An Introduction to Point – Set – Topology (Part II)
Professor Anant R. Shastri
Department of Mathematics
Indian Institute of Technology, Bombay
Week 12-Lecture 58
Homogeneity continued

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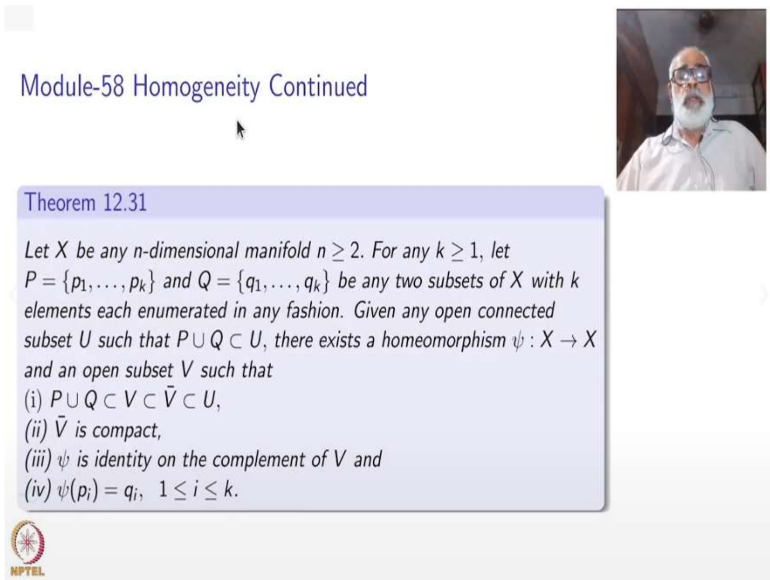

Hello welcome to module 58 of NPTEL NOC a course on points topology part II. So, we continue with the theme homogeneity. So, today we are reaping the final result of all our efforts last time. So, let X be any n-dimensional manifold $n \geq 2$. For any $k \geq 1$, take two k subsets $P = \{p_1, p_2, \dots, p_k\}$ and $Q = \{q_1, q_2, \dots, q_k\}$ take any two k-subsets inside X. Given any open connected subset U such that the union of P and Q is inside U, (connectedness of U is important, conclusion is that) there exists a homeomorphism from X to X and an open subset V of X such that $P \cup Q$ is inside V contained contained inside $\overline{V}$ contained in the original open connected subset U, $\overline{V}$ is compact and f is identity outside V, and f maps p_i to q_i, for $i = 1, 2, \dots, k$.

So, that is the k fold homogeneity. So let us see how one proves it. We have prepared all the required inputs already.

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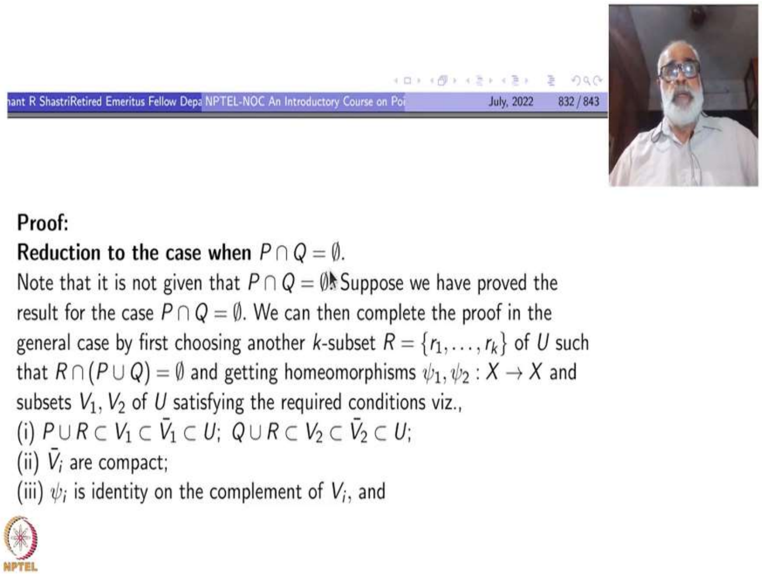

So, first I will take the case namely P and Q are disjoint sets. They are finite sets each with k elements. Logically this is not necessary but writing down the proof becomes very easy with this approach. So, you can have different ways of proving this also.

I choose this method, in which I first take the case when that P and Q are disjoint.

Suppose we have completed the proof for this case. Then first I will show you how to complete the proof in the general case. So, suppose we have proved this case namely, $P \cap Q$ is empty. To complete the proof, so I choose another k-subset $R = \{r_1, \ldots, r_k\}$ of X, inside U itself which is completely disjoint from P as well as Q.

Now, I apply the first case here namely P and R are disjoint, and the to Q and R, they are k-subsets. Therefore by this assumption I get a homeomorphism ψ_1 and another one of ψ_2 for Q, both of them are homeomorphism of X to X, such that there are subsets V_1, V_2 of U satisfying the required conditions. What are the conditions? $P \cup R$ is inside $V_1, \overline{V_1}$ contained inside U, similarly $Q \cup R$ is contained inside V_2 contained inside $\overline{V_2}$ contained inside U, both $\overline{V_i}$ are compact, ψ_i are identity on the complement of V_i, and $\phi_1(p_i) = r_i$ and similarly $\psi_2(q_i) = r_i$ for $i = 1, 2, \ldots, k$. Once we have this all that we have to do is take $\psi = \psi_2^{-1} \circ \psi_1$. Then of course, $\psi(p_i) = q_i$. Outside $V_1 \cup V_2$ both of them will be identity therefore the composition will be also identity. closure of $V_1 \cup V_2$ being the union of $\overline{V_1}$ and $\overline{V_2}$ is also compact. so, the conclusion for the general case follows.

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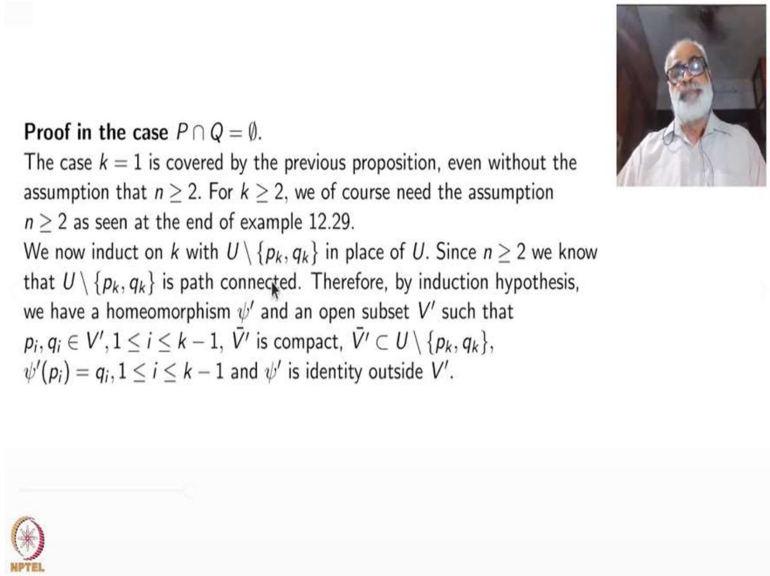

Now, the case $P \cap Q$ is empty. For this I have to use induction on k. When k is 1 we have already proved it in the previous proposition. Any point can be moved to any other point within a connected open set that we have proved already. And for this we do not need even the assumption that $n \geq 2$. Even for $n = 1$, we have proved this anyway.

For $k \geq 2$, we of course need the assumption that $n \geq 2$, because we have already seen examples to the contrary.

So, we assume $n \geq 2$ and make the make this induction hypothesis now. Assume that the statement holds for $(k-1)$ subsets. Starting with any connected open subset U, and disjoint k-subsets P and Q of U as above, I take the open subset $U' = U \setminus \{p_k, q_k\}$ that is again an an open subset and since $n \geq 2$, it is also connected. (Throwing away finitely many points from an open subset of $\mathbb{R}^n$ does not destroy its connectivity, only if $n \geq 2$. Same conclusion holds for ny open subset of any n-manifold also.)

So, you apply the induction hypothesis to $P' = \{p_1, \ldots, p_{k-1}\}, Q' = \{q_1, \ldots, q_{k-1}\}$ inside U' which gives you a homeomorphism that is denoted by ψ' and an open subset V' such that $P' \cup Q'$ is inside V' and its closure is compact and is contained in U' and $\psi'(p_i) = q_i$ for $i = 1, 2, \ldots, k-1$. Moreover ψ' is identity outside V'. In particular it will not disturb p_k and q_k, $\psi'(p_k) = p_k$ and $\psi'(q_k) = q_k$.

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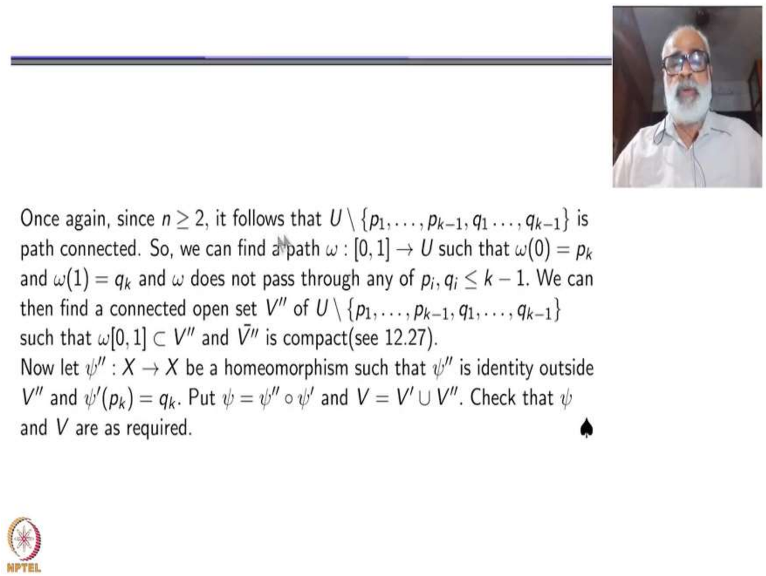

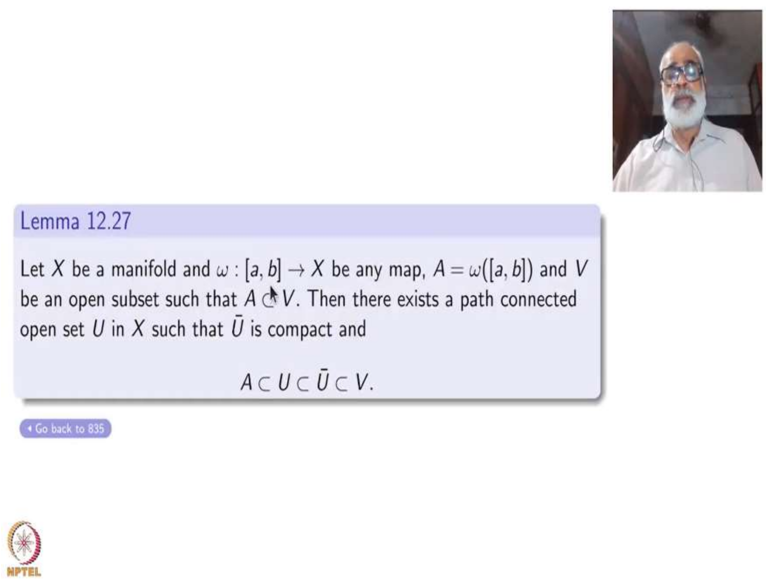

So, now how to complete the proof? Once again $n \geq 1$ is point to be used to see that $U \setminus (P' \cup Q')$ is connected open set so it is path connected). So, we can find a path ω from $[0,1]$ to U such that $\omega(0)$ is p_k and $\omega(1)$ is q_k and moreover, this ω does not pass through any of these other $2k-2$ points, p_i's and q_i's. We can then find a connected open set V'' of this connected subset such that the image of this path ω is contained inside this V'' and the closure of this V'' is compact and is contained in $U \setminus (P' \cup Q')$.

So, this is the other lemma which we have proved. So let me show you this lemma so that was the lemma. Starting with any path and contained inside an open subset V, actually you can find a path connected open subset of U such that $\overline{U}$ is compact and A contained in U contained in $\overline{U}$ contained in V.

You apply that, so we have got this thing. This V'' here. Now, apply the theorem for just those two points p_k and q_k to get ψ'' from X to X a homeomorphism such that ψ'' is identity outside this V'' and maps p_k to q_k that is all. Now, all that you have to do is: take ψ equal to $\psi'' \circ \psi'$. First ψ' will map p_i to q_i, i from 1 up to $k-1$ and then ψ'' will take over and maps p_k to q_k, whereas ψ'' is identity on all these earlier points. Therefore, the ψ will be the required map. Clearly, it is identity outside $V' \cup V''$ both of them contained inside U. So, that is a proof.

So this is a big theorem we have proved now, thanks to efforts last time we had already done all the preparations.

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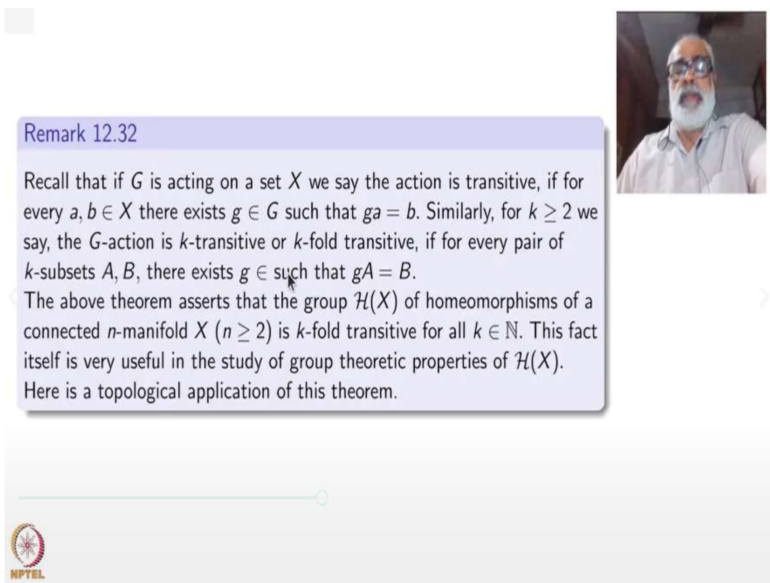

So, here is a remark. Recall that if G is a group acting on a set X, we say this action is transitive if given any two elements a and b in X, you can find an element g inside the group such that $g(a) = b$. Any two points can be mapped to each other. The same thing as saying that the orbit space of the action is the entire space X, there is single orbit.

Similarly, for $k \geq 2$, we say the G-action is k-transitive or k-fold transitive, if for every pair of linearly ordered k-subsets A and B of X, there exists one single $g \in G$ such that $g(A)$ equal to B.

That means if you have $A = \{a_1, a_2, \ldots, a_k\}$ here and $B = \{b_1, b_2, \ldots, b_k\}$ there, $g(a_i) = b_i$, for $i = 1, 2, \ldots, k$ (stronger that just saying that $g(A) = B$).

The above theorem asserts that the group $H(X)$ namely the group of all self-homeomorphisms of X on a connected n manifold X, $n \geq 2$, is k-transitive for all k. There is no restriction on k in the previous theorem. This fact is very useful in the study of group theoretic properties of $H(X)$. Now what I am going to do is? I am going to give you a topological application of this theorem. The application itself is useful that I will not be able to do.

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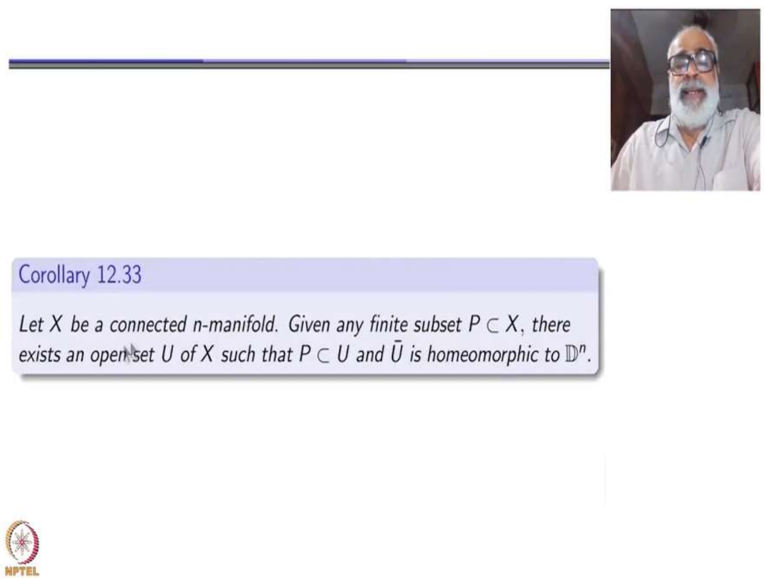

So, here is the corollary. Take a connected n-manifold, given any finite subset P of X, there exists an open subset U of X such that this P is contained inside U (that is very easy, but this additional part is stunning) and $\overline{U}$ is homeomorphic to $\mathbb{D}^n$. So, you think about this one. Any finite set scattered around the entire manifold, only condition is that X is connected n-dimensional manifold.

So, this is an easy corollary now to this theorem let us see how.

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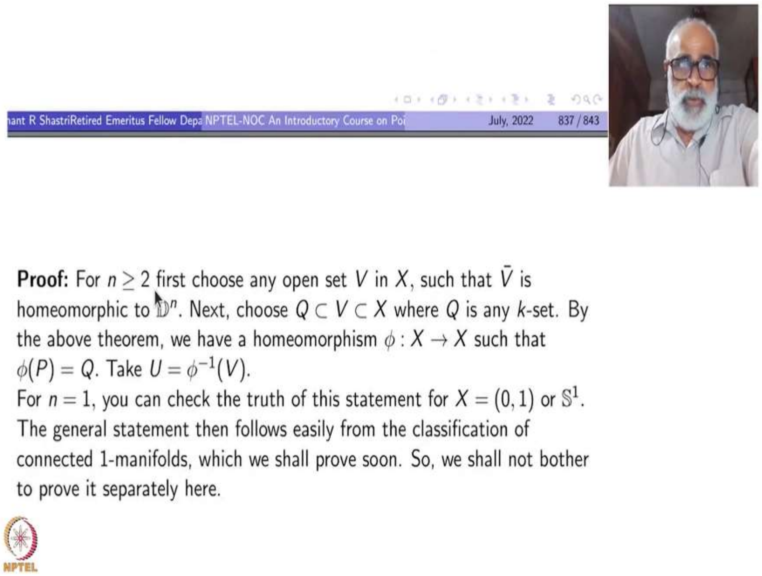

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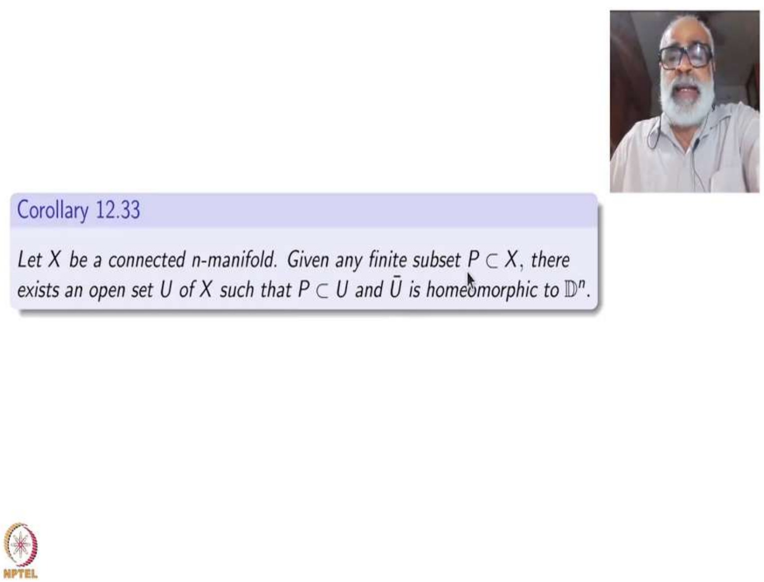

For $n \geq 2$, of course, first choose any open set V in X such that $\overline{V}$ is homeomorphic to $\mathbb{D}^n$. There are plenty of them. Next choose a subset Q of V. such that number of elements in Q is the same as that for P.

Now, the above theorem says that we have homeomorphism ϕ from X to X such that $\psi(P) = Q$. But what happens now take U equal to $\psi^{-1}(V)$. Then $\overline{U}$ is homeomorphic to $\overline{V}$ which is homeomorphic to $\mathbb{D}^n$, $\overline{V}$ is homeomorphic to $\mathbb{D}^n$, This is because, ϕ is defined on the whole of X Over. And clearly U contains P. So, for $n \geq 2$, you are done.

But I have stated this theorem even for $n = 1$. For that neither this theorem is going to help nor it is needed. But you need something else what is that? Namely, a result which you are going to prove. We completely classifying one-dimensional manifolds and show that a connected 1-dimensional manifold is homeomorphic to $\mathbb{R}$ or to a circle.

From that this will be easy. In case of $\mathbb{R}$, you can choose a closed interval to contain all the give points. In case a circle, you can delete any point other than the given ones.

So, that follows from classification of 1-manilfods, and therefore we will not separately prove it all right? The classification theorem will be proved soon.

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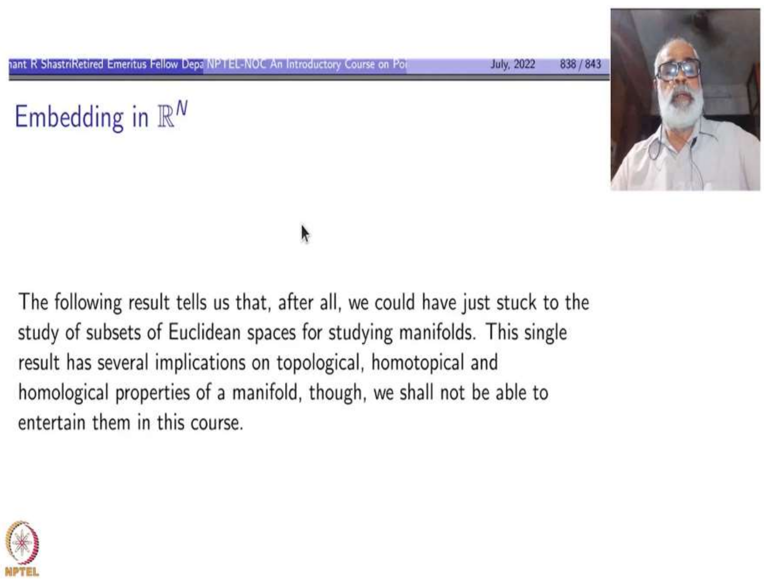

Now, I will give you something about these abstract manifolds. The following result tells us that after all, we could have just stuck to the study of you know subsets of Euclidean spaces with all those paraphernalia II-countability and Haursdorffness etc would have been automatic, including metrizability.

So, you could have just studied those spaces for manifolds, why? So, this is the theorem. I am going to state. This single result has several implications in topological, homotopical and homological properties of a manifold. Though we shall not be able to entertain them in this course. So, those things will be taken up in the subsequent algebraic topology courses.

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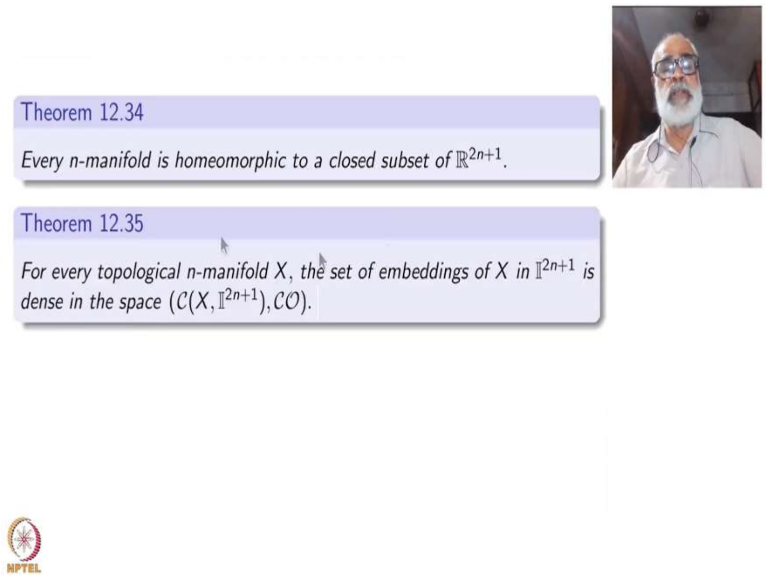

So, what is this theorem this is the theorem every n-manifold is homeomorphic to a closed subset of $\mathbb{R}^{2n+1}$.

There is a more elaborate statement here. Take any topological n-manifold X. The set of embeddings of X inside $[0,1]^{2n+1}$ is dense in the space of all continuous functions from X into $[0,1]^{2n+1}$ with the compact open topology.

So, that is the statement. Of course this is the more elaborate statement than the previous theorem. Take any continuous function, you can approximate it by an embedding. In particular, there will be an embedding, because you can take a constant function and approximate it by an embedding. That means that given any n-manifold, and any point in $[0,1]^{2n+1}$, any arbitrary small neighborhood of that point, will contain a copy of the given manifold.

So, that is the strong conclusion of this theorem, I will not be able to do this one this is very powerful result I will not be able to do even this one here because these proofs are quite lengthy. They do not need many more techniques than what we have done but there are some techniques needed but they are quite lengthy so I will not be able to do that.

So, that is the strong conclusion of this theorem, I will not be able to do this one. This is very powerful result. I will not be able to do even the earlier one here because these proofs are quite lengthy. They do not need many more techniques than what we have learnt so far but there are some techniques needed but they are quite lengthy. So I will not be able to do them.

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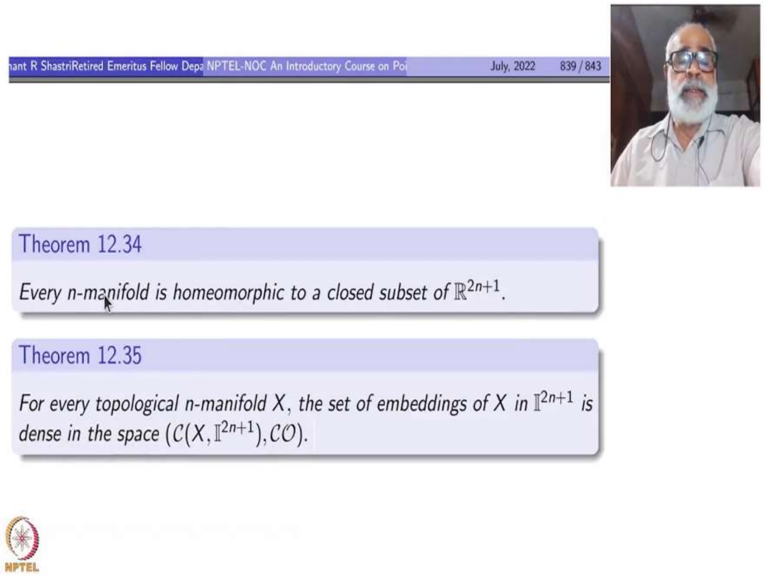
Remark 12.36

The proofs of these theorems are somewhat lengthy and hard. The smooth version of Theorem 12.34 goes under the name easy Whitney embedding theorems which you may read from many books such as [Shastri, 2011]. However, for the topological case, there are not many references available. You are welcome to see this in the excellent old book [Hurewicz and Wallman] (Theorem V-3). Or you may choose to read a nice proof of the embedding Theorem 12.34 from [Munkres,1975]. Here, we shall be satisfied with an easy proof of the following weaker version:

Theorem 12.37

Every compact manifold (with or without boundary) is homeomorphic to a closed subset of some Euclidean space.



However I will tell you a few things about it. Namely, the proof of this theorem are very length and hard. There is a smooth version of this one also which goes under the name easy Whitney embedding theorems, for that you have to do differential topology, so those proofs are much easier to do there. You may read them from many books such as for example my own book on Differential Topology. However, for the topological case, there are not so many easy references. You are welcome to see the excellent book of Hurewicz and Wallman from which I have taken this statement. So, I have given you a correct reference also, it is in fifth chapter theorem 3.

Or you may choose to read a nice proof of embedding theorem from Munkres book. Here we shall be satisfied with an easy proof of a weaker version, namely, the compact case.

Every compact manifold with or without boundary, (this time I allow it to have boundary also, that is why I specifically mentioned it here) is homeomorphic to a closed subset of some Euclidean space.

So, it can be embedded so I have relaxed the dimension of the ambient Eucliean space and put an extra condition on X, namely, it is compact. So that we will prove now, yeah here.

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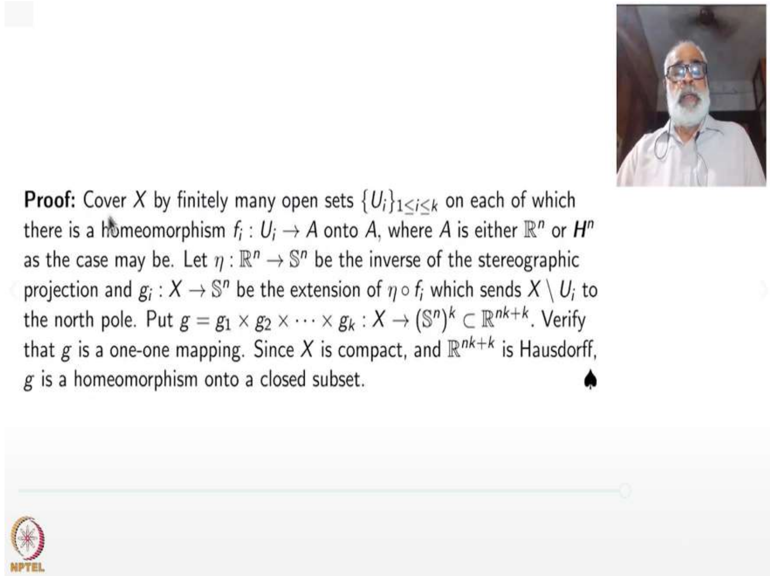

So, X being a compact space right? You cover X by finitely many open subsets $U_i,\ i = 1, \dots, k$ such that for each i, there is a homeomorphism f_i from U_i onto A where this A is either $\mathbb{R}^n$ or $\mathbb{H}^n$. See here I am taking ϕ_i is onto A and A could be either $\mathbb{R}^n$ or $\mathbb{H}^n$ to $\mathbb{R}^n$ or onto $\mathbb{H}^n$ since X may have boundary points. So, there are two different cases have to be allowed as the case may be.

Now, go to our standard map eta which is the inverse of stereographic projection, η is from $\mathbb{R}^n$ to $\mathbb{S}^n \setminus \{N\}$ the north pole. And then take g_i from X to $\mathbb{S}^n$ to be the extension of this $\eta \circ f_i$ where g_i maps $X \setminus U_i$ to the north pole.

See f_i's are defined only on U_i to A, what is A? it is $\mathbb{R}^n$ or $\overline{\mathbb{H}^n}$, $\overline{\mathbb{H}^n}$ is the closure of $\mathbb{H}^n$ in $\mathbb{R}^n$. So $\eta \circ f_i$ makes sense so we get a function from U_i to $\mathbb{S}^n$. Let g_i be the extension of that where the rest of the closed set $X \setminus U_i$ is sent to the north pole. You have to see that because of the surjectivity of f_i, this g_i is a continuous function, so that I will leave it to you, it is not very difficult.

So, all these g_i's, $i = 1$ up to k are now continuous functions from X to $\mathbb{S}^n$. What is its property on U_i? Each f_i is $1-1$ then you are composing with η so that is also $1-1$, so important property of these g_i is on U_i, g_i is $1-1$ map. When you take a $1-1$ map product with any number of them, it will still be $1-1$ map, so what happens is this $g = (g_1, g_2, \dots g_k)$ will be a $1-1$ mapping of the whole of X into the product of k copies of $\mathbb{S}^n$.

X is compact and the product of spheres is Hausdorff. Therefore, g is a homeomorphism on to a closed subset of the product space. Proof is over, okay?

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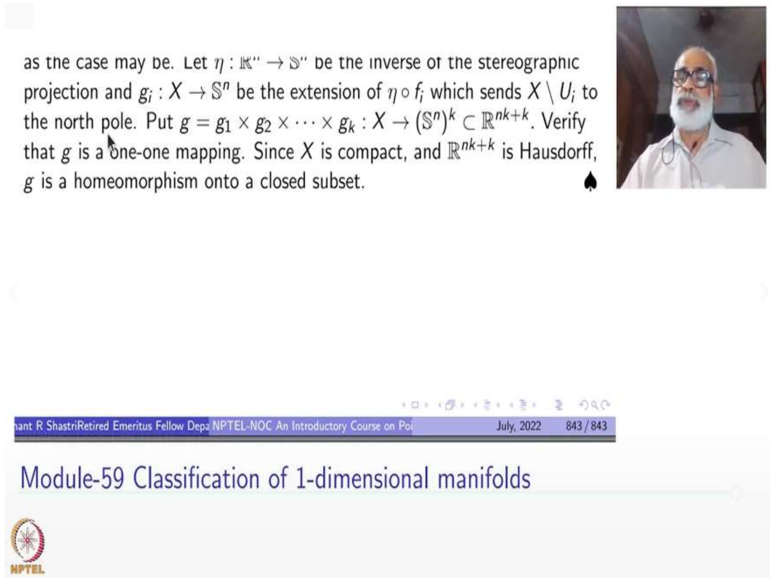

Next time we shall take up the classification of 1-manifolds, fine.