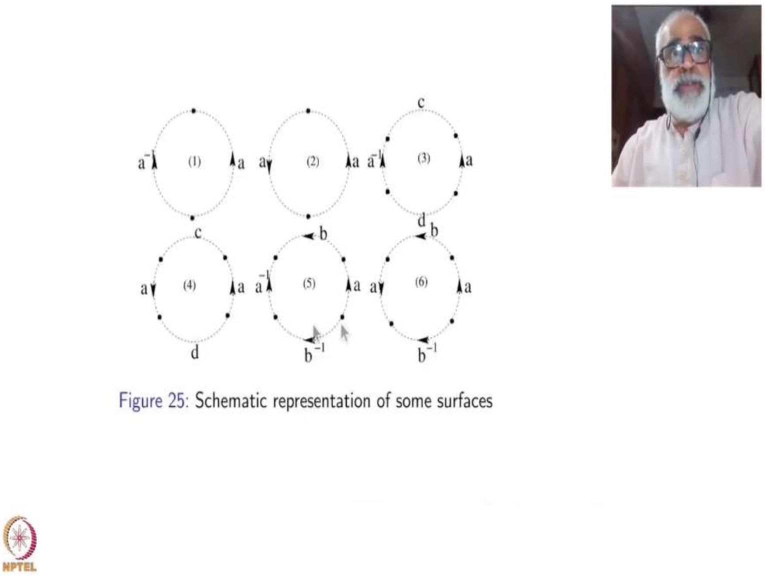

Now let me try to generalize this kind of thing. This is the preparation needed to perform the next step `edge identification'. I have just chosen some scheme, what I am going to do? I am going to perform some identification on the edges. how? Namely follow these instructions whatever they may be. Whenever an edge is repeated identify those two edges, how do you identify? Depending upon the exponents. If you have $\{aa\}$, in the same way. Remember you have to fix the orientation on the entire circle right in the beginning. Then any letter x for an edge indicates that you have to take that edge with the induced orientation and if it is x^{-1}, you have to take it with orientation opposite to the induced one.

So identify means what, you have to use a homeomorphism from one edge to the other which preserves the chosen orientations on each of them. So, only thing that matters is whether you finally have homeomorphism which preserves orientations or reverses orientations.

Some edges are never identified with another one in the scheme, because they are not repeated, so those edges are called free edges. So, what will happen to them when you perform all the identification? Those free edges will remain free, so they will be the boundary parts of the resulting surface. This is what we want to see now, let us see.

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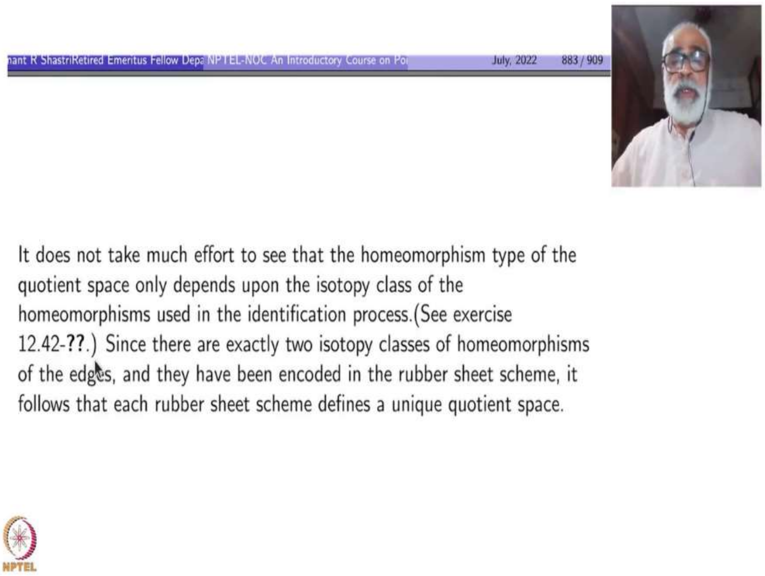

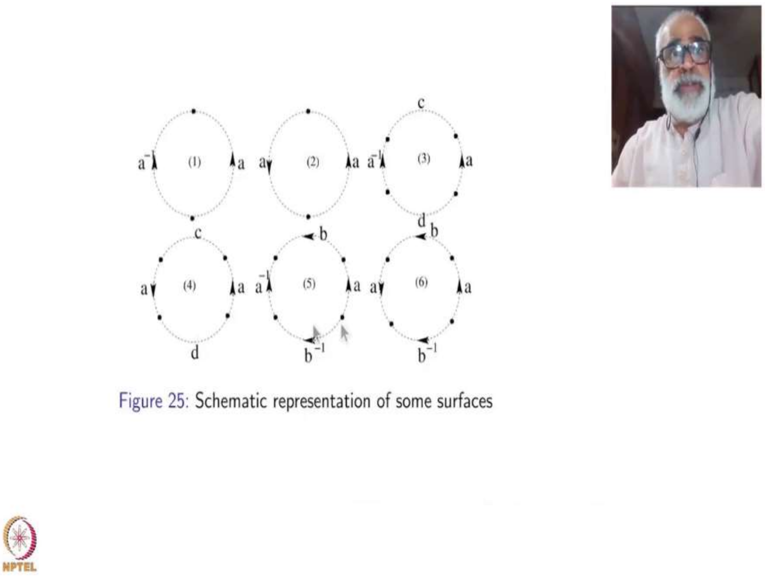

So it does not take much effort to see that the homeomorphism type of the quotient space only depends upon the isotopy classess of the homeomorphisms used in the identification process. So, this is what I meant by those exercises 12.42, 43, 44, 45 whatever. I hope you had some time to spend it, even if you do not solve them. What is the meanings of isotopy etc. You must have understood by now. Since there are exactly 2 isotopy classes of homeomorphisms from $[0, 1]$ to $[0, 1]$, you can call them orientation preserving and orientation reversing. They have been encoded in the rubber sheet scheme by what? by putting an exponent on some letters representing edges.

So it follows that each rubber sheet scheme defines a unique quotient space there is no ambiguity in the definition of a quotient space obtained by using a scheme. In other words for all matter, the scheme represents the entire topology of the quotient space. So, this is the underlying principle.

We are not going to prove any of these statements here, by the way. It does not take much effort to see that the homeomorphism type or quotient space only depends upon the isotopy class of homeomorphisms used in the identification process. So, I repeat just that but I do not prove it.

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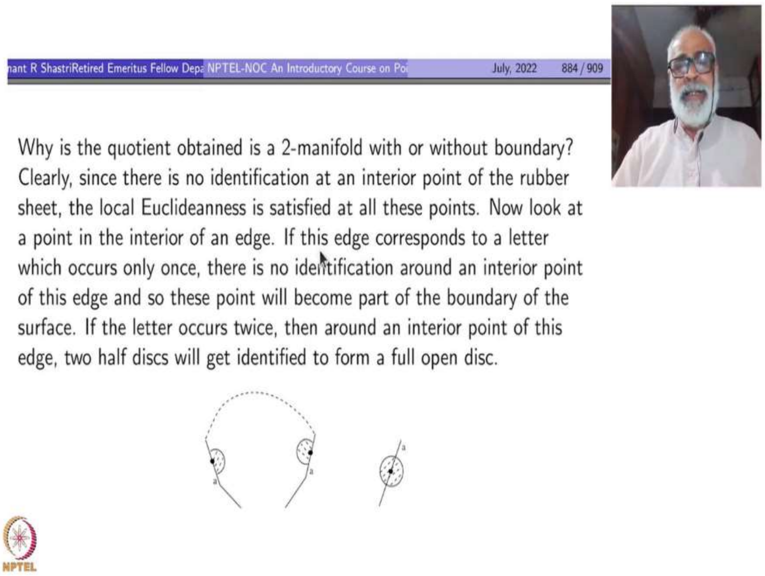

which occurs only once, there is no identification around an interior point of this edge and so these point will become part of the boundary of the surface. If the letter occurs twice, then around an interior point of this edge, two half discs will get identified to form a full open disc.

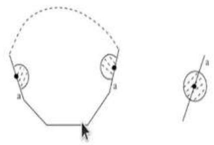

Figure 26: Local Euclideanness at the image of an interior point of an edge

Now why is the quotient so obtained is 2-manifold? That is a more serious more mundane question, that we should try to understand, manifold with or without boundary. So, I already told you the role of free edges. On a free edge, at the most the two endpoints, the vertices, may get identified, because at each vertex there are two edges incident, so, even if one of them is not free is not free that will affect the vertex. If in this process, the two end points of a free edge get identified, then the edge will become a circle in the quotient space. You can look at any interior point of a free edge. On one side there is the disc and on the other side there is nothing. So you can see that it is actually a boundary point. We will we will make it clearer.

Clearly since there is no identification at an interior point of the entire disc, the quotient map q restricted to the open disc is a homeomorphism onto an open subset of the quotient space. Therefore, at all those points our quotient space is locally Euclidean of dimension 2. So, now move to a point on the interior of an edge, like this. This is an edge but do not look at the vertices for the time being. Look at an interior point of the edge. At that point what happens? It is possible that this letter a is repeated in the scheme that means, this edge and the repeated edge are getting identified, what happens? This half disc neighborhood here and half disc this neighborhood there they will get identified only along the edge to give you a full disc neighbourhood around the those points, in the quotient space. So the image of that point becomes an interior point with an open neighborhood homeomorphic to an open disc inside $\mathbb{R}^2$. So, therefore, at these points also we have the local Euclidianness. What is left out? Image of all these vertices that is not very easy to handle. When you take the image of all these vertices and their neighbourhoods in the quotient space is it homeomorphic to again a disc or a half disc or neither? I merely assert that at these points also, we can show that the quotient space is locally Euclidean of dimension 2, without adding any extra condition on the rule of identifications. I will skip the proof for that.

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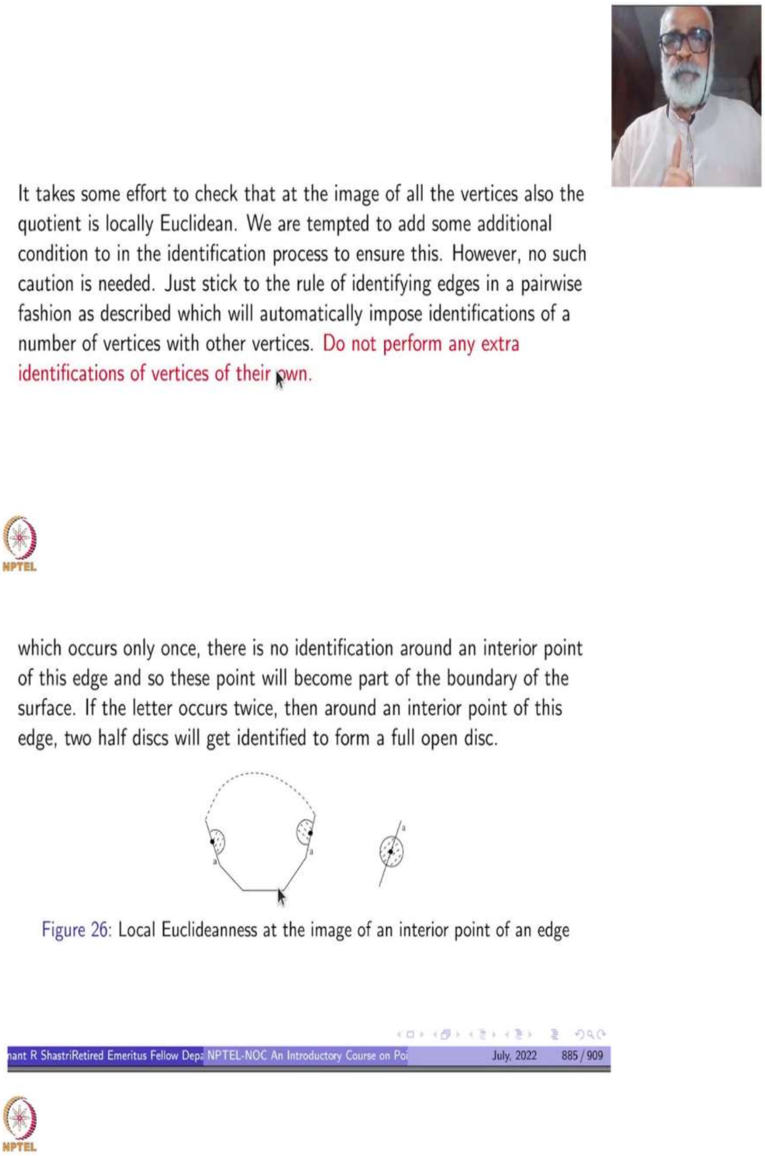

But I will assure you that you do not have to put any more conditions. My paper scheme is well defined, it will always produce a manifold with boundary provided there are free edges otherwise it will produce a manifold without boundary. So, I want to tell you that do not perform any extra identifications of vertices on their own, you are supposed just identify certain pairs of these as indicated. After doing all those identifications some vertices are automatically identified with some other vertices, because identification is taking place on the closed edges and not on open edges. So that is a point to be noted here.

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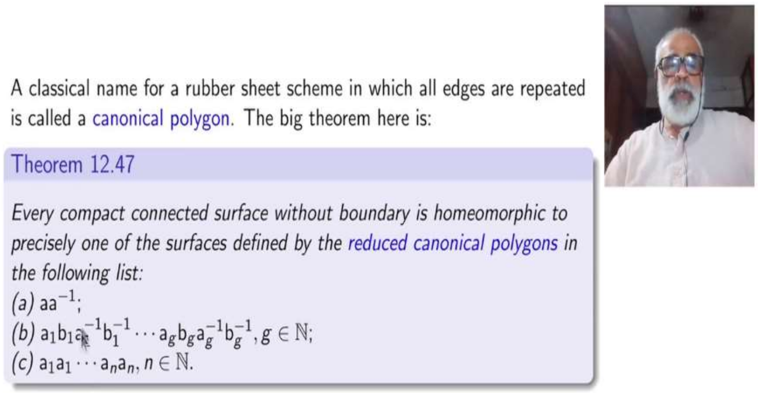

Now I want to state something. Actually, this is a big theorem now, immediately. A classical name for a rubber sheet scheme in which all edges are repeated is called a canonical polygon, everything is repeated, I mean repeated only once. So, aa^{-1}, cc^{-1} or something like that, no free edges are there, that is the meaning of that. Such a scheme will be called a canonical polygon. I have not much use for this terminology but I will use it because it is classical and I want to state something. Among these canonical polygon there are some which are called reduced canonical polygons. I will list them and the beauty is that this list is so complete that it will represent all the compact surfaces, compact, connected 2-manifolds, without boundary, because there are no free edges.

The first one in the list is a lonely element, it is a very typical element, it is like you have 0 before you list natural numbers. So, $\{a, a^{-1}\}$, you know what it represents. We have already done that $\{a, a^{-1}\}$ represents the 2-sphere. Then there is list consisting of an infinite sequence of members, indexed by g, where g runs over all natural numbers.

They are $a_1b_1a_1^{-1}b_1^{-1}a_{2b2}a_2^{-1}b_2^{-1}\ldots, a_gb_ga_g^{-1}b_g, g \in \mathbb{N}$. There is a definite purpose denoting this number by g. This g is short for `genus'. So that the standard name for the surfaces represented in this list being torus of genus g. Suppose I stop at $g = 1$, then I know that this is nothing but our standard Torus. So that Torus is called Torus with genus 1. You can call this member in (i) a Torus of genus 0. But nobody uses that kind of terminology.

So these are all closed surfaces that means they are compact and without boundary. Also they are all orientable. So, this list gives you all of them as g varies over $\mathbb{N}$.

The third list is again an infinite sequence indexed by natural numbers but here I use ordinary $n \in \mathbb{N}$. They are $a_1a_1a_2a_2, \ldots, a_na_n$. (Do not write $a_1^2a_2^2$ etc, that makes no sense. These are sequences there is no algebra here, third are not multiplied here). So, as n varies you know the first one a_1a_1 is nothing but the projective space.

So you may wonder where is the Klein Bottle? It is neither here nor here now? The story is that the Klein Bottle and all those non orientable surfaces are all hidden in the (iii)

So, $a_1a_1a_2a_2$ will represent the Klein Bottle, you understand. So, in general what is happening if you do not use reduced canonical polygons? As listed? Several paper models several rubber sheet models may represent the same surface. So, here the list is shortened and now the claim is that every member here is a distinct member up to a homotopy type, upto homeomorphism type, and actually, even upto diffeomorphism type. That is the strongest theorem here. Two of them will be different if they are listed in different sublists (a), (b), (c), first of all, and then, if two of them belong to the same list (b) or (c) then the different numbers here, will tell you that they are different homotopy types That is the strongest.

If two things are not homotopic to each other, they will not be homeomorphic to each other, if they are not homeomorphic to each other, they will not be diffeomorphic to each other. So, this classification is very strong classification. Up to homotopy type they are different, so that is the meaning of this one. So, let us stop here today, I will tell you a little more about how these things actually look like geometrically, how the Klein Bottle is hidden here, etc. So those things I will try to tell you next time. Thank you.