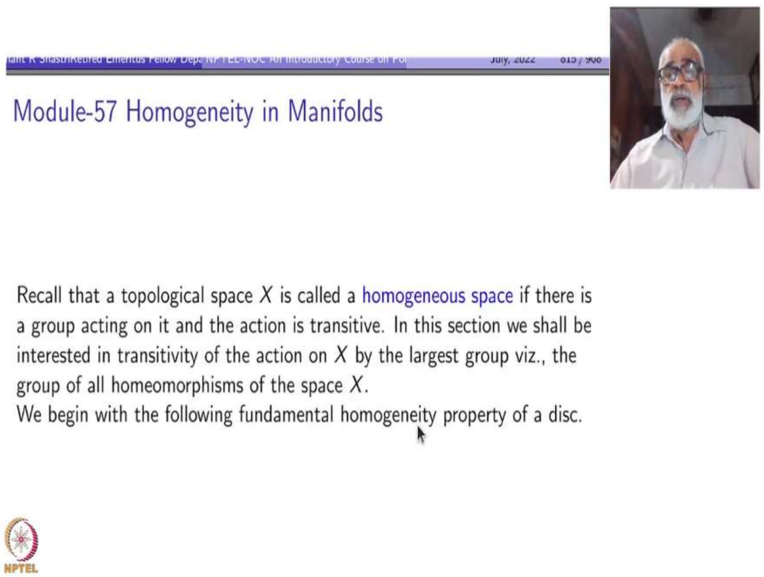
An Introduction to Point-Set-Topology (Part II)
Professor Anant R. Shastri
Department of Mathematics
Indian Institute of Technology, Bombay
Lecture 57
Homogeneity

(Refer Slide Time: 00:16)

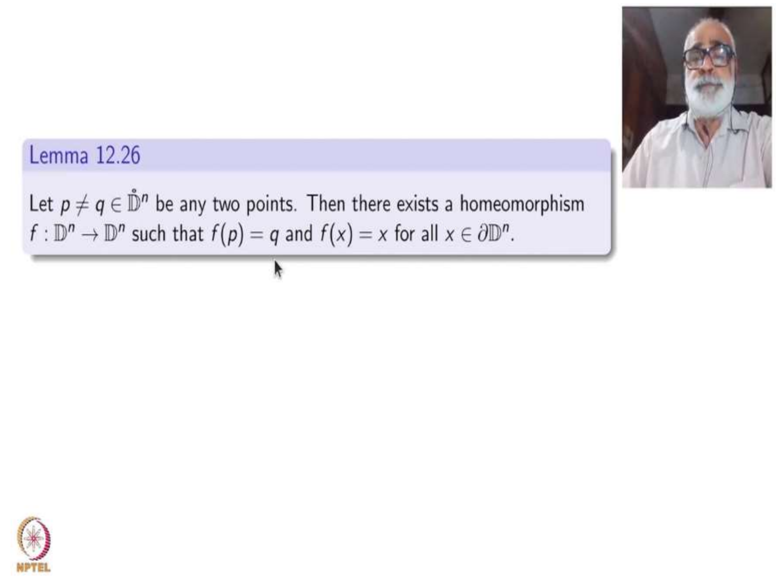


Welcome to module 57 of NPTEL NOC an introductory course on Point Set Topology Part II. Continuing with the study of manifolds, today we will take the topic homogeneity. You might have come across the word homogeneous in different contexts like group action on a set and or on a topological space and so on. So, here is something that I understand a topological space is called homogeneous space, if there is a group acting on it and the action is transitive.

So, transitive means what? There is one single orbit. Given any two points there is a group element G, which will map this point to that point, $g(x) = y$. That is the kind of thing. In this section we shall be interested in transitivity of the action on X, what is the group? the group is the largest group you can think of namely, the group of all self-homeomorphisms of the space X.

In a topological space, when you take group actions, you would like to take them through homeomorphisms namely, each multiplication by an element of G must be a homeomorphism (just continuous is enough, automatically it will be a homeomorphism of course). So, the best thing is to take the space of all homeomorphisms. Under the composition of functions, it will form a group. So, you can look at that group acting on the space X. Is it transitive is the question. The answer is, yes in the case of connected manifolds.

(Refer Slide Time: 02:31)



So, let us see how good it is. The transitivity has many other ramifications here. So, we begin with the simplest object namely an open disk. Our model for manifolds. Take any two distinct points p and q in the interior of $\mathbb{D}^n$, I have also denoted it by $\mathbb{B}^n$, the open ball of radius 1.

Then there exists a homeomorphism f from $\mathbb{D}^n$ to $\mathbb{D}^n$ such that $f(p)$ equal to q and $f(x)$ equal to x for all x on the boundary of $\mathbb{D}^n$. On the boundaries it is identity and p is mapped to q.

You might have heard such things in complex analysis. There you have such functions which are even complex differentiable. Here we are doing it for $\mathbb{D}^n$ for any n, and we do not have the strong structure of complex analysis on $\mathbb{D}^2$.

(Refer Slide Time: 03:49)

Proof: Given any $y \in \mathring{\mathbb{D}}^n$, let $\phi_y : \mathbb{S}^{n-1} \times (0,1] \to \mathbb{D}^n \setminus \{y\}$ be the function $\phi_y(v,t) = (1-t)y + tv$. Note that ϕ_y is a homeomorphism and $\phi_y(v,1) = v$, $\forall\ v \in \mathbb{S}^{n-1}$. Also note that $\lim_{t\to 0} \phi_y(v,t) = y$ for all $v \in \mathbb{S}^{n-1}$. Now take $\psi_{p,q} : \mathbb{D}^n \setminus \{p\} \to \mathbb{D}^n \setminus \{q\}$ to be $\phi_q \circ \phi_p^{-1}$. Check that

$$\lim_{x\to p} \psi_{p,q}(x) = q.$$

Therefore, we can extend $\psi : \mathbb{D}^n \to \mathbb{D}^n$ continuously by putting $\psi_{p,q}(p) = q$. Exactly the same way we can define the map $\psi_{q,p}$ which will be the inverse of $\psi_{p,q}$. Clearly $\psi_{p,q}(x) = x$ for all $x \in \mathbb{S}^{n-1}$. ♠

So, the proof is quite straightforward, namely, use is the convexity of $\mathbb{D}^n$. For any y inside $\mathbb{B}^n$, let us write ϕ_y from $\mathbb{S}^{n-1} \times (0,1]$ to this $\mathbb{D}^n \setminus \{y\}$, y is an interior point. You throw away that. So, take this function ϕ_y of a vector v, v is a vector inside $\mathbb{S}^{n-1}$, a unit vector, take (v,t) going to $(1-t)y + tv$. Note that this ϕ_y is a homeomorphism. Verify. Look at this RHS, when $t = 0$ it is y, which is anyway not in the codomain, I that I have thrown away y, Also $(v,0)$ is not there in the domain either, after all. But when $t = 1$ this is v.

So, RHS is gives you the formula for the line segment joining v and y, the open line segment, the point y itself is not there. You can actually write down the inverse map. Every point in $\mathbb{D}^n \setminus \{y\}$ has a unique expression like this. That follows from the convexity of $\mathbb{D}^n$. So, this phi is homeomorphism.

Also note that when t tends to 0, this $\phi_y(v,t)$ tends to y, for all $v \in \mathbb{S}^{n-1}$. So, I can extend ϕ continuously to the whole of $\mathbb{S}^{n-1} \times [0,1]$ the closed interval by sending $(v,1)$ to y. But of course, the extended ϕ will not be a homeomorphism. Because all the points $(v,1)$ have gone to the single point y here.

So, now take $\psi_{p,q}$ (depends upon two distinct points in $\mathbb{B}^n$, any two arbitrary points in the interior), from $\mathbb{D}^n \setminus \{y\}$ to $\mathbb{D}^n \setminus \{q\}$ to be $\phi_q \circ \phi_p^{-1}$. From here you come here and then come back here. So, automatically if you take x tending to p in the domain, then $\psi_{p,q}(x)$ will tend to q.

So, I can extend this ψ continuously to a map $\mathbb{D}^n$ to $\mathbb{D}^n$, by sending p to q, that will be a continuous extension.

Exactly same way we can define $\psi q, p$ also, namely by taking $(\phi_p \circ \phi_q^{-1})^{-1}$, it will have the property that q will be mapped to p. Indeed, this $\psi_{q,p}$ will be actually the inverse of $\psi_{p,q}$. Since we have verified that these two are, the continuous functions it will follow that $\psi_{p,q}$ is a homeomorphism.

Clearly on the boundary of $\mathbb{D}^n$ is mapped to the boundary. Actually $\psi_{p,q}$ is the identity map on the boundary.

So you know how to construct a homeomorphism like this.

(Refer Slide Time: 08:17)

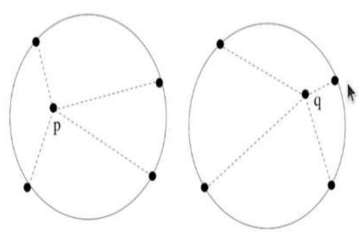
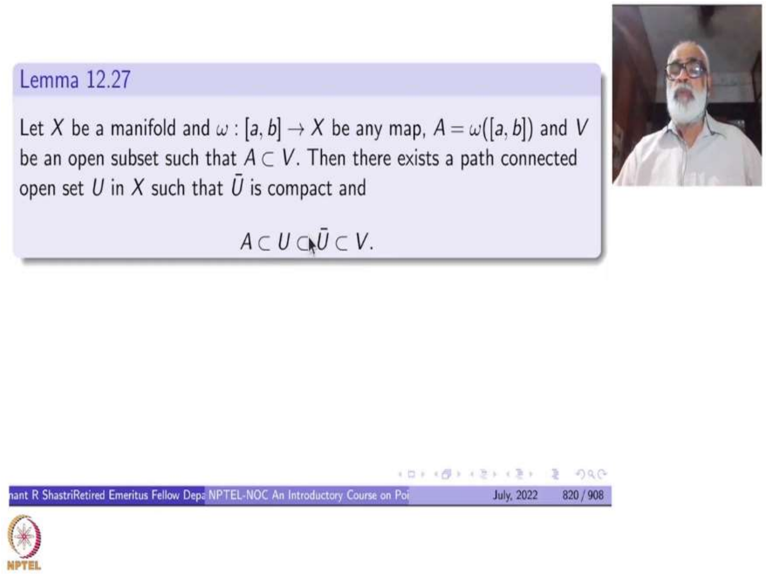


Figure 18: Homogeneity of a disc

So, here is a picture how it is done. Each line segment like this will be mapped onto a line segment like that. The points on the boundary, they are kept fixed. This is $\psi_{p,q}$.

So, this is the homogeneity of the disk, namely, any point in the interior goes to any other point in the interior by a homeomorphism of the entire disk which is identity on the boundary, that is an extra property. In particular, we have a homeomorphism of the open disk to itself also doing the same job. So, that extra property is going to help us very much now. Namely, the homeomorphism is identity on the boundary.

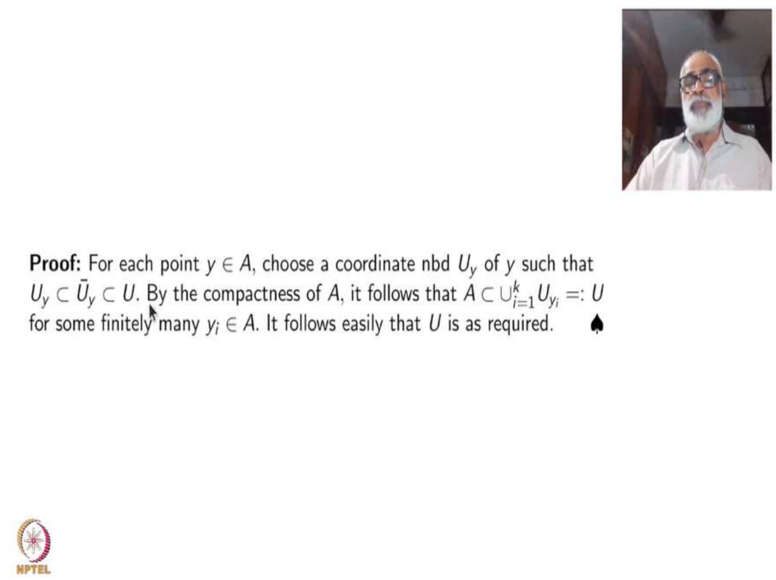
(Refer Slide Time: 09:16)



So next lemma is to take any manifold X and a path omega from $[a,b]$ to X. (Recall that any continuous function defined on a closed interval is called a path. That is all.) Take A to be the image of that path and V be an open subset which contains A. Then there exists a path connected open set to U in X such that $\overline{U}$ is compact and A is contained inside U contained inside $\overline{U}$ contained inside V.

So, do not worry much about path connectivity. This is just regularity, because A is compact being the image of a close interval under continuous function, and V is open. So, you would get this U by regularity. But how do you get this U as path connected? there you have to use that A is path connected. And what more? That X is a manifold (not just a regular space). X is a manifold means it is locally path connected.

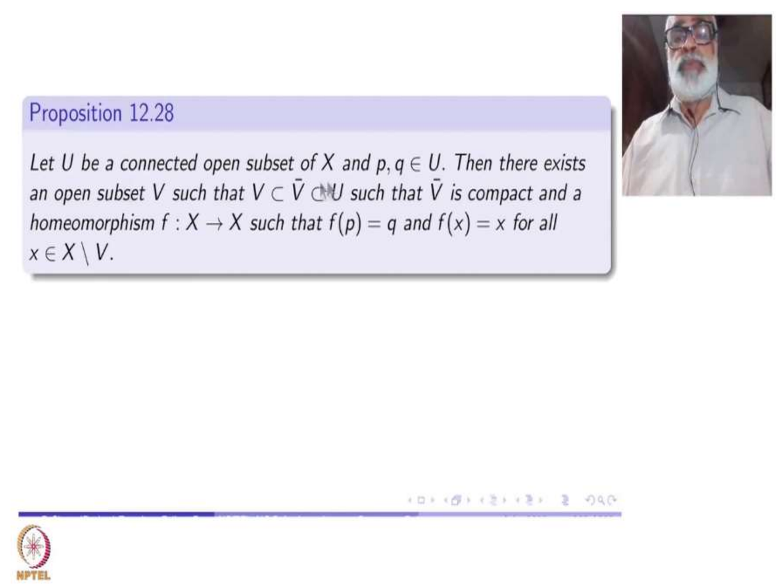
(Refer Slide Time: 10:37)



So, let us see the proof. For each point $y \in A$ choose a coordinate neighbourhood U_y of y such that U_y contained inside $\overline{U_y}$ contained inside V. By the compactness of A it follows that A is contained inside finitely many U_{y_i}'s. So, call that union U. So, this U is union of finitely many coordinate neighbourhoods. Coordinate neighbourhoods are homoeomorphic to open discs or open the whole of $\mathbb{R}^n$ whatever you mat say.

Therefore, all of them are path connected. Why is the union is path connected? You can rewrite U as a finite union of W_i, where $W_i = U_{y_i} \cup A$ for each i. Then each W_i is path connected and they all have A as a subset which is path connected. So their union is path connected.

(Refer Slide Time: 12:32)

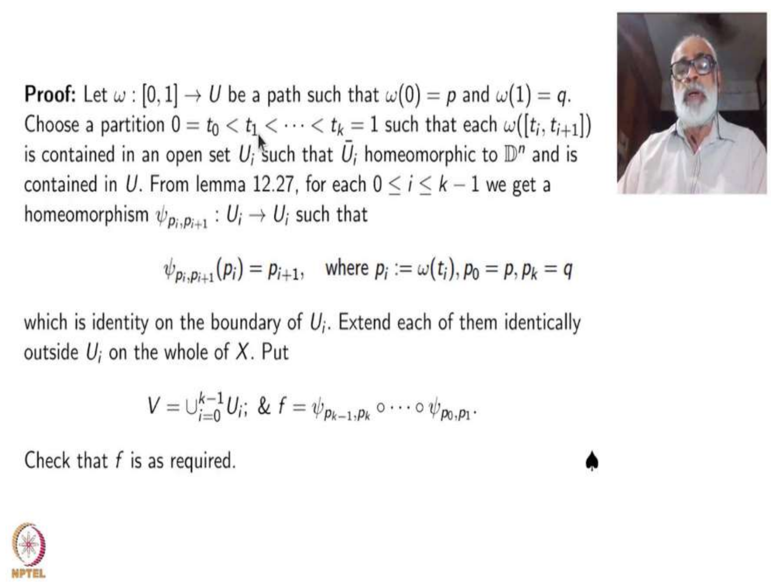


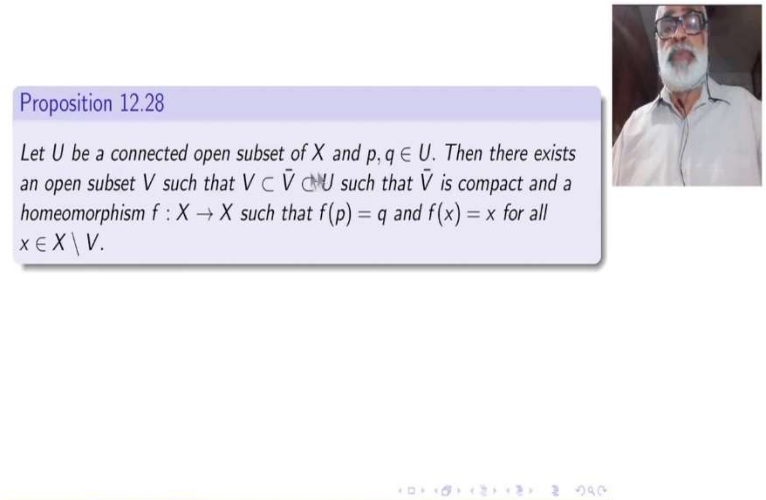
Let U be a connected open subset of X and p and q belongs to U. Then there exists an open subset V such that V is inside $\overline{V}$ inside U such that $\overline{V}$ is compact and a homeomorphism f from X to X, such that under f, p goes to q and f is identity outside of V.

Now, in this now, so many things are combined together. Two points are taken in the manifold, inside a connected open subset. They will be mapped one to the other by a homeomorphism which is identity outside a smaller open subset. And this smaller open subset of course contains both p, q and you can assume that it is relatively compact.

More generally, such homeomorphism are called homeomorphism with compact support. Support means what here? Closure of the set of all points wherein $f(x)$ is not equal to x. So, this is the proposition, it is not very difficult now, because we have made the two important lemmas here which are actually what we can say, preparatory results.

(Refer Slide Time: 14:20)





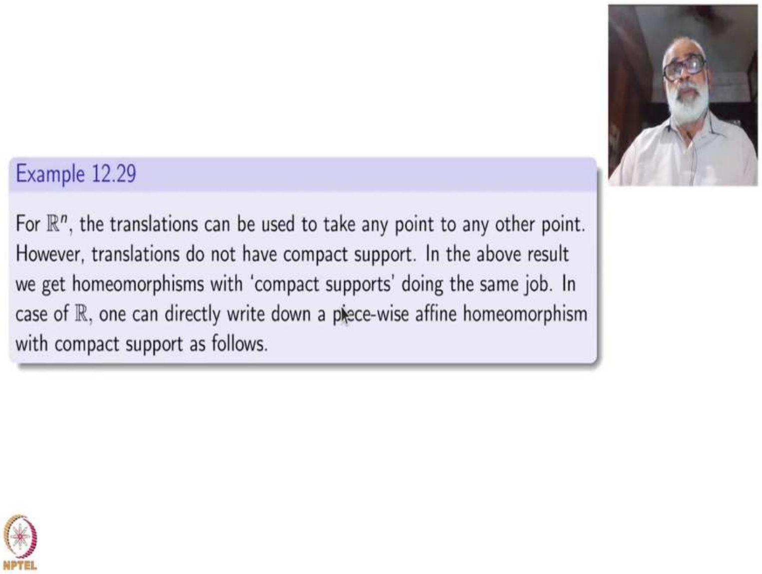
So, this is also a preparatory result you may say, but it builts upon the two earlier ones.

So, Start with a connected open subset U of X where X is a manifold. X is locally path connected therefore, U is path connected. So, start with a path joining p and q inside U. Then cut down this path into finitely many portions by taking a partition $0 = t_1 < t_2 < \cdots < t_k = 1$ such that each segment $\omega([ti, t_{i+1}])$ is contained in an open subset U_i such that $\overline{U_i}$ is homoeomorphic to $\mathbb{D}^n$. To begin with you have such coordinate neighbourhoods covering $\omega([0,1])$ which is compact and so there will be finitely many of them just like, you can make it into a partition like this, such that each segment is contained in one of them. This $\overline{U_i}$ are homoeomorphic to $\mathbb{D}^n$ and all of them happening inside is open subset U. You are not going outside the connected open subset U here. From the previous lemma for each $0 \leq i < k-1$, we get a homeomorphism $\psi_{p_i, p_{i+1}}$ from $\overline{U_i}$ to $\overline{U_i}$, which maps p_i to p_{i+1}, where $p_j := \omega(t_j)$. Because these $\overline{U_i}$'s are homoeomorphic to $\mathbb{D}^n$. You can take a homeomorphism f_i from $\overline{U_i}$ to $\mathbb{D}^n$, get the corresponding hoemeomorphism from $\mathbb{D}^n$ to $\mathbb{D}^n$ and come back via f_i^{-1}. So you get these $\psi_{p_i, p_{i+1]}}$ which is identity on the boundary of $\overline{U_i}$. Extend each of them identically outside U_i to a homeomorphism from X to X.

After that, take V equal to union of these U_i's and f equal to the composition $\psi_{p_0,p_1}, \psi_{p_1,p_2}, \ldots, \psi_{p_{k-1},p_k}$ in the reverse order.

So, this will map p_0 (first to p_1, the second one to p_2 and so on the last one) to p_k. Since $p_0 = p$ and $p_k = q$, we are done.

(Refer Slide Time: 17:36)

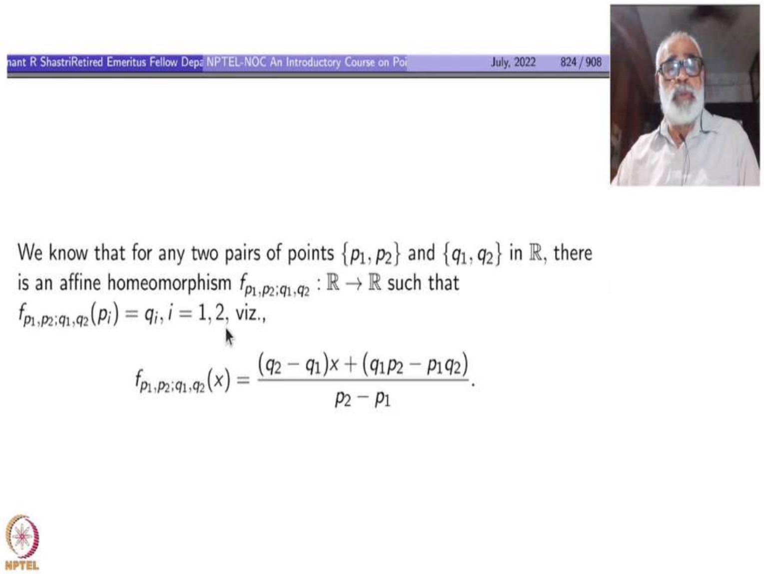


Here is an example. For $\mathbb{R}^n$, the translations can be used to take any point to any other point. If you have two distinct vectors u, v, the translation by $u - v$ will take v to u. That is all. But translation do not have compact support. They will translate everything or they will not translate anything, that is identity. In the above result we get homeomorphism with compact supports doing the same job.

There is no extra assumption of course, it is true for $\mathbb{R}^n$ also, for any connected manifold you have this proof. In case of $\mathbb{R}$, one can directly write down a piecewise affine homeomorphism with compact support as follows.

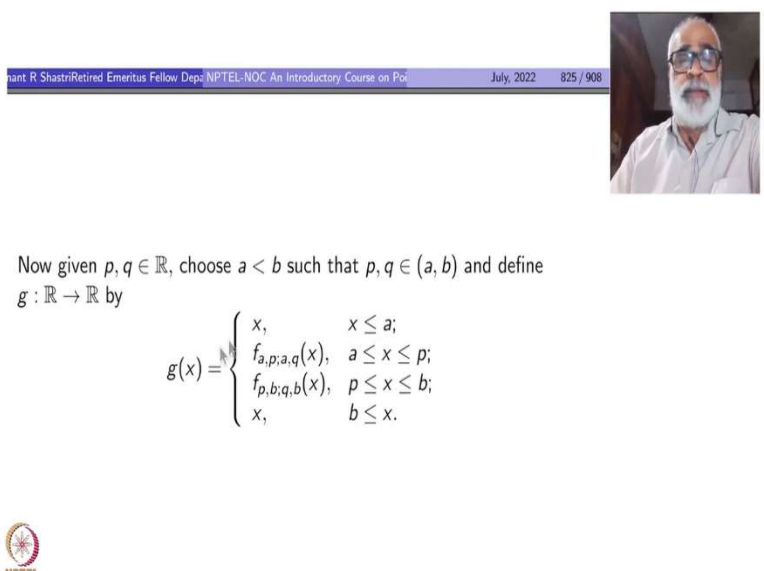
So, I want to do this one for $\mathbb{R}$, by hand, not by using any theorem and that will be very useful and instructive.

(Refer Slide Time: 18:49)



We know that for any two pair of distinct points $\{p_1, p_2\}$ and $\{q_1, q_2\}$ in $\mathbb{R}$, there is a unique affine homeomorphism taking p_1 to q_1 and p_2 to q_2. This we have used already in the proof of a theorem earlier. So I index this map by $f_{p_1,p_2;q_1,q_2}$. It is given by a formula here. So, you see, all that you have to do is to send x to $(q_1 - q_2)x + q_1p_2 - p_1q_2$ the entire thing divided by $p_1 - p_2$. It is a polynomial map.

You know this. There are much more or stronger statements viz., by interpolation formula of Lagrange etc., there are polynomial maps and so on. This is just an elementary verification here that this the linear map does the job. (Refer Slide Time: 20:03)



We know that for any two pairs of points $\{p_1, p_2\}$ and $\{q_1, q_2\}$ in $\mathbb{R}$, there is an affine homeomorphism $f_{p_1,p_2;q_1,q_2} : \mathbb{R} \to \mathbb{R}$ such that $f_{p_1,p_2;q_1,q_2}(p_i) = q_i, i = 1, 2,$ viz.,

$$f_{p_1,p_2;q_1,q_2}(x) = \frac{(q_2 - q_1)x + (q_1 p_2 - p_1 q_2)}{p_2 - p_1}.$$

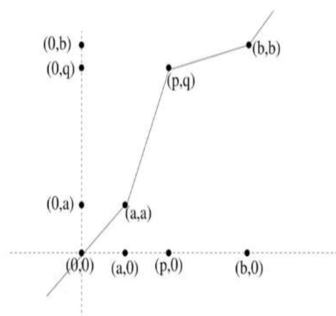
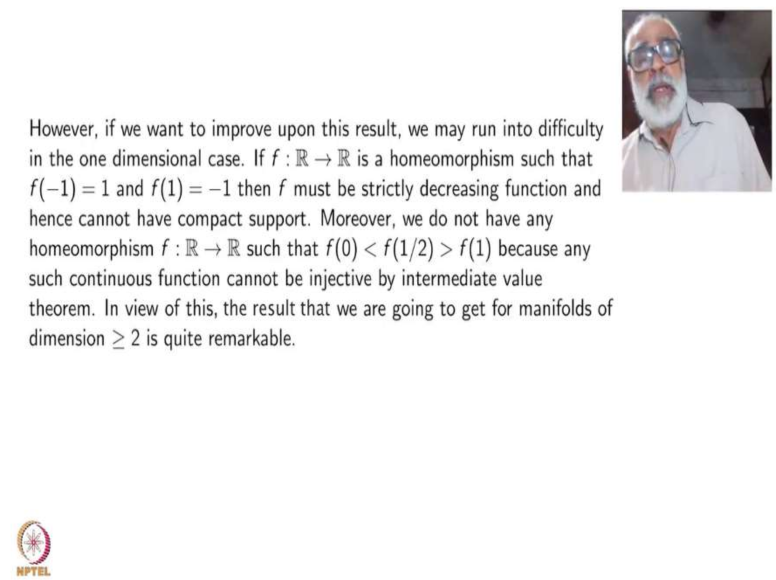


Figure 19: Homogeneity in $\mathbb{R}$

Now given two arbitrary p, q in $\mathbb{R}$, choose $a < b$ such that p and q are inside (a, b). And define g from $\mathbb{R}$ to $\mathbb{R}$, instead of one single affine linear parameterization what I am going to do, I will break $\mathbb{R}$ into four parts and take different maps on each of them. Because I want g to have compact support and affine linear maps do not have compact support. So first cut off all x for each $x \leq a$ or x is greater that or equal to b. Take g to be identity there. Now let me see what I am going take g from $[a, b]$ to $[a, b]$. Already I am forced to take $g(a) = a$ and $g(b) = b$. I also want p going to q. Look at the probable graph of g.

Beyond this $[a, b]$, it is the graph of the identity map and hence part of the main diagonal in $\mathbb{R}^2$. Here over $[a, b]$, I have the point (p, q) which must lie on the graph. So, the easiest this is to join the points (a, a) to (p, q) and (p, q) to (b, b) by line segments. That gives you the formula for the entire map g.

(Refer Slide Time: 22:19)

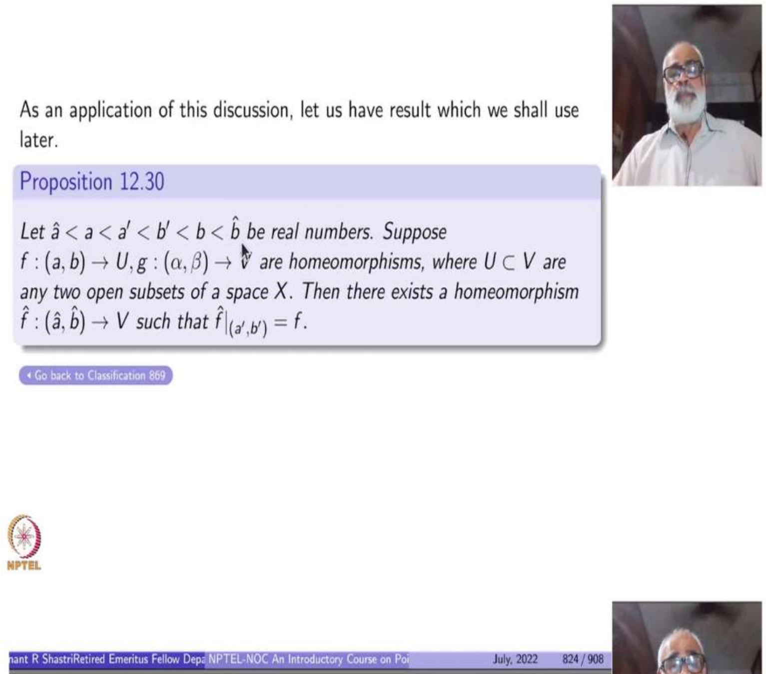


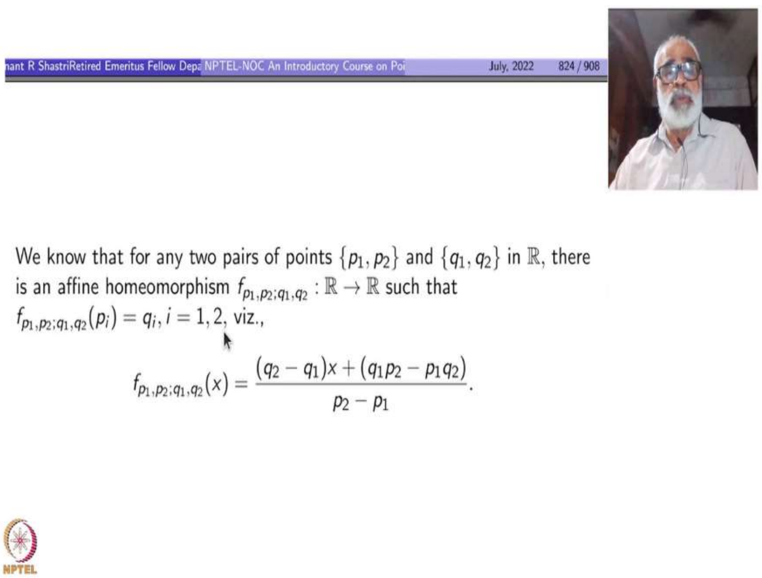
So, if you want to improve upon this result we may run into difficulty in this 1-dimensional case. Improve upon this one means instead of asking one point to be mapped onto another, suppose you have two 2-point sets, can you map a homeomorphism mapping one set to another? If f is a homeomorphism such that $f(-1) = 1$ and $f(1) = -1$, then f is strictly decreasing function and hence f cannot have compact support

All that I am using is the fact that a homeomorphism of $\mathbb{R}$ to $\mathbb{R}$ has to be monotonically increasing or decreasing. Therefore, if you start with two sets having more than one point each, you will have to take them in a particular order. You cannot just shuffle them, a, b, c cannot be mapped onto to a', b', c' unless either a, b, c are strictly increasing and the other one is strictly decreasing or increasing both of them is possible. But it has to be in a particular order.

If it is increasing here and decreasing there you will have to take some monotonically decreasing homeomorphism, otherwise you will have to take monotonically increasing function that is all. Of course in the former case, you will not get compact support. So, the moment you have to deal with sets with more than one points, there are further restrictions imposed.

(Refer Slide Time: 24:25)





So, whatever it is, this kind of patching up of affine linear maps of this nature as discussed above, has to be employed. So, this can technique can be used to obtain a lot of things. So let me have a result here, which we will use later on, in the classification of 1-manifolds.

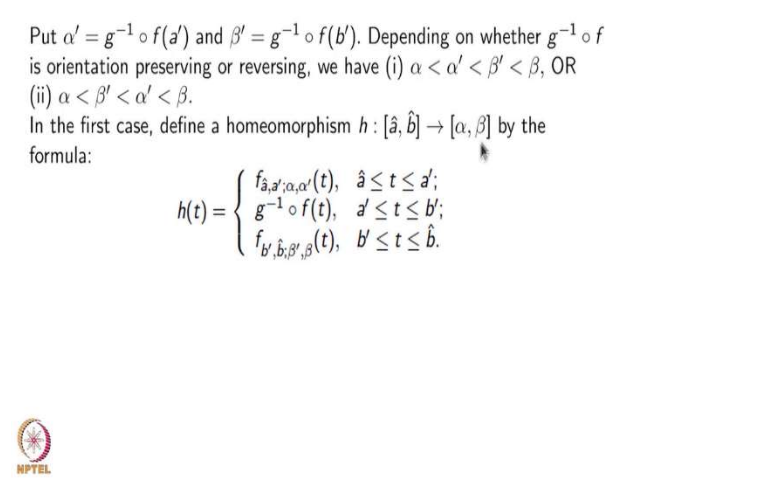
So start with these six numbers, arrange them in the increasing order. I have denoted them by $\hat{a}, a, a', b', b$, and $\hat{b}$. You can use any other notation $a_1, a_2, a_3, a_4, a_5, a_6$ as well but right now stick to this notation.

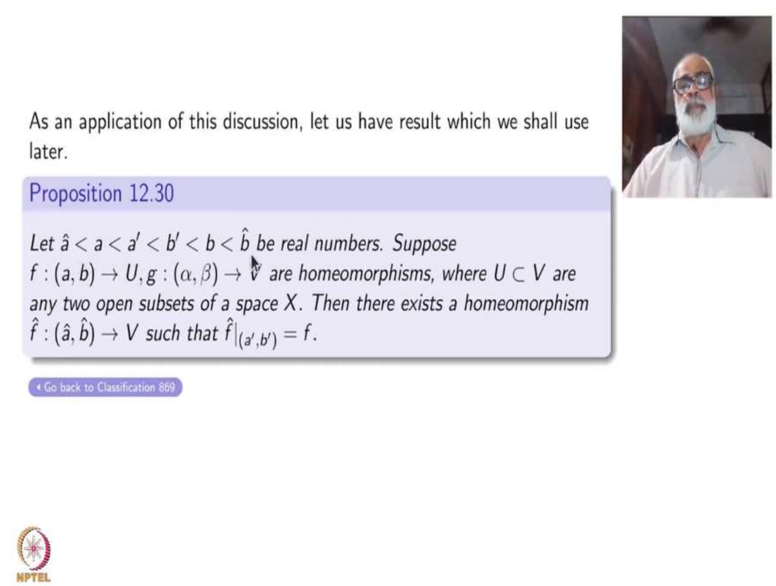
Suppose, you have a homeomorphisms f from (a, b) onto U, g from (α, β) onto V where U and V are some subsets of some space X, (one is contained inside the other is the only condition and both of them are homoeomorphic intervals one is parameterized by this f, another parameterized by g and there is no relation between (a, b) and (α, β) etc.)

Now what is the conclusion? There exists a homeomorphism from $\hat{f}$ from $(\hat{a}, \hat{b})$ onto V, such that restricted to (a', b'), it is f.

Start with this f it is covering only U which is a subspace of V. Now, the new map $\hat{f}$ from $(\hat{a}, \hat{b})$ will cover the whole of V and is an extension of f. That is the important point to note.

(Refer Slide Time: 27:38)





So, let us see how it is done, not very difficult at all. Take α' equal to $(g^{-1} \circ f)(a')$ and β' equal to $(g^{-1} \circ f)(b')$. Notice $f(a', b')$ is a smaller open subsets smaller than $f(a, b) = U$. On that I do not want to change this f at all. In any case they are inside V and so g^{-1} those points makes sense, and will be between α and β. Depending upon whether $g^{-1} \circ f$ is orientation preserving or reversing, we have two cases, namely, α' is less than β' or β' is less than α', I have no control over that.

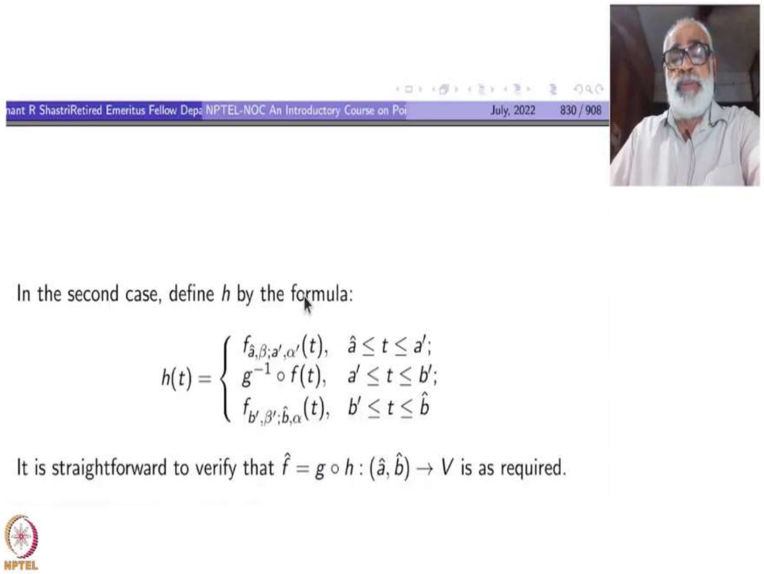
But in either case, they are in the interval (α, β). I have taken first f and then taken g^{-1}. So, they are in the domain of g viz., the open interval (α, β). So, there are two cases. Accordingly I will define two different homeomorphism $\hat{f}$.

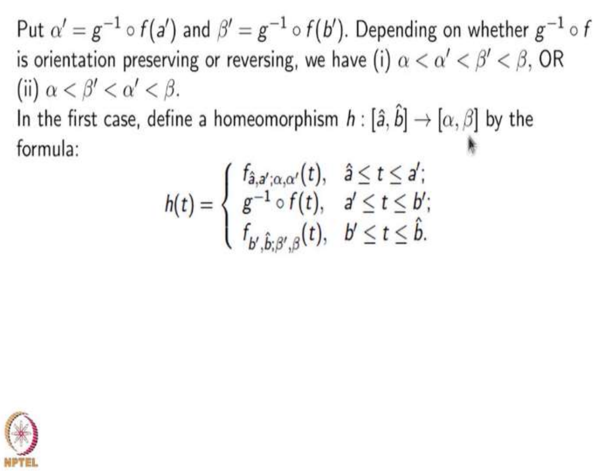
So, first let us define a homeomorphism h from $(\hat{a}, \hat{b})$ to (α, β). Let us do the work inside $\mathbb{R}$ first and then go to these arbitrary space X.

So, this is situation is what we are familiar with, h from $(\hat{a}, \hat{b})$ to (α, β) by cutting the domain and codomain in into three parts; The first part is $[\hat{a}, a']$ to $[\hat{\alpha}, \alpha']$, the affine linear homeomorphism. The second part is $[a', b']$ to $[\alpha', \beta']$ and the map is $g^{-1} \circ f$ and finally from $[b', \hat{b}]$ to $[\beta', \beta)$ another affine linear homeomorphism.

So, this map is between $\hat{a}$ to a' the second one is just $(g^{-1} \circ f)(t)$ wherein I do not want to change my f namely on (a', b'). So, it is between (a', b'), it is $(g^{-1} \circ f)(t)$. The third one is again this is cutting off things we have to use $f(b')$ to β, β' and $\hat{b}$ to β, so endpoint is going matching here, here endpoints matching on this side, this part was the in between things. So, that will happen β' less than about t less than $\hat{b}$. So, they agree because of the definitions of these things. So, h is a homeomorphism, this h is from here to here. So, homeomorphism. Now, all that I do is.

(Refer Slide Time: 30:49)





Since on the common endpoints the two definitions agree, so the map h is well defined and is a homeomorphism. Now, all that I do is to take $\hat{f}$ to be $g \circ h$. On this middle interval, what happens g and g^{-1} and it f. So, $\hat{f}$ is f on this interval.

So, there is another case here. What you have to do is to change the definitions on the frt and the third intervals, because we need them to be monotonically decreasing.

So, in the middle interval, you have to take it to be $g^{-1} \circ f$ only. So, in the first part take the affine linear homeomorhism from $[\hat{a}, a']$ to $[\alpha', \beta]$ which is orientation reversing i.e., decreasing. And in the third part, again take the affine linear homeomorphism from $[b, \hat{b}]$ to $[\alpha, \beta']$ which is again orientation reversing. You map $\hat{a}$ to β and a' to α' here. Similarly, here b' should be mapped to β' and $\hat{b}$ to α. So, change the order that is all.

So, we will continue the study of homogeneity next time. We will do something really marvellous next time for general manifolds. Using these ideas from real numbers. Thank you.