An introduction to Point-Set-Topology (Part- 2)
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Lecture No 60
Classification of 1-dm. Manifolds continued

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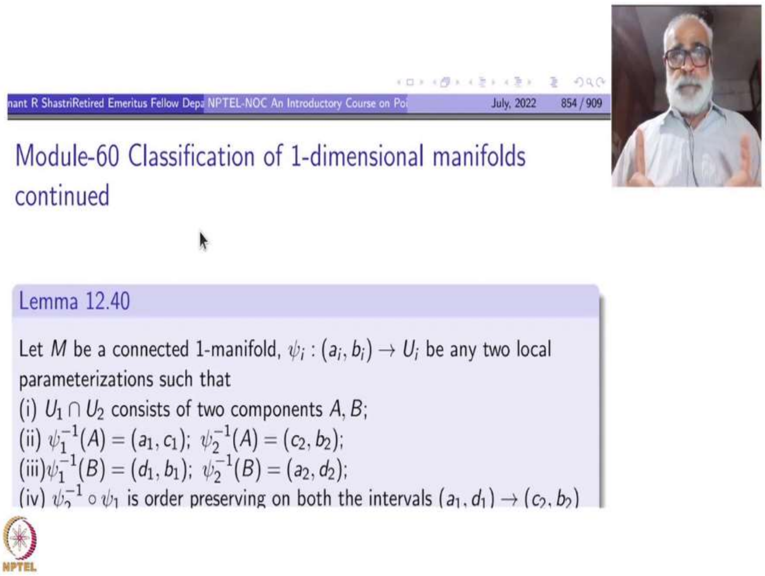

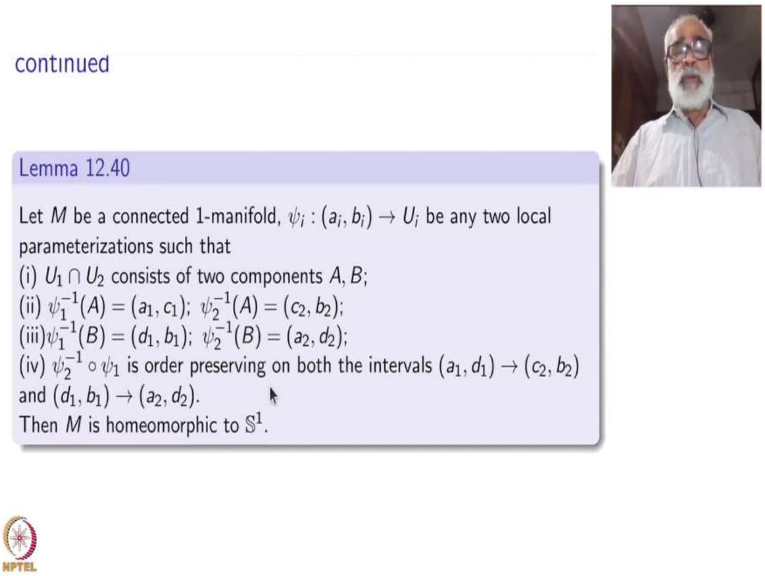

Hello. Welcome to Module 60 of NPTEL NOC an introductory course on Point-Set-Topology Part-II. So, today we shall continue our classification of 1-dimensional manifolds which we started last time. Beginning with any manifold with or without boundary the first thing we did was to reduce the proof to the case of manifolds without boundary. And now, look at a manifold X without boundary.

Take a cover by coordinate neighborhoods, as soon as there is an open cover there will be a countable subcover because the manifold X is II-countable. Next thing we want to to is how these members of this cover, which are all homeomorphic to open intervals intersect each other.

So, we made two list of things which we want to happen and one of them we consider namely, when the two open intervals like this, they actually intersect in a very nice way like this, then we could get a map to the union from open interval which is a homeomorphism. So, the union itself was an open interval with the homeomorphism. So, this was a nice case. The next case that we want to consider today is that two of them have intersection consisting of two connected components and they intersect like this properly not like that, or that and so on.

So, the two things are coming this way, just the way in the first case and the other one also coming nicely like that. So, this is the case we want to understand now, and this is another desirable case. So, that the first one. After that, we will see that these are the only two cases possible. So, that will allow us to complete the classification.

So, start with any connected 1-manifold M, let ψ_i from (a_i, b_i) to U_i be any to local parameterizations, such that

(i) $U_1 \cap U_2$ consists of two components. I am just labeling the two components A, B.

(ii) The first one $\psi_1^{-1}(A)$ is open interval (a_1, c_1). (See the domain of ψ_1 is (a_1, b_1). So, this $\psi_1^{-1}(A)$ is (a_1, c_1) is a non trivial assumption.) And $\psi_2^{-1}(B)$ is (c_2, b_2). So, here it is the left end and there in the other one, it is the right end. So, that is the hypothesis, these are all hypotheses.

(iii) The third part is that $\psi_1^{-1}(B)$ (exactly the opposite here) is (d_1, b_1) and $\psi_2^{-1}(B)$ is (a_2, d_2).

So, this is the other end right end and this is the left end and so on.

(iv) The fourth condition is, now look at $\psi_2^{-1} \circ \psi_1$, starting from two parts of the interval (a_1, b_1) go via $\psi_1(U_1 \cap U_2)$, then follow by ψ^{2-1} back to the part of the interval $(a, 2, b_2)$, so, from interval to interval, this is a homeomorphism, that must be order preserving on both the intervals, namely, the first portion coming from A and second portion coming from B, first portion will be (a_1, d_1) to (c_2, b_2) and the other one will be from (d_1, b_1) to (a_2, d_2). So, both of them should be order preserving. Then M is homeomorphic to $\mathbb{S}^1$.

The conclusion is of course quite strong. There are four different conditions. I have assumed here. In the earlier case, we had assumed that none of U_1, U_2 cover the entire union, which is the same as saying one does not contain the other. Remember that. In this case, the assumption that the intersection consists of two components automatically gives you that one does not contain the other.

So, there is no need to separately state it. So, the conclusion is that the union will be now homeomorphic to the circle itself, the entire M is circle. In other words, whenever such things happen, there is no other open subsets needed to cover M. The whole M will be $U_1 \cup U_2$. So, this is the whole idea. Let us see how the proof goes. This proof is not all that difficult, once you have understood the previous one.

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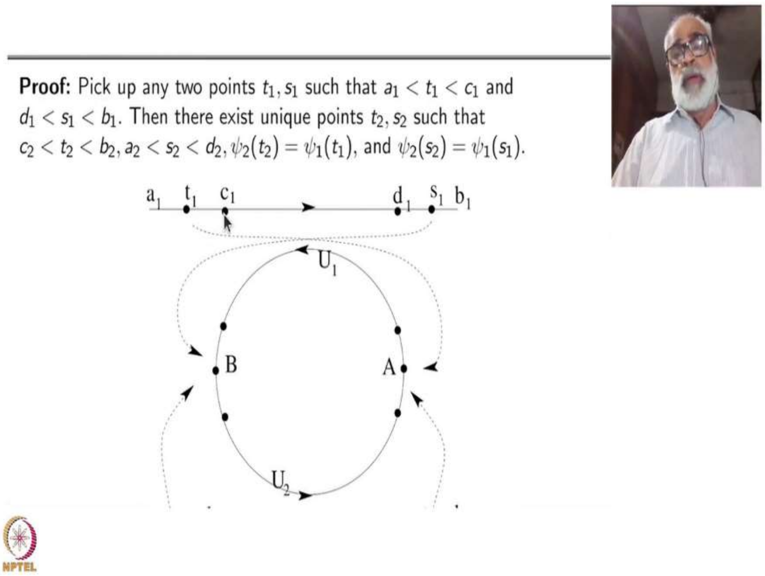

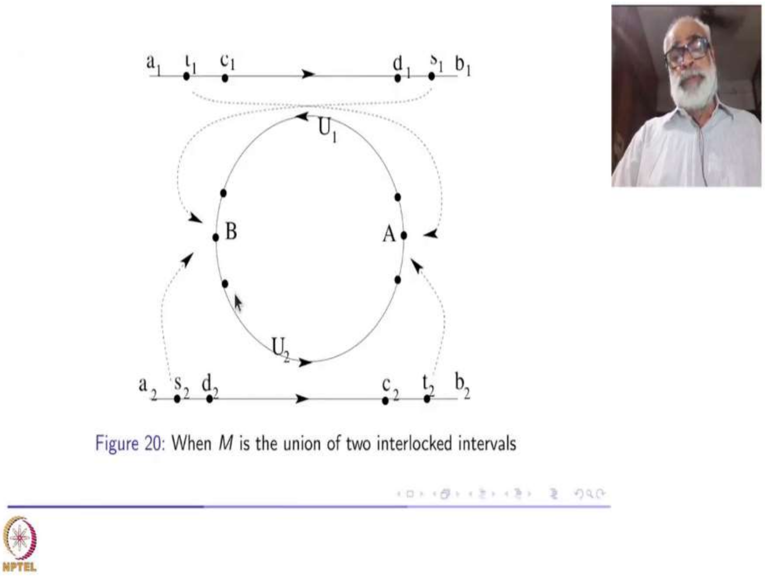

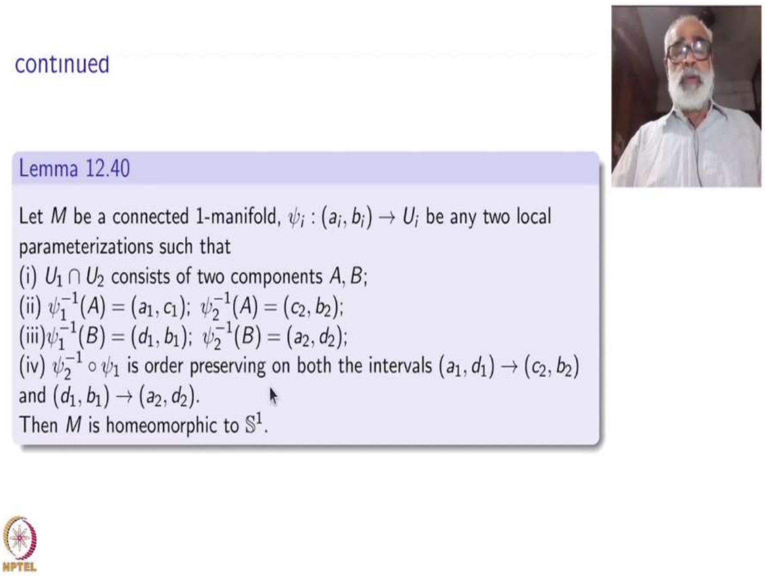

So, here, just look at the picture. (a_1, b_1) to U_1, you have one coordinate neighborhood ψ_1. And here (a_2, b_2) to U_2, you have another ψ_2. Of course, the final picture, I have put it nicely, but right now you have to assume that this is some manifold that is all. So, this is U_2 part below homeomorphic to that one and that U_1 part above, and the intersection is this $A \cup B$. So, U_1 comes from here to all the way here up here and U_2 is from this point to that point.

So, these two are the components of the intersection A, B. Now, $\psi_1^{-1}(A)$ is this part (a_1, c_1), $\psi_1^{-1}(B)$ is (d_1, b_1). Similarly, $\psi_2^{-1}(A)$ is (c_2, b_2) and $\psi_2^{-1}(B)$ is (a_2, d_2). Not only that, when you come from here to here by ψ_1 then take ψ_2^{-1} to here, So, you get a homeomorphism from (a_1, c_1) to (c_2, b_2), that must be order preserving. Similarly, you start from here (d_1, b_1) with ψ_1, come here to B and then go by inverse image of ψ_2, you get a homeomorphism from (d_1, b_1) to (a_2, d_2) that must be also order preserving. So, these are the assumptions that I have made. The conclusion is that this $U_1 \cup U_2$ is a circle, not only that, once you have that one there is nothing else M is a connected manifold. So, it is the whole of M. This is what we have to see

So, pick up any two point t_1 and s_1 as shown here.

Namely, $a_1 < t_1 < c_1$ and $d_1 < s_1 < b_1$. Then there exists unique points t_2 and s_2. What are they? Look at the image of s_1 here and that is the image of something here because these are homeomorphism anyway. So, there is a unique s_2 here which comes to the image of s_1, viz., $\psi_1(t_1)$ and $\psi_1(s_1)$, they are also equal to $\psi_2(s_2)$ and $\psi_2(t_2)$, respectively.

So, you have started picking up these points. So, clearly these t_2 and s_2 will be obviously lie in (a_2, d_2) and (c_2, b_2), respectively. that is all. After that what you do?

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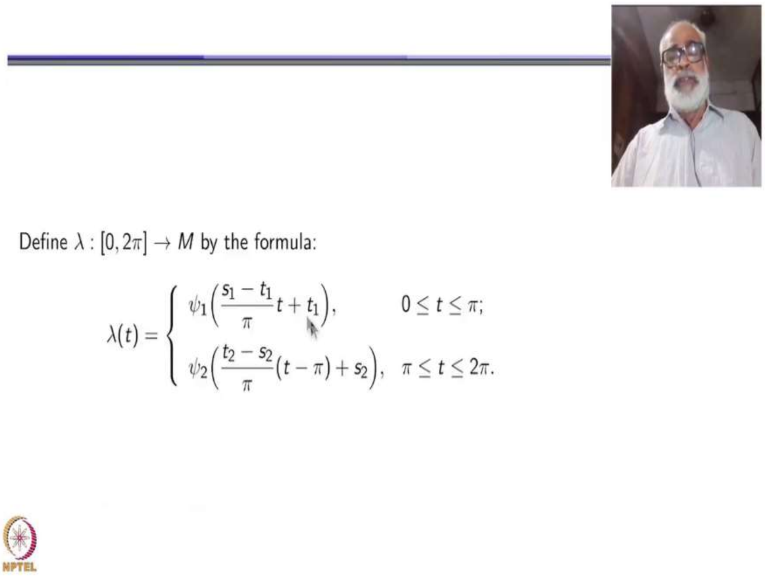


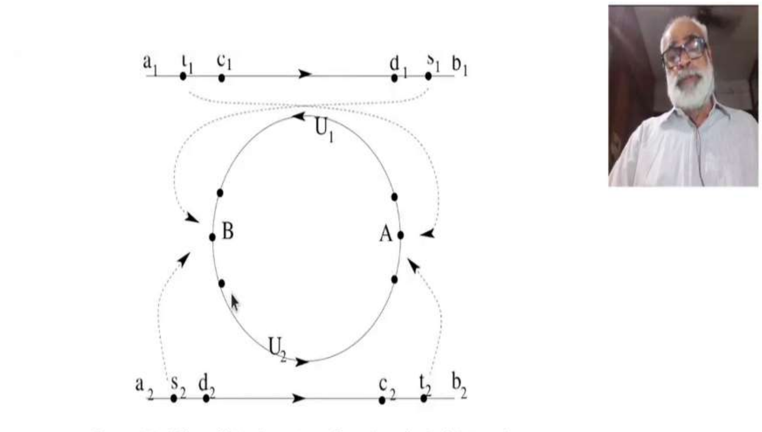
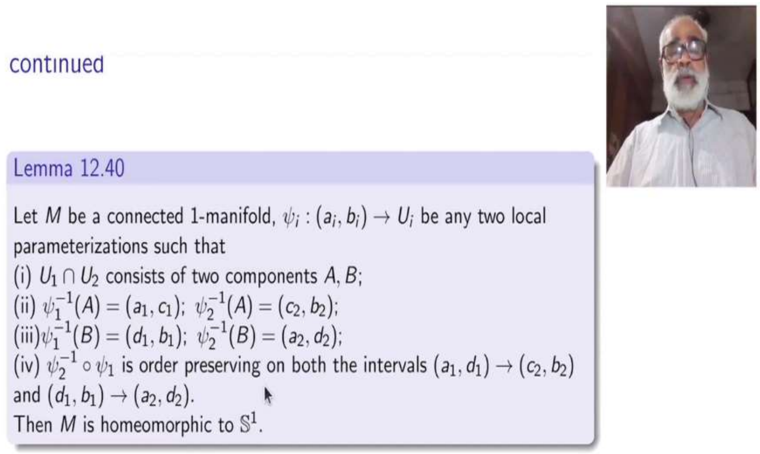


Figure 20: When M is the union of two interlocked intervals



Now, you just define the map λ from $[0, 2\pi]$ closed interval to M by the formula:

In the first part, it is ψ_1, in the second part it is ψ_2. The only thing is you have to adjust the whole thing by re-parameterizing the intervals. And where do you break up? First take t upto π, I am going to take ψ_1, from π to 2π, I am going to take ψ_2. Here in the picture it is clear. So, from this point t_1, I will map up to this point s_1, using ψ_1. For π to 2π, I will use ψ_2 from s_2 to t_2. I will ignore the rest of these overlapping parts. But $[s_2, t_2]$, I will re-parameterize by $[\pi, 2\pi]$. Similarly, $[t_1, s_1]$ is parameterized by $[0, \pi]$. So, map will be like this. So, this part is coming at this part coming here. So, the arrow is here, this arrow is here this way, so you have to understand. So, this is counterclockwise since I have taken here.

So, all the time after reparameterising I just take $t \mapsto e^{it}$, or equivalently $t \mapsto (\cos t, \sin t)$. In particular, t_1 is being mapped first to 0 and then to $(\cos 0, \sin 0) = (1, 0)$. Go all the way up till here and then pick up ψ_2 here. So, the idea is clear and formula is also clear. So, take the order preserving affine linear homeomorphism $[0, \pi]$ to $[s_1, t_1]$ and similarly from $[\pi, 2\pi]$ to $[s_2, t_2]$. So, that is all idea. So, this are the maps.

So, the first one is $t \mapsto (s_1 - t_1)(\pi(t + t_1))$. When $t = 0$ this is t_1 and $t = \pi$, (π above π below cancel out), plus t_1 and minus t_1 cancel out you get s_1. Similarly, when $t = \pi$ here, this is s_2 and when $t = 2\pi$ this will become t_2. So, take ψ_1 of this, and ψ_2 of that.

When these two points are the same, namely for $t = \pi$, there are two formulas you have to see they are the same. Why they are same? Because ψ_1 of this point and ψ_2 of that point are the same. Therefore lambda is well defined and is continuous.

Similarly, $\lambda(0)$ is equal to $\lambda(2\pi)$, viz., at the two ends, λ is given by $\psi_1(t_1)$ and $\psi_2(t_2)$ respectively and they are the same. So, that is what you have to see. So, what happens is first of all on $[0, 2\pi]$, it is continuous because on the common point they agree, so it is a continuous function. In each interval $[0, \pi]$ and $[\pi, 2\pi]$ is given by a homeomorphism.

So, λ is injective because they are mapped into different components, different parts of the codomain. And here, here comes the important thing that they are order preserving. So, there is a common portion here namely I have taken this part. So, this part is covered by ψ_2 also. I have covered this part, this part have been covered by ψ_2.

Also, on the intersection they are order preserving, therefore, the leftover parts are also covered by other map. This argument is similar to what we have seen in the first case, yesterday. So, λ is surjective onto $U_1 \cup U_2$.

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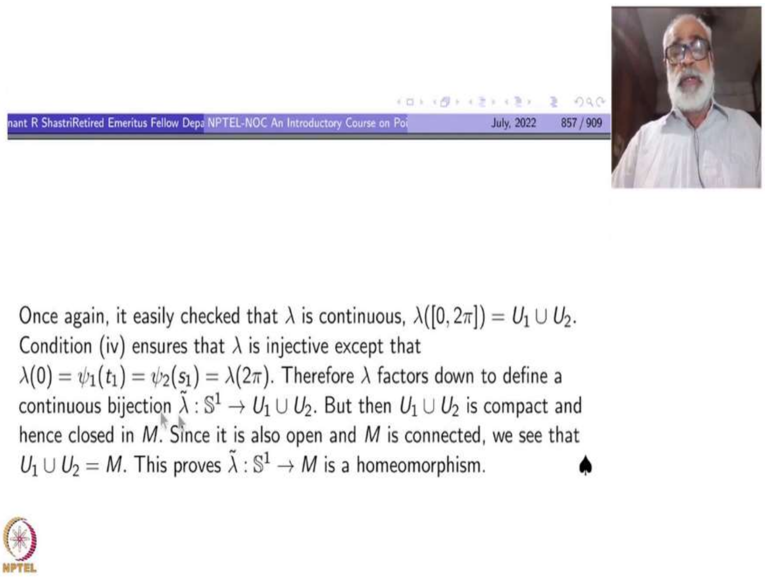

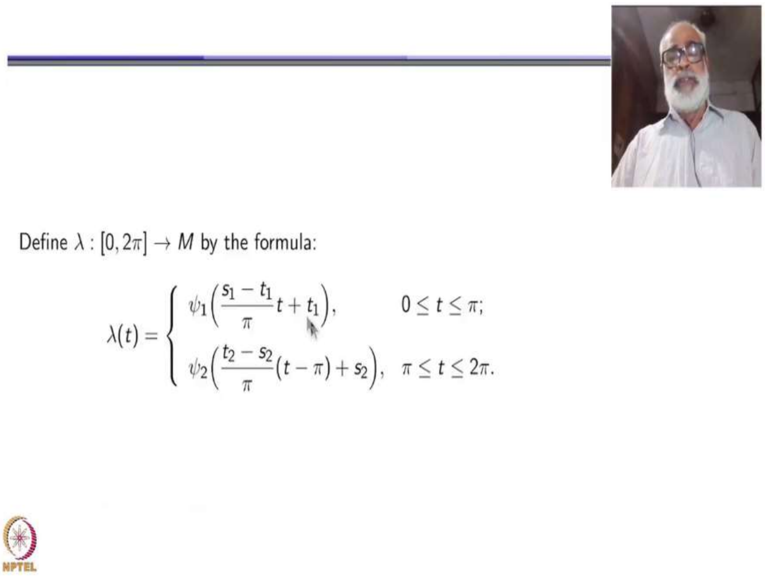

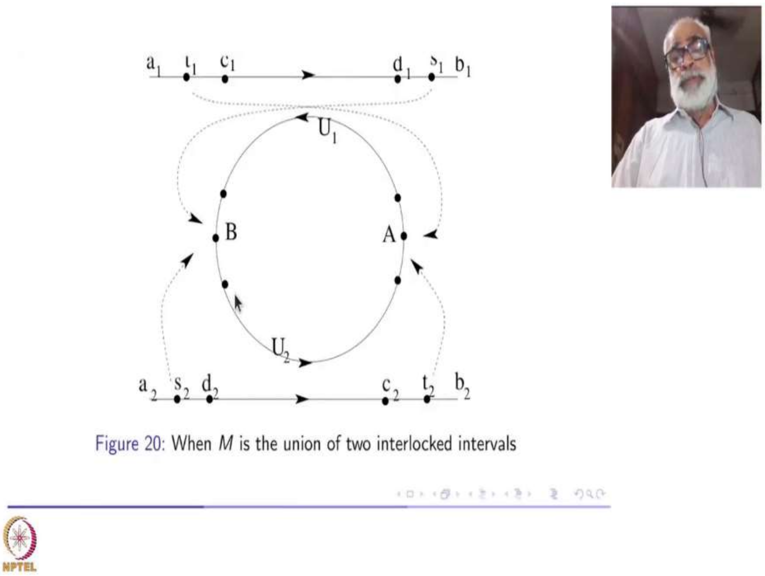

Figure 20: When M is the union of two interlocked intervals

So, the entire lambda is injective except that the two end points 0 and 2π are mapped to the same point. That is well and good, that is precisely what we wanted here. Therefore, λ will λ factor down to a continuous bijection $\tilde{\lambda}$ from $\mathbb{S}^1$ to U_1 and U_2, under the quotient map $t \mapsto e^{it}$ from $[0, 2\pi]$ from $[0, 2\pi]$ to $\mathbb{S}^1$.

So, under that, this will give you a homeomorphism now, continuous bijection from the circle to into M, $\mathbb{S}^1$ is compact, M is Hausdorff. It is surjective onto $U_1 \cup U_2$. So, it is a homeomorphism of $\mathbb{S}^1$, onto its image and the image is $U_1 \cup U_2$ that much we know. But then, $U_1 \cup U_2$ is open as well as being compact, it is closed also in M. Therefore, it must be the whole space because M is assumed to be connected.

So, every bit is used here.

So, we have completed the proof that M is actually homeomorphic to the circle in this case.

So, two important cases which will produce the two different members of the list have been covered. So, now, the claim is that there is no other case.

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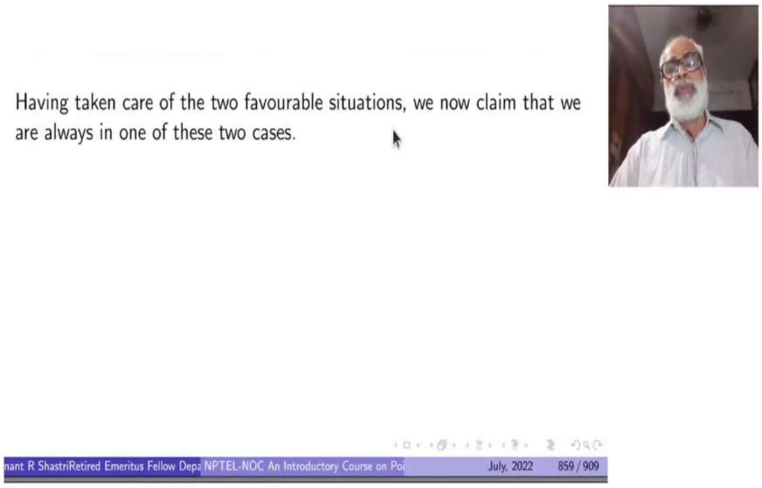

Having taken care of these two favorable situations, we now claim that we are always in one of these two cases. What is the meaning of that?

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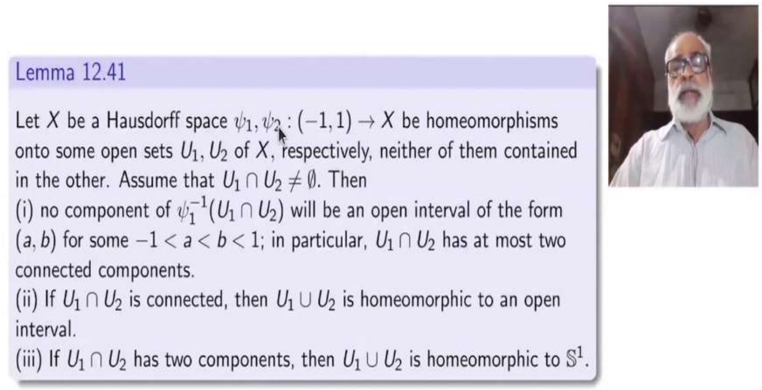

Let me elaborate. Start with a Hausdorff space X, ψ_1 and ψ_2 from $(-1, 1)$, (I have changed the notation for the domain intervals, this does not matter, you could have taken any open intervals) to X be homeomorphisms onto some open subsets U_1, U_2 of X, respectively, neither of them contained in the other. Assume that the intersection is non-empty. Then these are the only things that can happen:

(i) No component of $\psi_1^{-1}(U_1 \cap U_2)$, (which I have assumed, is non-empty), will be an open interval of the form (a, b) where, $-1 < b < 1$. Notice that first of all, connected components of an open subset of $(-1, 1)$ are open intervals. The claim is that those open intervals none of them will be able to avoid both the endpoints of $(-1, 1)$, they will not be away from both the endpoints. That is the meaning of saying that $-1 < b < 1$ does not occur.

Then what else can happen? It can happen that only if one of the endpoints must be there in the interval. So, I assume -1 belong to a connected component. If then 1 is also there, it is will be the whole space that is not allowed. So, it is of the form $(-1, b)$ for some $b \in (-1, 1)$. Similarly, it can be $(a, 1)$ for some $a \in (-1, 1)$. So, there are only two possibilities. Therefore, the conclusion is, in particular $U_1 \cap U_2$ has at most two connected components. This is obvious because connected components of $U_1 \cap U_2$ are in $1-1$ correspondence with the connected components of $\psi_1^{-1}(U_1 \cap U_2)$ because ψ_1 is a homeomorphism. See, this U_1 and U_2 are sitting in the topological space X, but here we have come inside an open interval. So, there you can see that there are only two possibilities, at the most you have two components. So, this is the strongest thing you have to observe.

(ii) Next, if $U_1 \cap U_2$ is connected, (that means only one component case) then $U_1 \cup U_2$ is homeomorphic to an open interval.
(This is our first case. So, that is what I have to verify.)
(iii) Lastly, if $U_1 \cap U_2$ has two components, then $U_1 \cup U_2$ is homeomorphic to $\mathbb{S}^1$. This is the second one.

So, this is precisely the meaning of saying that there is no other possibilities. Indeed, the argument for this has been already used by us. Any way let me elaborate on this one.

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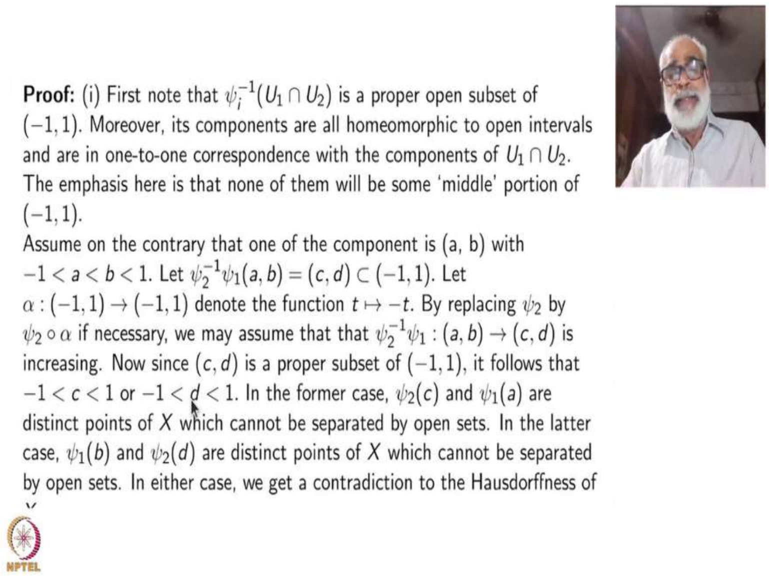

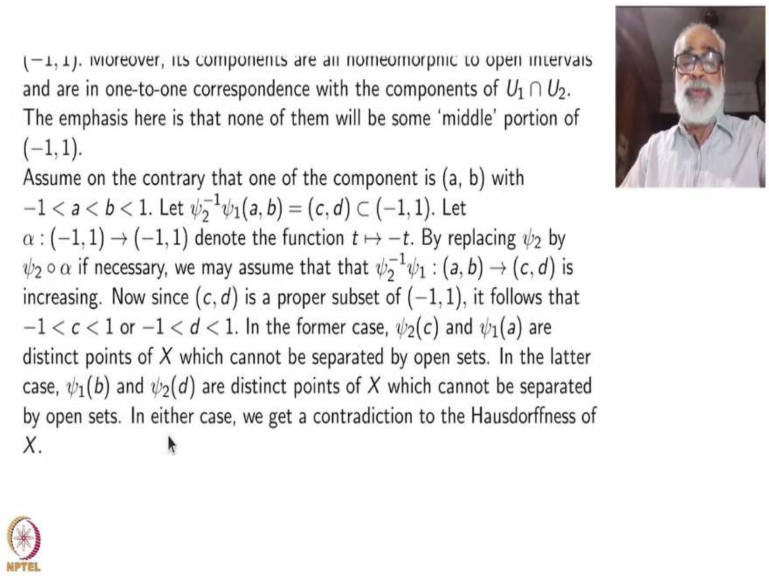

First of all, note that $\psi_i^{-1}(U_1 \cap U_2)$ is a proper open subset of $(-1,1)$. It cannot be the whole space because neither U_1 nor U_2 is contained in the other, so $U_1 \cap U_2$ cannot be the whole of $U_i,\ i = 1, 2$.

Moreover, its components are all homeomorphic to open intervals and they are in one-one correspondence with the components of $U_1 \cap U_2$ because ψ_i is a homeomorphism.

The emphasis here is that none of them will be equal to some middle portion of $(-1,1)$, i.e., (a,b) such that $-1 < a < b < 1$. So that is not possible is the claim.

Assume on the contrary that one of the components of $\psi_1^{-1}(U_1 \cap U_2)$ is of the form (a,b), with $-1 < a < b < 1$. Suppose it has happened, we want to say it does not happen so we must get a contradiction.

Then look at $\psi_2^{-1} \circ \psi_1(a, b)$. That is equal to an open interval (c, d) contained in $(-1, 1)$. Here, I do not know what are c and d. I do not care either. Only thing you know is that (c, d) is not the whole of $(-1, 1)$. Let us see. That may be useful or it may not be useful.

So, let now α from $(-1, 1)$ to $(-1, 1)$ denote the function $t \mapsto -t$. The reflection. By replacing ψ_2, I do not want to change the ψ_1, with which something bad has happened, replacing ψ_2 by $\psi_2 \circ \alpha$, if necessary,
we may assume that $\psi_2^{-1} \circ \psi_1$ from (a, b) to (c, d) is increasing, i.e, order preserving. The other possibility was that it is order reversing, and then only take this composite with α. If it is order preserving already, I do not replace ψ_2. Do not change it unnecessarily, that is all. When if it is increasing do not do anything, if it is not increasing that must be increasing because there are only two possibilities for a homeomorphism of intervals to intervals.

So, we have $\psi_2^{-1} \circ \psi_1$ is increasing. Since (c, d) is a proper subset of $(-1, 1)$, it follows that $-1 < c < 1$, or $1- < d < 1$ (or both). There are only two possibilities, the last being covered by both.

So, at least one of c or d must be strictly inside the interval $(-1, 1)$ and that is going to cause us problems. Namely, consider the former case: suppose c is in the middle of the interval $(-1, 1)$. What happens? $\psi_2(c)$ and $\psi_1(a)$ they are coming very near, but they are not identified, in X. So, $\psi_2(c)$ and $\psi_1(a)$ are two distinct points of X which cannot be separated by open sets in X. In the latter case the same thing happens with $\psi_1(b)$ and $\psi_2(d)$. So, in either case you have got a contradiction to the Hausdorffness of X.

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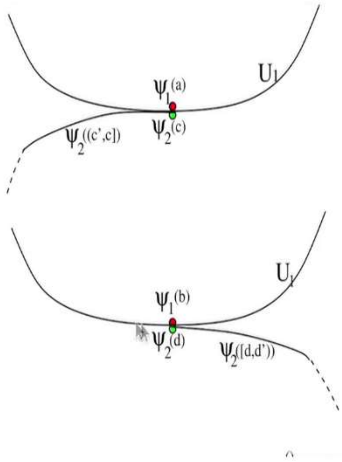

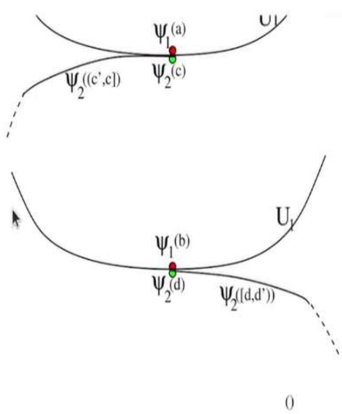

Figure 21: Failure of Hausdorffness

Let X be a Hausdorff space $\psi_1, \psi_2 : (-1, 1) \to X$ be homeomorphisms onto some open sets U_1, U_2 of X, respectively, neither of them contained in the other. Assume that $U_1 \cap U_2 \neq \emptyset$. Then
(i) no component of $\psi_1^{-1}(U_1 \cap U_2)$ will be an open interval of the form (a, b) for some $-1 < a < b < 1$; in particular, $U_1 \cap U_2$ has at most two connected components.
(ii) If $U_1 \cap U_2$ is connected, then $U_1 \cup U_2$ is homeomorphic to an open interval.
(iii) If $U_1 \cap U_2$ has two components, then $U_1 \cup U_2$ is homeomorphic to $\mathbb{S}^1$.

So, here is the picture fully explaining the situation. This is your U_1 and that is the portion of U_2. Other portion I have drawn I do not care what is happening. The ψ_1 a coming like this ψ_1 part is coming like this, a is some in between -1 to coming here and ψ_2 to c just comes here, this portion and this portion will be in every neighborhood of this a as well as b as well as c. The a and c are not, a or c are distinct points.

Sorry, not this portion. This, the other portion because they have to continue for afterwards. This is not the endpoints of the intervals. This was very common. So, for every open neighborhood of $\psi_1(a)$ inside the U_1 there will be some portion every common with ψ_2 of it. This is the intersection part on the left-hand side. Similarly, here what happens there will be on the left-hand side there will be intersection part.

These portions are distinct fine, but as soon as you hit it a, $\psi_1(a)$, $\psi_2(b)$. So, they are the image, they are there inside our X. But they are distinct points they are in different ones, they are not assumed to be an intersection, they are open intervals. So, these two points contradict the Hausdorffness of the interval. So, some c' to c on this part we will be coming there, that is all. So, I do not know how many I have drawn the rest of them.

So, d' to d that will be hitting, that is the whole idea. So, they are in this part of this one. Rest of the part here they will be common. So, here all of them will be common. So, you cannot separate them by disjoint open subsets. So, that completes the proof of this first claim that no interval can be of this form. Once you have that, you have only at the most two components for intersection.

Now, I have to say that all hypotheses of the first case is covered or all the hypotheses for a second case is covered that is all. Then these (ii) and (iii) are the only possibilities. That is the second part and third part here.

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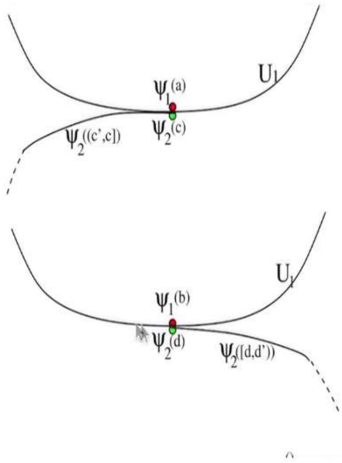
(ii) From (i) it follows that for each $i = 1, 2$, $\psi_i^{-1}(U_1 \cap U_2)$ is of the form either $(-1, a_i)$ or $(b_i, 1)$. There are four cases to be considered, but they are all symmetrical. So, for definiteness, consider the case when these intervals are respectively of the form $(-1, a_1)$ and $(-1, a_2)$. We claim that $\psi_2^{-1} \circ \psi_1 : (-1, a_1) \to (-1, a_2)$ has to be decreasing. Otherwise, it follows that every nbd of $\psi_1(a_1)$ and every nbd of $\psi_2(a_2)$ will intersect each other, contradicting the Hausdorffness of X. Therefore, $\psi_2^{-1} \circ \psi_1 : (-1, a_1) \to (-1, a_2)$ is decreasing. Now consider $\psi_1'(t) = \psi_1(-t)$. Then the two homeomorphisms ψ_1', ψ_2 fit the hypothesis of lemma 12.39 and so we are done.



So, second part a, the first part from whatever we have seen it follows that for each $i = 1, 2$, $\psi_i^{-1}(U_1 \cap U_2)$ is of the form $(-1, a_i)$ or $(b_i, 1)$.

So, here for ψ_1, there are these two cases, and for ψ_2 there are two, so all in all there are four cases possible, but they are all symmetrical. So, for definiteness you just cover any one of them, argument will be same in other cases. So, consider the case when these intervals are of the form $(-1, a_1)$ and $(-1, a_2)$ for ψ_1 and ψ_2 respectively. We claim that ψ_2^{-1} of ψ_1 from $(-1, a_1)$ to $(-1, a_2)$ is decreasing. (Recall we are under case (ii) when we assume that $U_1 \cap U_2$ has only one component.)

If on the contrary, it is increasing what happens? $\psi_1(a_1)$ and $\psi_2(a_2)$, they will be coming very close to each other, but they are different. So they will violate Hausdorffness.

Now, consider $\psi_1' = \psi_1 \circ \alpha, \psi_1'(t) = \psi(-t)$. Then the two homeomorphism ψ_1' and ψ_2 fit the hypothesis of lemma 12.39. Now, they will be exactly like and we can join them. No need to worry about what is happening here. So, we are done. So, that is the case (ii).

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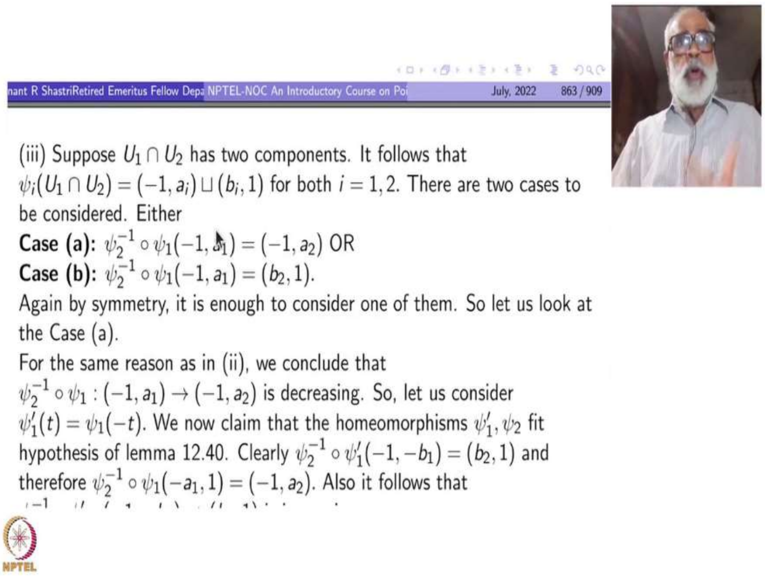

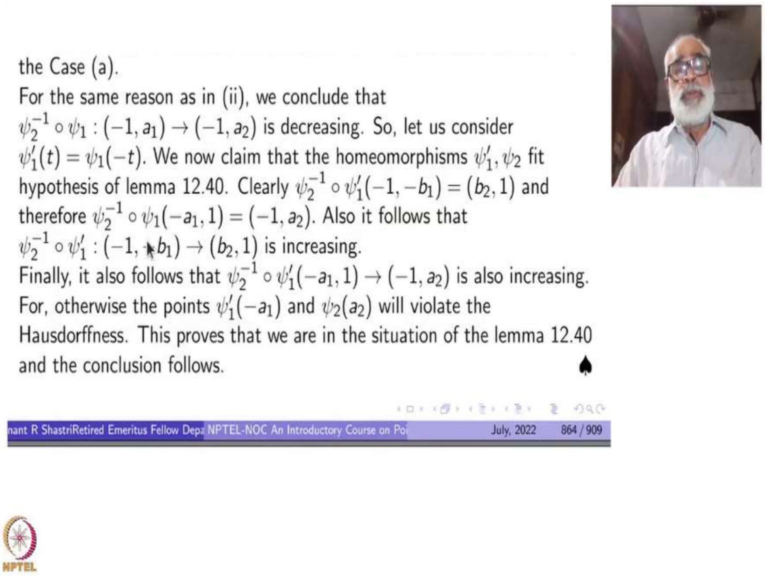

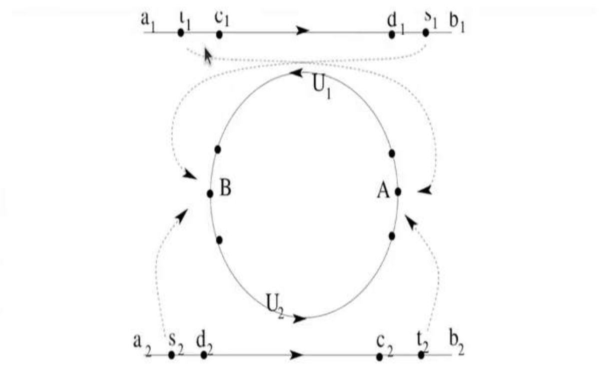

Figure 20: When M is the union of two interlocked intervals

Lemma 12.41

Let X be a Hausdorff space $\psi_1, \psi_2 : (-1,1) \to X$ be homeomorphisms onto some open sets U_1, U_2 of X, respectively, neither of them contained in the other. Assume that $U_1 \cap U_2 \neq \emptyset$. Then
(i) no component of $\psi_1^{-1}(U_1 \cap U_2)$ will be an open interval of the form (a, b) for some $-1 < a < b < 1$; in particular, $U_1 \cap U_2$ has at most two connected components.
(ii) If $U_1 \cap U_2$ is connected, then $U_1 \cup U_2$ is homeomorphic to an open interval.
(iii) If $U_1 \cap U_2$ has two components, then $U_1 \cup U_2$ is homeomorphic to $\mathbb{S}^1$.

The case (iii) is also similar, but I have to show that we are inside the second case correctly. Now, we are assuming that $U_1 \cap U_2$ has two components. It follows that $\psi_i^{-1}(U_1 \cap U_2)$ must be again end intervals i.e., of the form $(-1, a_i) \sqcup (b_i, 1)$ for i equal to 1 and 2. There are two subcases to be considered again here, namely, (a) $\psi_2^{-1} \circ \psi_1(-1, a_1) = (-1, a_2)$ or (b) $(b_2, 1)$.

As soon as you select one of them, soon as this happens the other one namely the image of $(b_1, 1)$ will be also fixed, no choice for it. You have freedom only to choose where one of the component goes. The other component has to go to the remaining component there is no choice.

So, I have only these two cases here, the first component here goes to the first component there or the first component goes to second component there, so these are two cases. Again, by symmetry I have just see what happens in the first subcase. So, let us take case (a).

For the same reason as in (ii) we conclude that $\psi_2^{-1} \circ \psi_1$ has to be decreasing from $(-1, a_1)$ to $(-1, a_2)$.

So, let us change ψ_1 by composing with reflection so as to make it increasing here. But then a change will occur for the other part $(b_1, 1)$ also. The point is that you can choose c the order only on one of them by changing the sign, the order on the other part will be automatically decided, though the intersections have two different components. As soon as you adjusted the first one correctly, you may or may get the second one correct. So, that is why I have written. In this picture, I have shown this portion coming here this portion coming here. So, now, if you change this interval, so that the both of them are like this, then you are in a nice shape that is all.

Whether you want to do it or not it just depends upon you, but here we assume that we replaced ψ_1 with ψ_1'. We now claim that the homeomorphism ψ_1' and ψ_2 fit the hypotheses of lemma 12.40 completely.

Clearly ψ_2^{-1} of ψ_1' from $(-1, b_1)$ the initial segment goes into the far end $(b_2, 1)$. So, both of them will be increasing. Finally, it follows that $\psi_2^{-1} \circ \psi_1$ is increasing on both the parts. For, otherwise you will have $\psi_1'(a_1)$ and $\psi_2(a_2)$ will be violating the Hausdorffness. So, that is why we are in the nice situation of second lemma there.

Therefore, the conclusion is that, in this case, the entire manifold has to be $\mathbb{S}^1$. Not really. We do not need that right now, we just do not, because I have not assumed that X is connected here. So, we only conclude that $U_1 \cup U_2$ is $\mathbb{S}^1$, that is fine. This is the lemma in which, which just says that, whatever we desire only that will happen, that is all. Now, let us complete the proof of the theorem.

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Completion of the proof of the theorem 12.38:
Recall that we started with a connected 1-dimensional manifold. By second countability, we get countable cover $\{U_i\}$ of X by open sets and homeomorphisms $\phi : U_i \to (-1, 1)$.
Step-I Inductively, we define a finite or infinite increasing sequence $\{W_k\}$ of open sets of X such that each W_k is connected and $\cup_k W_k = X$, as follows: Put $W_1 = U_1$. Having defined W_k, let

$$S_k = \{i \in \mathbb{N} \ : \ U_i \not\subseteq W_k \ \& \ U_i \cap W_k \neq \emptyset\}.$$

$S_k = \emptyset$ implies that $W_k = X$, using the facts $X = \cup_i U_i$ and X is connected. In that case, we have achieved our goal and nothing more to be done. Otherwise, let n_k be the least element of S_k and put $W_{k+1} = W_k \cup U_{n_k}$.

Recall that we started with a connected 1-dimensional manifold. By II-countability, we get a countable cover U_i of X by open sets, all of them homeomorphic to $(-1, 1)$, say say ϕ_i from U_i to $(-1, 1)$ are the homeomorphisms. Inductively, we define a finite or infinite, we do not know, but countable, increasing sequence of open sets W_k in X such that each W_k is connected, union of W_k is the whole space X.

That is all first we have to do this way. But we will do the same thing in a much more elaborate way. So, I will describe it as follows. So, how do we do that?
Start with $W_1 = U_1$. So, you have got a countable cover. So, you have indexed them in some manner with positive integers. Never mind, but that indexing may not be very good. So, we are going to do some changes here. So, start with $W_1 = U_1$. Having defined W_k, what I am going to do? Look at the set S_k of all i such that U_i is not contained W_k and U_i intersects W_k. So, for $k = 1$, so W_1 is here is defined. What is this S_1? S_1 consists of all $i \in \mathbb{N}$, such that U_i is not contained inside U_1 and $U_i \cap U_1$ is non-empty. Now, suppose S_1 is empty. That means either all other U_i are contained in U_1 or none of the them intersect U_1. In the first case we have nothing more to prove. The second case contradicts the hypothesis that X is connected.

Therefore, we may assume S_1 is non empty. There is a minimal number n_1 and define $W_2 = U_1 \cup U_{n_1}$. Exactly similarly, having defined W_k, if S_k is empty then we have arrived at our goal. Otherwise define $W_{k+1} = W_k \cup U_{n_k}$, where n_k is the least element of S_k.

Clearly we get an increasing sequence of open sets $\{W_k\}$ finite or infinite such that their union is X, each W_k is connected and W_{k+1} is $W_k \cup U_{n+k}$ where U_{n+k} is a coordinate neighbourhood. Note that several of U_i may be left out because they are covered by the union of others.

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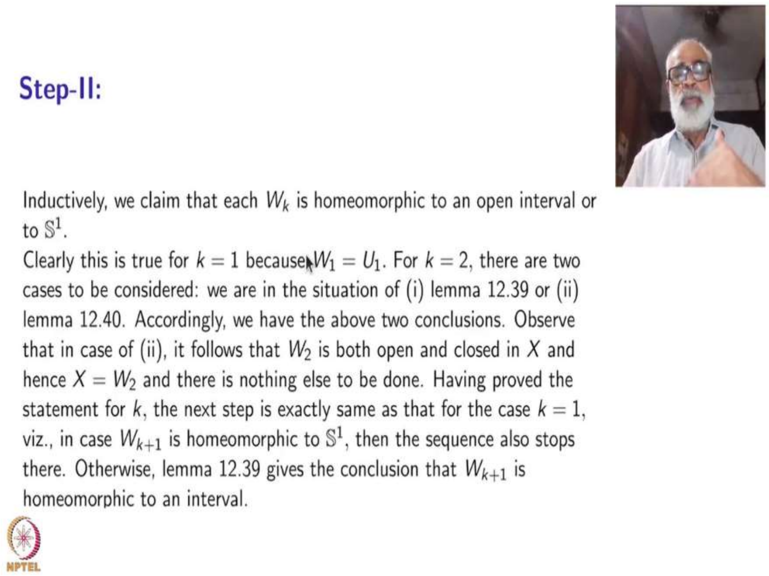

Now, inductively, we claim that each W_k is homeomorphic to an open interval or it is $\mathbb{S}^1$. There are two cases at each stage. As soon as it is $\mathbb{S}^1$, we know that we have come to an end. So, what is the other case? The other case is that each time you get W_k is an interval. There, we have not yet completed the proof. Each time it is homeomorphic to an interval, if you have stopped at a finite stage, then it is okay it is an interval, you have completed the proof, but it may not stop it may be an infinite sequence.

So, in that case, you have to write a small proof there, that is all. So, let us see why this happens. Clearly for $k = 1$ because W_1 is U_1 there is nothing to prove, U_1 is already homeomorphic to an open interval. For a general k, there are two cases to be considered. What are they? By the previous lemma, we are in the situation of lemma 12.39 or lemma 12.40.

Accordingly, we have the above two conclusions. See W_k is an interval by induction hypothesis and U_{n_k} is also homeomorphic an interval. So, number of connected components of their intersection is at most two. If it is one, then their union will be homeomorphic to again an interval. If it is two, then the union will be $\mathbb{S}^1$ and the sequence stops. So, those are two cases.

Suppose now the sequence does not stop. Well, why does it keep going on? All the W_k are are homeomorphic to open intervals. And we have got an infinite sequence of ever increasing. Why the entire union is homeomorphic to an open interval? This is what we have to show.

Remember, these things are not happening inside $\mathbb{R}$, of course, that is the final conclusion yet to be proved. (I mean finally, it is so. But right now you are working on an abstract 1-manifold. abstract manifolds, you want to show that the entire thing is homeomorphic to an open interval, which is same thing as showing that homeomorphic to $\mathbb{R}$. So, that is the last part here.

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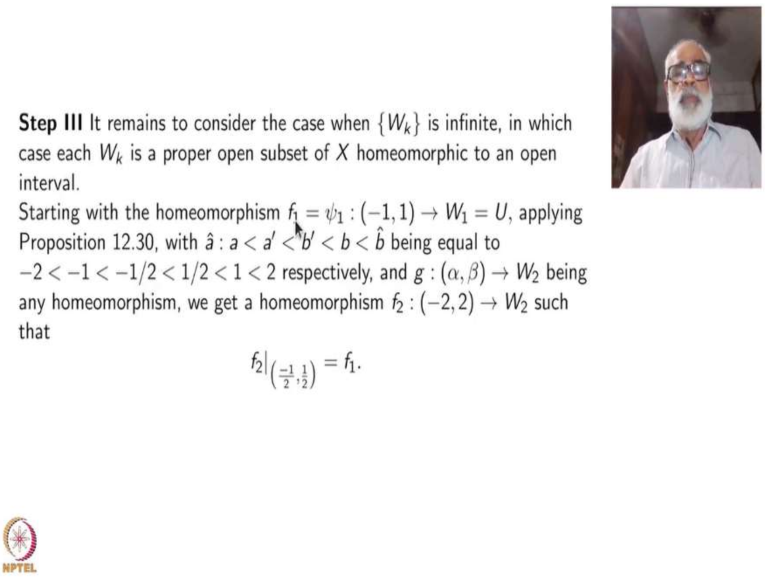

Starting with the homeomorphism $f_1 = \psi_1$ (I am renaming ψ_1) from $(-1,1)$ to $U_1 = W_1$.

Apply proposition 12.30 with these $\hat{a} < a < a' < b' < b$, (remember this proposition) being respectively equal to $-2 < 1 < 1/2 < 1/2 < 1 < 2$. And take g from (α, β) to W being any homeomorphism, (here I do not know what it is, does not matter) we get a homeomorphism f_2 from $(-2,2)$ to W_2, such that f_2 restricted to the smaller interval $(-1/2, 1/2)$ is your f_1. f_1 is defined on $(-1,1)$ but on the larger one, the homeomorphism is modified so as to coincide with f_1 only on this smaller open interval.

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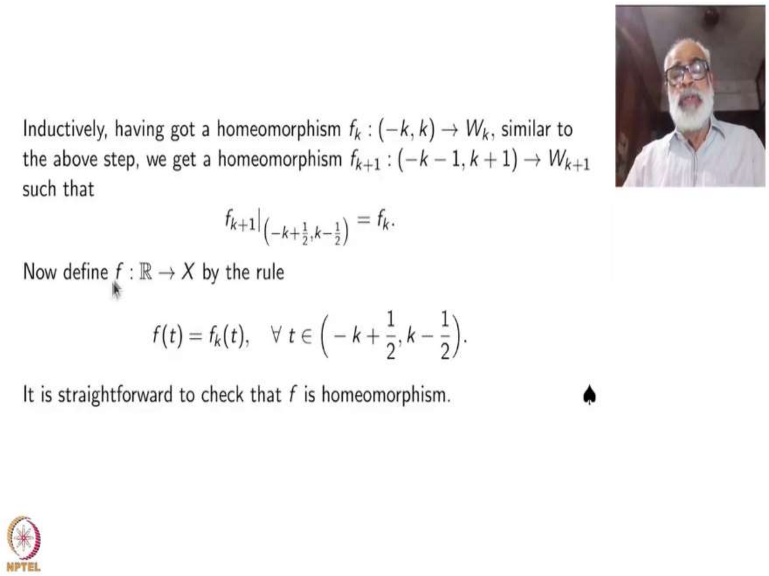

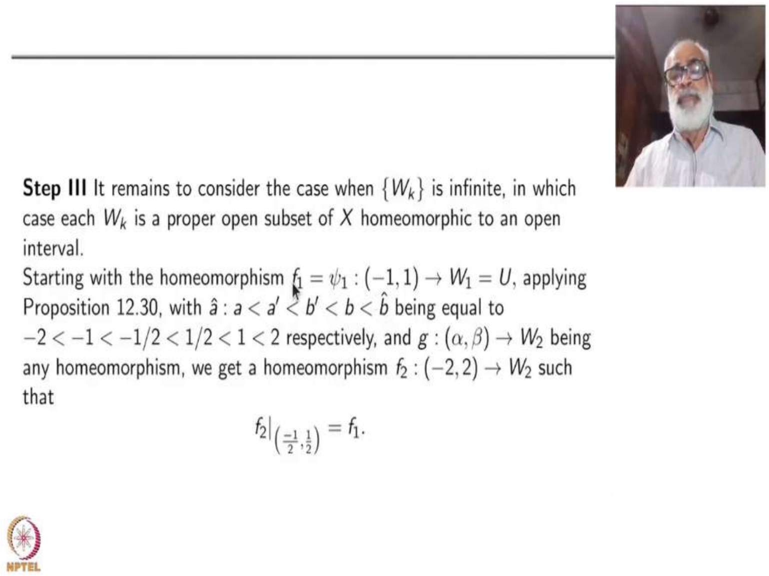

Inductively, having got a homeomorphism f_k from $(-k, k)$ to W_k, so this is an inductive constructuion, f_1, f_2, f_3 and so on, f_k is from $(-k, k)$ to W_k, similar to the above step, knowing that W_{k+1} is also homeomorphic to an interval, we get a homeomorphism f_{k+1} from $(-k-1, k+1)$ to W_{k+1} such that restricted to a smaller interval $(-k+1/2, k-1/2)$, (reduce interval by half on both sides), this $f_{k+1} = f_k$.

So, now you define f from $\mathbb{R}$ to X by the rule: $f(t) = f_k(t)$ whenever t belongs to this interval, $(1/2 - k, k - 1/2)$. Given any t inside $\mathbb{R}$, it must be in one of these intervals.

Once t is in this interval for some k, even if you take a larger value for k, the value for $f_k(t)$ on this interval is the same thing because f_{k+1}, f_{k+2} they are all equal to $f_k(t)$ here. Therefore, $f(t)$ is well defined. It is straightforward to check that f is a homeomorphism, all that you have to do is to show that f is continuous bijection and an open mapping directly.

To show that f is an open mapping you can show that for any small open interval (a, b) , plus epsilon the image is open, you do not have to show the that image of all open sets is open. Image of every suuficiently small interval is open, image is open then the whole thing will be open mapping. So, I will leave that one to you, but learn this method, how to patch up homomorphisms.

If things are arbitrary homeomorphisms not agreeing with each other then you will have a problem. So, when you have inductive steps like this, it is possible to patch it up to get a homeomorphism on the entire thing. Why it is surjective here? Because union of all W_k is the whole of X.

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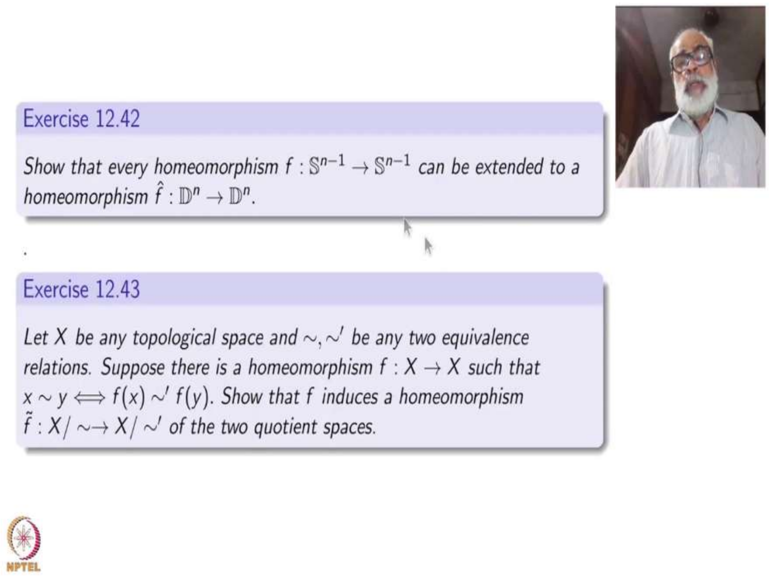

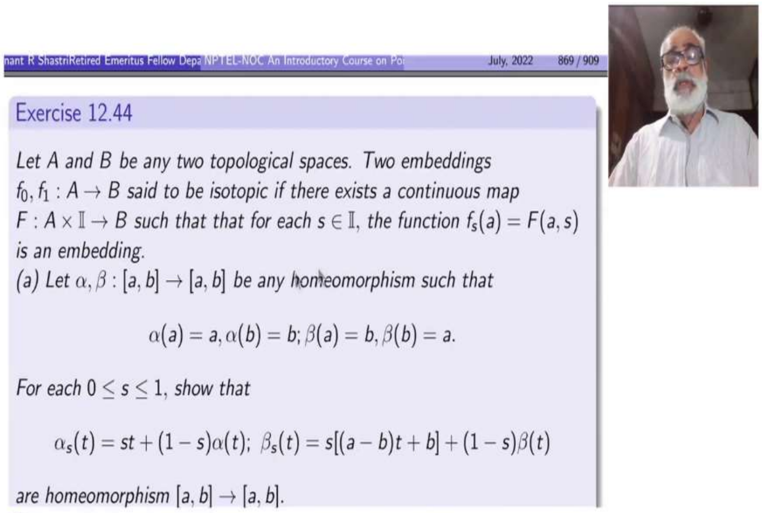

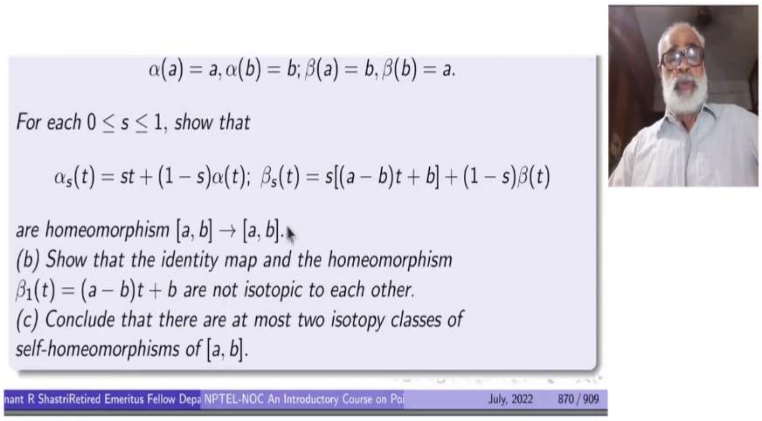

So, here are a few exercises which will help you to understand the next topic. So, take some time to think about them even if you do not solve them completely. With those words, I will just leave it to you to keep looking at them so that you have some time to think about these. So, here are two of them. One is on quotient spaces, then this is our old friend about what is happening to homeomorphism from one interval to another interval.

So, this is an old topic which I have been discussing several times here. So, there is some new concept here called isotopy. It is just like homotopy. You might not have come across this earlier. So go through that now.

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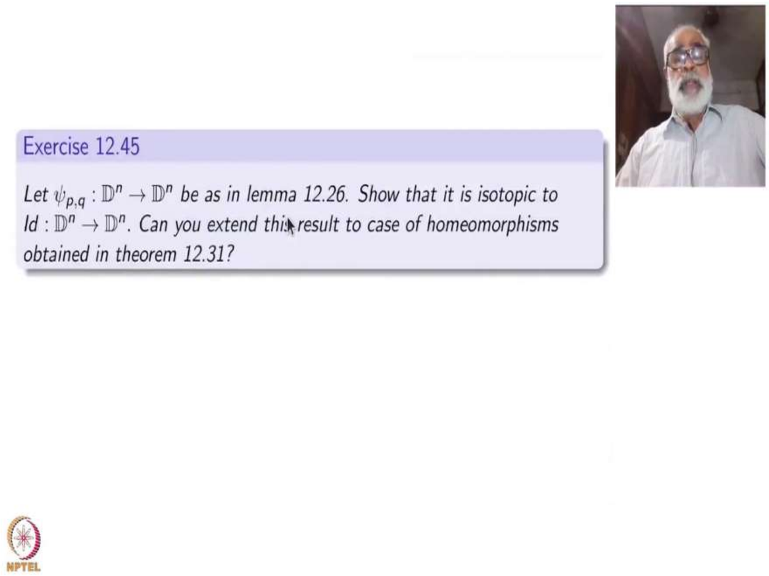

Then we have this transitive action. So, you have got these ψ_{pq}'s. So, can you see that they are also isotopic to each other? This is what one has to think about. So, next time we will study a little bit about surfaces. So, two more lectures on that. All right, thank you.