An Introduction to Point Set Topology Part II
Professor Anant R. Shastri
Department of Mathematics
Indian Institute of Technology, Bombay
Lecture 54
Groups of Homeomorphisms

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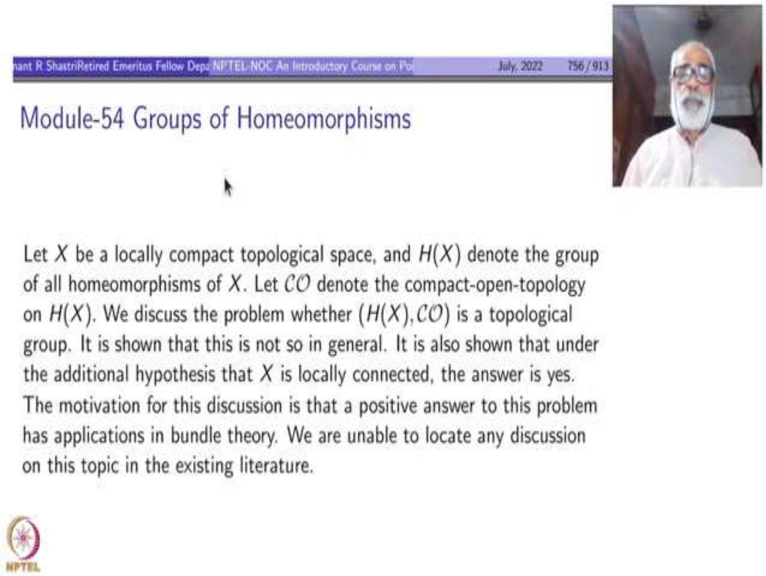

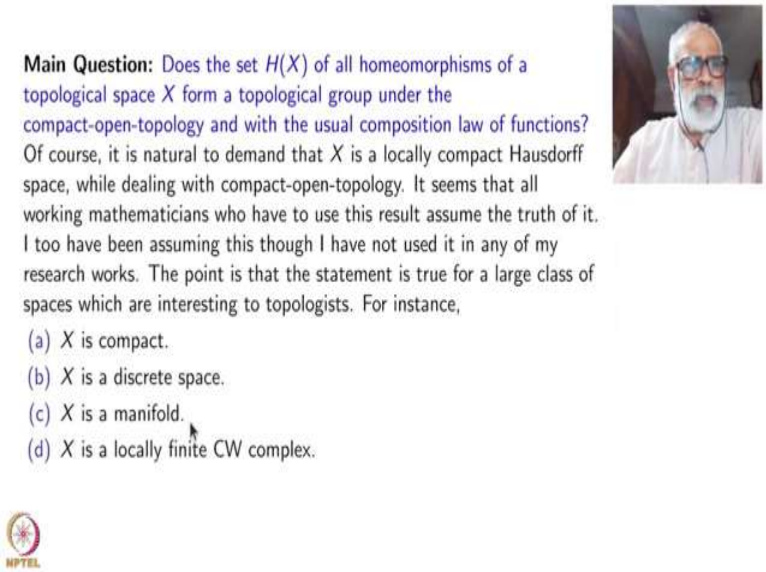

We first notice that the proof of an affirmative answer to the above question, in case the space is compact, is quite easy. So, naturally, our first attempt to prove the same for a non compact, locally compact Hausdorff space X is to go to the one-pt-compactification X^* and try to come back. Note that every element $f \in H(X)$ has a unique extension $\hat{f} \in H(X^*)$, where $\hat{f}(*) = (*)$. Also note that the association $f \mapsto \hat{f}$ can be used to identify the group $H(X)$ with a subgroup of $H(X^*)$. Therefore, we are lead to ask:

Subquestion: Does the compact open topology on $H(X)$ coincide with the subspace topology from $H(X^*)$?

However, it turns out that this question is harder than the original question.

The answer to our main question itself is not always affirmative. The example that we have is obtained by removing the origin from the standard deleted-middle-one-third Cantor set in the unit closed interval. On the positive side, we have the following result:

Theorem 11.16

Let X be a locally compact, locally connected, Hausdorff space and $H(X)$ denote the group of all self homeomorphisms of X. Let $\mathcal{CO}$ denote compact-open topology on $H(X)$. Then $(H(X), \mathcal{CO})$ is a topological group.

On the way to prove this theorem, we shall see that the result holds for all compact Hausdorff spaces, without the assumption of local connectivity.

Hello, welcome to NPTEL NOC an introductory course on point set topology Part II module 54, groups of homeomorphisms. Remember this is a part of our study of compact open topology. So for the rest of this talk we will always assume that X is a locally compact Hausdorff topological space though we may not mention it again and again. And let $H(X)$ denote the group of all homeomorphisms of X, self-homeomorphisms.

Let $\mathcal{CO}$ denote the compact open topology on $H(X)$. We discussed the problem whether this $H(X)$ with the compact open topology is a topological group or not. It is shown that this is not so in general. In general means just under this hypothesis namely X is locally compact Hausdorff. It is also shown that under the additional hypothesis that X is locally connected, the answer is yes.

An example to illustrate that local connectivity is not always necessary, is included that is not correct thing local connectivity is necessary is included.

The motivation for this discussion is that a positive answer to this problem has applications in bundle theory. We are unable to locate any discussion on this topic in the existing literature.

Main question is the following: Does the set $H(X)$ of all self-homeomorphisms of a topological space X form a topological group under the compact open topology with the usual composition law of functions? Whenever you take self-homeomorphisms or self automorphisms of any kind the group law is always the composition law.

Of course it is natural to demand that X is locally compact Hausdorff space, because we are dealing with compact open topology. It seems that all working mathematicians who have to use this result assume the truth of it. I too have been assuming this though I have not used it in any of my research works. The point is that the statement is true for a large class of spaces which are interesting to topologists.

For instance if:

(a) X is compact, we will see an easy proof of the assertion here;

(b) X is discrete, you can do it by hand and see that this is also okay,

(c) X is a manifold, this case is not easy;

(d) X is a locally finite CW complex that is also not easy.

So these are the things, some sort of proofs seems to be there in the literature but nobody has written down it anywhere. So whenever people have used it especially for manifolds apparently they are sure of that the result is true somewhere so that is the whole situation.

So it turns out that this question seems to be harder than the original question, indeed one can more or less see that the two questions are equivalent at least these harder in the sense it is stronger in a sense, if this is true then you can get an easy proof this question, this main question.

We first noticed that the proof of an affirmative answer to the above question in the case when X is compact is quite easy. So naturally, our first attempt to prove the same for a non-compact locally compact Hausdorff space is to go to the one point compatication X^*, use the theorem there and try to bring back, try to come back to the original space X.

Note that every element f belonging to $H(X)$ has a unique extension $\hat{f}$ which is also homeomorphism of the one-point compactification, such that the extra point $\hat{f}^*$ is just the star that point at infinity whatever you want to call X^* is the one point complete equation here. Also note that the association f going to $\hat{f}$ can be used to identify the group $H(X)$ with a subgroup of $H(X^*)$, namely those which fix the point at infinity.

Therefore we are led to ask the following question. Does the compact open topology on $H(X)$ coincide with the subspace topology from $H(X^*)$? Start with $H(X^*)$, give the compact open topology to it, you can take the subspace topology on $H(X)$ because it is a subset now you can treat this actually as a subgroup also. But the question is whether that subspace topology is the same as the compact open topology on X.

So it turns out that this question seems to be harder than the original question. Indeed one can more or less see that the two questions are equivalent. The second question is stronger in the sense if it is true then you can get an easy proof the original main question.

So, the answer to our main question itself is not always in the affirmative. The example that we have is obtained by removing origin from the standard deleted middle one third Cantor's set in the unit closed interval. On the positive side we have the following result. Let X be a locally compact locally connected Hausdorff space and $H(X)$ denote the set of all self-homeomorphisms of $H(X)$. Take the compact open topology on $H(X)$. Then this $(H(X), \mathcal{CO})$ is a topological group.

On the way to prove this theorem we shall see that the result holds for all compact Hausdorff spaces without the extra assumption of local connectivity, so we do not have to prove that one separately.

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Proof of theorem 11.16

Proof: Recall that the compact-open-topology is generated by subbasic open sets of the form

$$\langle K, U\rangle := \{f \in H(X) \ : \ f(K) \subset U\}.$$

For a locally compact Hausdorff space, by (b) of theorem 11.5, the continuity of the composition map

$$H(X) \times H(X) \to H(X); \quad (f, g) \mapsto f \circ g$$

is a consequence of the continuity of

$$H(X) \times H(X) \times X \to X; \quad (f, g, x) \mapsto f \circ g(x). \qquad (38)$$

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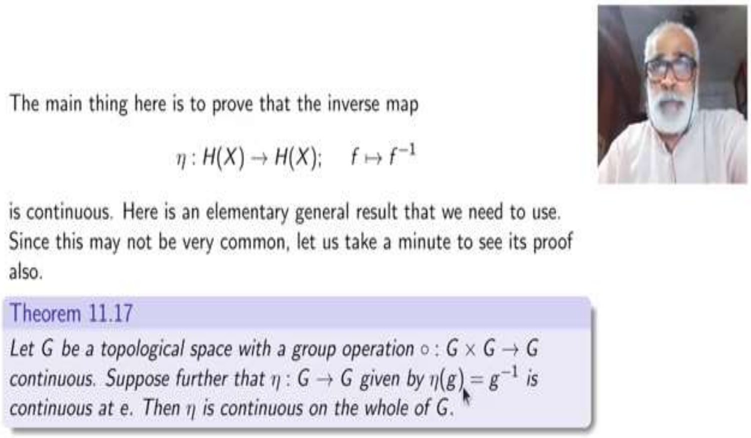
To verify the continuity of (38), fix f, g, x such that $f \circ g(x) \in U$, where U is open in X. Choose an open set V such that $g(x) \in V$, $\bar{V}$ compact and $f(\bar{V}) \subset U$. Then choose an open set W such that $x \in W$, $\bar{W}$ is compact and $g(\bar{W}) \subset V$. It follows that $\langle \bar{V}, U\rangle \times \langle \bar{W}, V\rangle \times W$ is a neighbourhood of (f, g, x) which is mapped inside U by the map (38). Thus the continuity of the composition is established.

In particular, it follows that for every $f \in H(X)$, the left and right compositions, $L_f, R_f : H(X) \to H(X)$ given by

$$L_f(g) = f \circ g; \quad R_f(g) = g \circ f$$

are continuous.



So let us start proving it. But there will be many interruptions in between, so let us hope we can start proving. Recall that the compact open topology is generated by subbasic open sets of the form, $\langle K, U\rangle$ is nothing but the set of all f belonging to $H(X)$ such that $f(K)$ is contained inside U. You remember this notation we used for all continuous functions from X to Y, when we are dealing with space of all continuous functions. Here I have restricted this notation to only homeomorphisms of X to X.

Therefore, the same notation I will use to mean certainn subsets of $H(X)$ only. So, I have redefined it carefully here. This is actually the old notation $\langle K, U\rangle$ intersected with $H(X)$. Pay attention to it.

In any case $H(X)$, is a subspace of all continuous functions from X to X, so this is subspace topology. Therefore if you take subbasic open sets in the bigger space, and intersect them in with the given subset, they will form a subbase for the subspace topology, so there is no problem. So this is the subspace topology from the compact open topology on all continuous functions.

For a locally compact Hausdorff space, from part (b) of our earlier theorem namely exponential correspondence theorem, the continuity of the composition map $H(X) \times H(X)$ to $H(X)$, viz., (f, g) mapsto $f \circ g$, is a consequence of the continuity of the corresponding function under evaluation map and so on, viz, (f, g, x) mapsto $(f \circ g)(x)$ from $H(X) \times H(X) \times X$ to X. This is a function into X.

We have used this criterion several times now, we are going to use it now also.

So how to verify that this (f, g, x) mapto $f \circ (x)$ is continuous? From the product space $H(X) \times H(X)X$ to X?

So to verify this, fix f, g and x and an open set U such that $(f \circ g)(x)$ is in U. Then I have to produce a neighborhood of f cross a neighbourhood of g cross a neighborhood of x such that under this function the entire neighborhood goes inside U. So that is the continuity at this point (f, g, x).

So, start choosing an open neighbourhood V of $g(x)$ such that $\bar{V}$ is compact and $f(\overline{V})$ is contained inside U. So this I can do becuase f is continous, $f(g(x))$ goes inside U, and X is locally compact.

Similarly once you have chosen this V, now $f(x)$ goes into V, therefore you will get an open subset W such that x is inside W, $\overline{W}$ compact and $g(\overline{W})$ inside V. Now it is straight forward verification to see that angled bracket $\bar{V}$, U, (so this $\bar{V}$ is compact and this V is open, so the set of all homeomorphisms taking $\bar{V}$ into U), that is a neighborhood of neighbourhood of f because f is one of them.

Similarly, angled bracket $\overline{W}$ comma is a neighbourhood of g. So the product of these two with W is a neighbourhood of (f, g, x). Now take any arbitrary element (f', g', y) in this open set. The g' of y belongs to V and f' of that belongs to U.

This proves the continuity of the composition law.

So in particular it follows that once you have proved that the composition is continuous you can fix f and look at g going to $f \circ g$, so that is the left multiplication by f, fix g, then f going to $f \circ g$, that is the right multiplication by g, both of them will be continuous from $H(X)$ to $H(X)$). That is all standard stuff whenever you have topological semigroup. So, the left multiplication L_f and right multiplication R_f both of them are continuous, from $H(X)$ to $H(X)$.

The main thing here is to prove that the inverse is also continuous. Once you prove that it follows that the group laws are continuous, therefore the $H(X)$ is a topological group. So this is what we have to prove.

Here is an elementary result that we want to use. Since this does not seem to be very common, let me just take a few minutes to state it clearly and then prove it also. This is a general result about group operations on topological spaces.

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Proof: We first note that for each $f \in G$, the left and right multiplication $L_f, R_f : G \to G$ are continuous. Since $(L_f)^{-1} = L_{f^{-1}}$, and $(R_f)^{-1} = R_{f^{-1}}$, it follows that each L_f, and each R_f is a homeomorphism. Now let $f \in G$, be any element. We check that

$$\eta \circ L_f = R_{f^{-1}} \circ \eta : G \to G.$$

Since $R_{f^{-1}}$ is continuous on the whole of G, it follows that $R_{f^{-1}} \circ \eta$ is continuous at e. Therefore $\eta \circ L_f$ is continuous at e. Since L_f is a homeomorphism, this clearly implies that η is continuous at $L_f(e) = f$. ♠

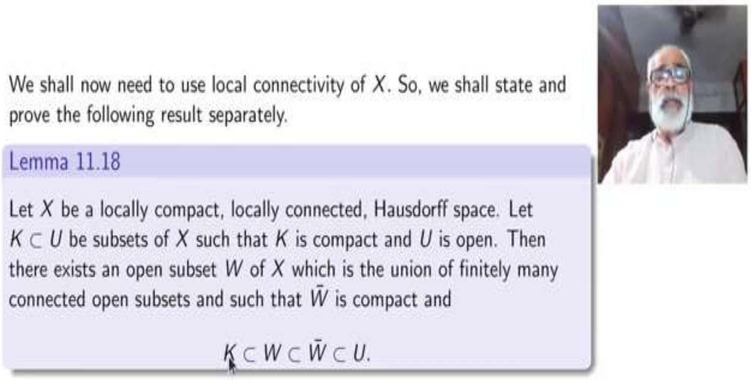
Continuing with the proof of theorem

Thus, it suffices to prove that $\eta : H(X) \to H(X)$ is continuous at $Id_X \in H(X)$.
Since $\eta^{-1} = \eta$, from the above remark, we need to show that sets of the form $\eta(\langle K, U\rangle)$ are neighbourhoods of Id_X, where $K \subset U$, K compact and U open in X.

Now

$$\begin{aligned} f \in \eta(\langle K, U\rangle) &\iff f^{-1} \in \langle K, U\rangle \\ &\iff f^{-1}(K) \subset U \\ &\iff K \subset f(U) \\ &\iff K^c \supset f(U)^c = f(U^c) \\ &\iff f \in \langle U^c, K^c\rangle \end{aligned}$$

Therefore we need to prove that sets of the form $\langle U^c, K^c\rangle$, where U is any open set and K is a compact subset with $K \subset U$, are neighbourhoods of Id. (Of course, we need to consider only the case when K is non empty and U^c is non compact. In particular, the proof of the theorem is over if X itself is compact. Note that we have not used local connectivity of X so far.)



Let G be a topological space with a group operation $G \times G$ to G being continuous. Only assumption is that the multiplication is continuous. Suppose the function η from G to G given by $\eta(g) = g^{-1}$ (that is called the inversion map) is continuous at just at e in G, the identity element. Then η is continuous on the whole of G.

This kind of results we must have used in any topological group itself. Like if you have homomorphism from one topological group to another which is continuous at one point, it will be continuous on the whole of group. Similarly for differentiable functions on $\mathbb{R}$. If they are group homomorphisms then differentiability at a single point will imply differentiability everywhere. Such kind of results I am sure you are familiar with. But in all of them, you already have a topological group.

The same kind of technique is being used here to prove continuity of the inversion itself, only under the assumption that the group composition is continuous. So that is why this is not all that common, so let me tell you the proof.

We first note that for each g, the left multiplication L_g and right multiplication R_g are continuous, because the multiplication itself is continuous. Since L_f^{-1} is equal to $L_{f^{-1}}$ and R_f^{-1} is $R_{f^{-1}}$, it follows that L_f and R_f each of them is a homeomorphism. First of all they are continuous but same reason $L_{f^{-1}}$ is also continuous, because f^{-1} is just another element in G.

So now let f belonging to G be any element. We want to check the continuity of η at that point. So first you check the following thing.

The inversion $\eta \circ L_f$ is nothing but $R_f^{-1} \circ \eta$ as a function from G to G. This is very straight forward verification. This is true in all groups.

Since R_f^{-1} is also continuous on the whole of G, it follows that $R_f^{-1} \circ \eta$ is continuous at e, because η is continuous at e. So when you take the composite with another continuous function, the composite will be also continuous at e.

So this side now $\eta \circ L_f$, we know is continuous at e. But now L_f is a homeomorphism. Therefore it will follow that η is continue at $L_f(e) = f$. So all that you need is to use viz., L_f is an open mapping, so this will be easy.

So, let us use this result now, namely continuity at one point, namely at identity is enough to conclude the continuity on the whole thing. Therefore come back to this special case now. η the inversion map from $H(X)$ to $H(X)$, I have to prove that it is continuous at the identity element of this group $H(X)$.

Note that the inversion, by the very definition, inverse of η is η itself and $\eta(e) = e$. η^{-1} an involution, η^{-1} is η.

So showing that it is continuous is the same thing as showing that it is open. We need to show that the image under η of $\langle K, U\rangle$ are neighborhoods of Id_X, where K is compact and U is open and such that K is a subset of U, (Id_X belongs to this set means that) are open in $\mathcal{CO}$. These are subbasic open sets and therefore that is enough for continuity of $\eta(Id_X)$.

So just do this elementary algebra now. Here f is a point of $\eta(\langle K, U\rangle)$ means f^{-1} is inside $\langle K, U\rangle$, that means $f^{-1}(K)$ is inside U that is same thing as saying K is inside $f(U)$ because f belonging to $\langle U^c, K^c\rangle$.

So starting with a subbasic open subset you have to show that this is an open subset in the definition of compact open topology. Notice such sets are not there in the subbase. You see to allow this as a subbasic open set, I need this one to be compact and this one to be open. Luckily since K is compact, it is closed, so the compliment is open, but why U^c must be compact?

U is just an open subset. That is why this is a non trivial problem.

Any way we not need to show that this actually open. We have to only show that this is a neighborhood of identity that is enough.

Therefore, we need to prove that sets of the form $\langle U^c, K^c \rangle$ where U is any open set and K is compact, K contained inside U, are neighborhoods of identity of X. Of course we need to consider only the case when K is non-empty. If K is the whole space K^c will be empty, and if K is empty, K^c will be the whole space so this will be automatically satisfied.

And we need to consider only the case when U^c is non-compact. If U^c is compact then this set is a subbasic open subset, there is no problem. So in particular if X itself is compact then the U^c will be compact and hence there is nothing to do. Therefore, η is continuous when X is compact. Note that so far we have not used local connectivity of X at all. Just assuming that X is compact Hausdorff space is good enough.

Now, we want to use local connectivity of X.

So here is one more elementary Lemma, so we should state and prove this separately instead of including it in the run of the proof.

Let X be a locally compact, locally connected, Hausdorff space. Let K contained inside U subsets of X, such that K is compact and U is open. Then there exists an open subset W of X such that this W is a union of finitely many connected open sets, each of which is actually compact also, (and hence in particular), $\overline{W}$ is compact and K is contained inside W contained inside $\overline{W}$ contained inside U.

The statement is that begin with K is compact inside U which is open. In between you can always get another open set W contained in $\overline{W}$ by regularity. But what I want is that this W itself is the union of finitely many connected subsets and each of them such that their closure is compact so that $\overline{W}$ will be also compact. So that is the extra thing and that extra thing comes from local connectivity. The rest of the statement is because of comapctness and Hausdorffness.

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Proof: For each $x \in K$, choose an open subset V_x such that

$$x \in V_x \subset \bar{V}_x \subset U$$

and such that each V_x is connected and $\bar{V}_x$ is compact. This is possible since X is locally compact as well as locally connected. Since K is compact, there exists a finite subset $\{x_1, \dots, x_k\} \subset K$ such that

$$K \subset \cup_{i=1}^k V_{x_i} =: W \subset \bar{W} \subset U.$$

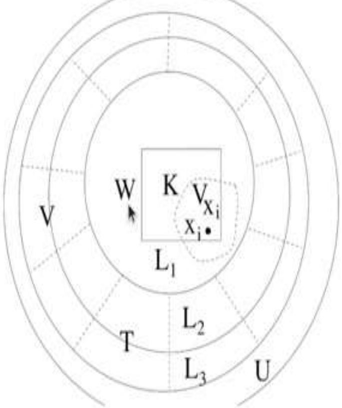
This W has the required properties. ♠

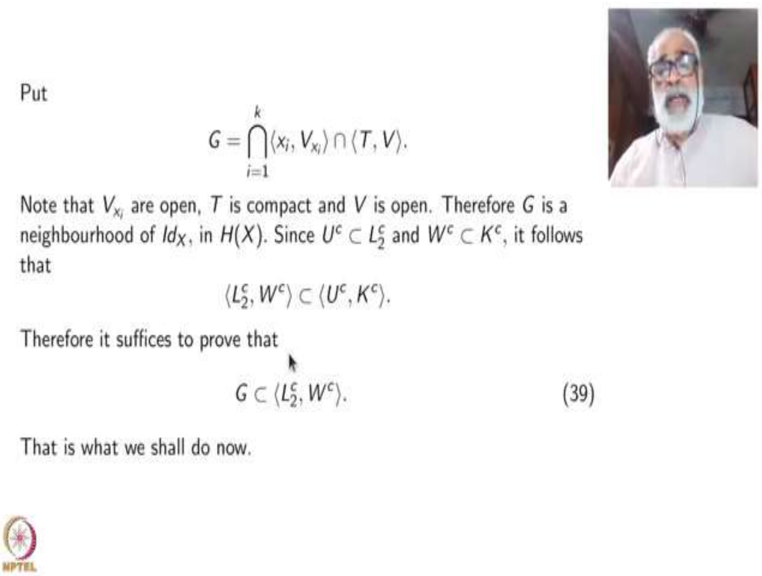
Continuing with the proof of theorem 11.16

Let V_{x_i}'s, W etc. be as in the above lemma 11.18. Put $L_1 = \bar{W}$. Using regularity of X, choose compact subsets L_2, L_3 such that

$$L_1 \subset \overset{\circ}{L_2} \subset L_2 \subset \overset{\circ}{L_3} \subset L_3 \subset U.$$

Put $V = \overset{\circ}{L_3} \setminus L_1$, and $T = L_2 \setminus \overset{\circ}{L_2} = \partial L_2$.





Let us go through proof this. It is not a difficult thing.

For each point $x \in K$, because x belongs to U and U is open and because the space is locally compact and locally connected, I can choose open V_x such that x belongs to V_x contained inside $\overline{V_x}$ contained inside U such that this V_x is connected and $\overline{V_x}$ is compact. First you choose U_x such that $\overline{U_x}$ is compact and inside U. Now using local connectivity, you can choose another open set V_x which is connected and its closure will be contained inside the closure of V_x, which is compact and so it will be also compact. that is the way you can prove this one. This is possible because X is locally compact first and then locally connected also.

Since K is compact you can get a finite cover for K, viz., there exist $x_1, x_2, \ldots, x_n$ all of them in K such that K is contained inside the union of V_{x_i}'s, i ranges from 1 to n. I am denoting that union by W. So W satisfies the assertions of the lemma. Over.

So we will use this lemma.

Now let us continue with the proof of theorem. So let V_{x_i}'s and W etc. be as in the previous lemma that we have proved just now. I will put L_1 equal to $\overline{W}$. I am starting some some kind of induction here. Not very deep.

Using the regularity of X, remember this $\overline{W}$ is compact and contained inside U, so I choose this $L_1 = \overline{W}$, it is contained inside U, so in between I will choose another L_2 which is a compact neighborhood of L_1, i.e. interior of L_2 contains L_1, and L_2 is contained inside U. So, similarly in between L_2 and U again another one denoted by L_3 such that L_2 contained in the interior of L_3, L_3 is compact and contained in U. So these are all compact subsets now L_2 and L_3, their interiors contain the previous one, so L_1, L_2, L_3 are all contained inside U.

Now you take V equal to interior of $L_3 \setminus L_1$. By the way this kind of things we have done several times while studying paracompactness etc. Whenever you have Hausdorff and locally compactness and so on so you are using kind of normality, here, a strongly regular property because things are compact and so on.

So take this V to be interior of L_3 setminus this closed set L_1, so this is an open subset. And put T equal to just the boundary of this L_2; L_2 is a compact set, throw away the interior of L_2, so that is T which is boundary of L_2.

So here is a picture of what we have done so far. We started with this rectangle here K contained inside this large open set U here. In between the first thing was to take a W such that W is the union of finitely many connected open subsets such that closure of each is compact. So those are the V_{x_i}'s, which cover the whole of K and is contained inside U. That is all. After that I have chosen the notation $\overline{W}$ to be L_1, so there is a compact set contained in U, then I choose L_2 then I choose L_3. So that is all the picture is about. What is this V? This V is this interior of $L_3 \setminus L_1$, so that is an open subset. And then T is just the boundary of L_2. So I have drawn some nice pictures here. My pictures may not be all that nice. There I have drawn a circle here and so on, this T need not be connected etc. The only thing is this W is a finite union of connected sets. There is no other properties. K may not be connected, U may not be connected, nothing. There is no connectivity assumption anywhere else.

Now you take G equal to the intersection of $\langle x_i, V_{x_i}\rangle$ for all $i = 1, 2, \ldots, n$ further intersected with $\langle T, V\rangle$. So this is a finite intersection of subbasic open sets and hence is a basic open set. Since x_i is inside V_{x_i} for all i and T is contained in V, therefore identity map of X belongs to G.

Therefore G is a neighborhood of identity of X in $H(X)$.

Now look at the complement of U. This is contained in the complement of L_2 also because L_2 is inside U. And complement of W contains complement of K, because K is contained in W. So I am just taking De Morgan law that is all. It follows that $\langle L_2^c, W^c\rangle$ is contained in $\langle U^c, K^c\rangle$. This is just a set theoretic property of this angled brackets. That is all.

Therefore it suffice to prove (we want to prove what?) we wanted to prove that this neighbourhood G of Id_X which we have selected, is contained in this open subset $\langle U^c, K^c\rangle$. So instead of that we can just show that this G is contained inside this smaller set, that is a stronger statement. So let us prove that G is contained inside $\langle L_2^c, W^c\rangle$.

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Then, for any $f \in G$, since f is a homeomorphism,

$$X \setminus f(T) = f(\overset{\circ}{L_2}) | f(L_2^c) \tag{40}$$

is also a separation. Since $x_i \in V_{x_i} \subset \overset{\circ}{L_2}$, we have $f(x_i) \in f(\overset{\circ}{L_2})$. Also note that $f(T) \subset V$ and hence does not intersect any of V_{x_i} and hence $V_{x_i} \subset X \setminus f(T)$. Since V_{x_i} is connected, and $f(x_i) \in V_{x_i} \cap f(\overset{\circ}{L_2})$, it follows that $V_{x_i} \subset f(\overset{\circ}{L_2})$. This this true for all i, we get $W \subset f(\overset{\circ}{L_2})$. Therefore, using (40), we get,

$$W^c \supsetneq f(\overset{\circ}{L_2})^c \supset f(L_2^c).$$

This proves (39) and hence the theorem. ♠

Remark 11.19

Attempt to find a justification for locally connectedness assumption in the above theorem, initially lead us naturally to the topologists sine curve. Recall that this is the subspace of $\mathbb{R}^2$ which is the union of the graph of $\sin \frac{\pi}{x}$ on the interval $(0, 1]$ together with the closed interval $0 \times [-1, 1]$. We delete the point $p = (0, 1)$ from this to obtain the space X which is non compact, locally compact, Hausdorff, and connected but not locally path connected. We can then think of the original space as the one-pt-compactification of X. Alas! it turns out that we do not understand the group structure of $H(X)$ sufficiently well. In particular, the subquestion above applied to this particular case also remains unsettled either way.

A Counter Example

Let C denote the standard deleted-middle-one-third Cantor set in the closed interval $[0, 1]$. A subspace A of C which is the closure of

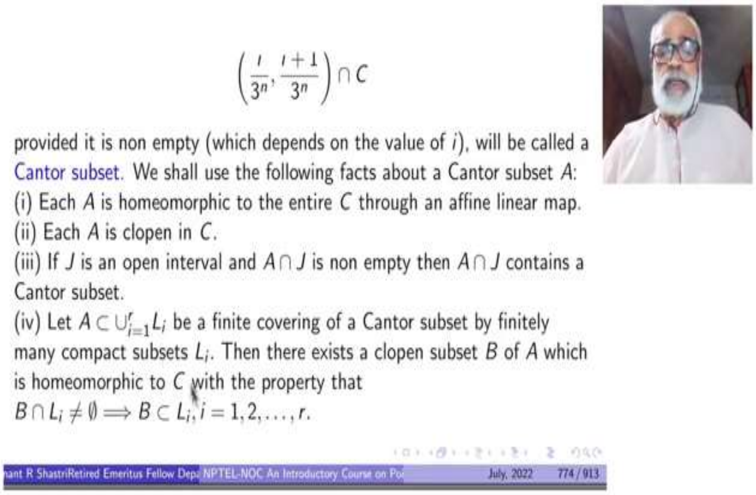
$$\left(\frac{i}{3^n}, \frac{i+1}{3^n}\right) \cap C$$

provided it is non empty (which depends on the value of i), will be called a Cantor subset. We shall use the following facts about a Cantor subset A:

(i) Each A is homeomorphic to the entire C through an affine linear map.

(ii) Each A is clopen in C.

(iii) If J is an open interval and $A \cap J$ is non empty then $A \cap J$ contains a Cantor subset.



Now consider T which is a closed subset, and look at $X \setminus T$. T was the boundary of L_2. Therefore its complement is the union of interior of L_2 and the the complement of L_2. Both are open, they are disjoint and the union is precisely $X \setminus T$, boundary of L_2. This is a standard way of getting a separation, this is nothing very special about L_2.

Now take any element of G, a homeomorphism of X. (Any homeomorphism will do for the conclusion that follows, but we are interested in what is happening for elements of G.) Since f is a homeomorphism, f of this separation will be another separation, of course of the space $X \setminus f(T) = (f(L_2^o))|f(L_2^c)$. A separation gives rise to separation under a homeomorphism.

Now look at all the points x_i's which are inside V_{x_i}'s respectively, and each V_{x_i}'s is connected and contained in L_2 interior because they are inside L_1 itself. So we have $f(x_i)$ are inside $f(L_2^o)$. Also $f(T)$ is contained inside V in this picture if you remember f belongs to G and G is contained inside $\langle T, V\rangle$, and so, $f(T)$ remains in V, and hence does not intersect any of the V_{x_i}'s because V_{x_i}'s are inside L_1 interior. So I have subtracted the entire of L_1 from L_3 interior so that is V, and hence all the V_{x_i}'s must be inside $X \setminus f(T)$.

But each V_{x_i} is connected, this is a separation, therefore each V_{x_i} must be in one of them, either $f(L_2^o)$ or $f(L_2^c)$. On the other hand $V_{x_i} \cap f(L_2^o)$ is non empty because $f(x_i)$ is in both of them. Here again we use the definition of G and the fact that f is in G. Therefore all the V_{x_i} are completely contained $f(L_2^o)$. So the entire W which is the union of V_{x_i}'s is contained inside $f(L_2^o)$.

But then look at the complements. We get W^c contains the complement of $f(L_2^o)$ which contains the complement $f(L_2)$. So what we have proved? $f(L_2^c)$ is inside W^c, what does that mean? f is inside the $\langle L_2^c, W^c\rangle$. Started with an element f in G and showed that it is here, so the proof is over.

Indeed the next thing namely producing a counter example. If at all you think that the above proof was difficult, then that will be little more difficult. So let us come to the task of producing the counter example.

Before that I will tell you some history, this history is very recent. After all it is just about three years old so whatever I remember I have put it here already.

Attempt to find a justification for locally connectedness assumption in the above theorem, initially led us naturally to the topologists' sine curve. Recall that this topologists' sine curve is a subspace of $\mathbb{R}^2$, it is a compact also and which is the union of the graph of $\sin \pi/x$ on the interval $(0, 1]$, together with the closed interval $0 \times [-1, 1]$ on the y-axis. I cannot go into the full study this space now but I am just recalling that this being a compact space, if you compact open topology on the set of self homeomorphisms of this space, that will be automatically a topological group as seen during the proof of the above theorem.

So what we want to do is destroy the compactness by removing a point. So do that, namely, take this very special point $(0, 1)$ (or equivalently $(0, -1)$). These are two special points there. So remove one of them and obtain the topological space X which is now non compact, it is locally compact, it is a subspace of $\mathbb{R}^2$ and so it is Hausdorff also. And it is connected, but not locally connected. It is not locally path connected, it is not locally connected either.

We can then think of the original space as the one point compactication of X, so I was trying to prove (or disprove actually I was trying to prove) that in this case there is an assertion, namely, an affirmative answer that $H(X)$ is a topological group. But it turns out that understanding the group structure of $H(X)$ is very important here, and we do not know it very well yet. So I have to abandon this example.

But this leads to a sub question here namely (this attempt is what is important here)

going to one point compactification, instead of just trying to work out this example, try to do it in general. So that is why we have this sub-question. So the point is that I wanted to say that this attempt was not successful, yet, I have not given it up completely of course.

So the next attempt is with, you can guess, what is it: the Cantor set. So Cantor set did help. So let us study the Cantor set now. Cantor set is also compact, remember that. So I want to destroy the compactness by removing a convenient point, you can perhaps do it with any point but most convenient is remove the point 0.

So recall that the deleted middle one third Cantor set C is obtained by deleting opne middle one-third of the interval $[0, 1]$ and keep repeating this operation on each of the resulting closed intervals.

So if you take some open interval of this length 1 by 3 power n, such as i/3^n to i+1 by 3^n) and look at its intersection with C, that may be empty because this interval may be already contained in one of the deleted parts. But if it is not empty, then it will a carbon copy of the Cantor set again.

Any subspace of A of C, which is of this nature, namely homeomorphic to C will be called a Cantor subset. Instead of starting with the open interval $(0, 1)$. I could have started with any interval and performed the same kind of operation of deletion on it. That is all. So we shall use the following facts about a Cantor subset, so Cantor subset I am denoting by A.

(i) So each A, each Cantor subset is homeomorphic to the entire Cantor set C, through an affine linear map. An affine linear map which will send the end points of the starting interval to endpoints of $(0, 1)$, that will automatically give you a homeomorphism of any Cantor subset with C.

(ii) Each A is clopen in C open as well as closed,

(iii) If J is a non empty open interval and $A \cap J$ is non empty, then $A \cap J$ contains a Cantor subset. In each open interval we have such a thing. That is the property of the Cantor set C. It re-appears inside every open interval intersected with C.

(iv) Finally, this may need a little more difficult to prove, similar to our lemma about local connectivity and so on, but see here this is a different kind of game. But the proof is not difficult.

Take any Cantor subset A contained in the finite union of some compact subsets L_i. Then there exists a clopen subset B of A which is homeomorphic to C, (that means B is also a Cantor subset) such that $B \cap L_i$ is non empty implies B is completely contained inside L_i.

There are finitely many of L_i's. Look at L_1 say. $B \cap L_1$ is non empty then the entire B is contained in L_1. If it is empty it is fine; so some of them may not intersect B. moment they intersect they will completely contain B, so that is the meaning of this one. This property is crucial and is used in a peculiar way, You will see.

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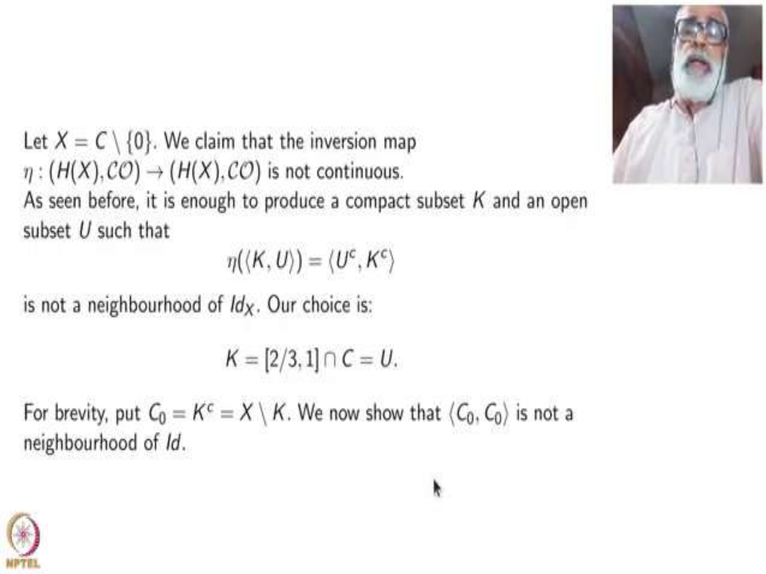

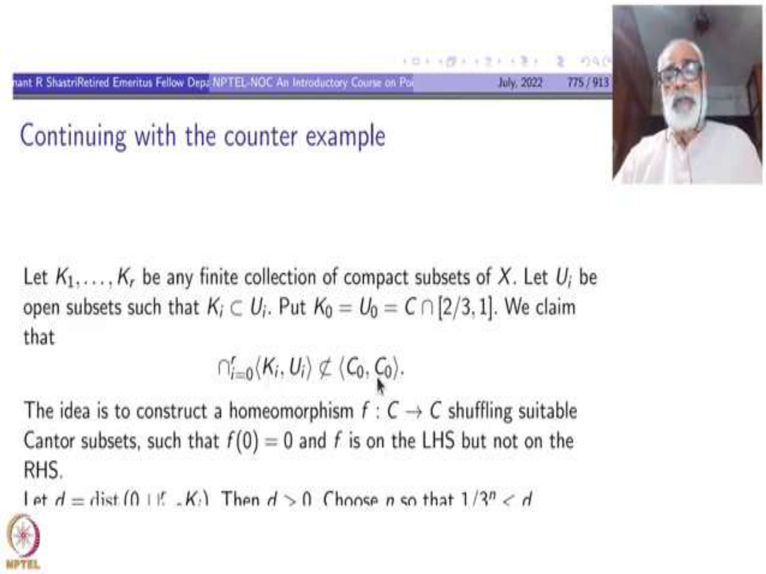

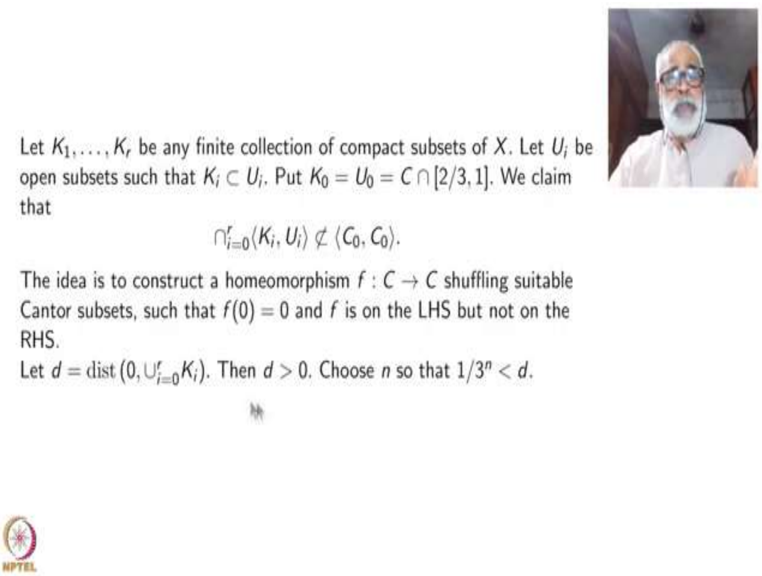

Now I take X to be the Cantor set $C \setminus \{0\}$. We claim that the inversion map η from $H(X)$ to $H(X)$ is not continuous. So $H(X)$ cannot be a topological group under the compact open topology.

As seen before it is enough to produce a compact subset K and an open subset U such that η of $\langle K, U \rangle$ which is equal to $\langle U^c, K^c \rangle$ (this we have seen already) is not a neighborhood of identity of X. Over.

So our choice is K the closed interval $[2/3, 1] \cap C$. So this is another Cantor subset. First, $[0, 1/3] \cap C$, which contains 0. I have thrown away that point. Now I am taking the part of C inside $[2/3, 1]$. That is my K which is a clopen in C. So I take that itself as U as well. It is compact, it is open, so I can take $\langle K, U \rangle$ which is a neighbourhood of Id_X. Now I have to show that $\langle U^c, K^c \rangle$ is not a neighborhood of identity of X.

So for brevity, we will just put C_0 equal to K^c which is $X \setminus K$. Remember where I am taking the complement here, in the space X. We now show that this $\langle C_0, C_0 \rangle$ is not a neighborhood of identity.

The statement is clear. So far there were only notations. So now the proof starts.

Let $K_1, K_2, \dots, K_r$ be any finite collection of compact subsets of X, let U_i be open subsets containing K_i's. One more K_0 you have to take, namely put $K_0 = U_0 = C \cap [2/3, 1] = K$. You include this also in the list.

We claim that the intersection i ranging from 0 to r of $\langle K_i, U_i \rangle$ is never contained in $\langle C, C_0 \rangle$. These are all the basic neighbourhoods of Id_X, i.e., any open neighbourhood of Id_X will contain an open subset of this form, because these are basic neighbourhoods of Id_X. Taking intersection with the extra open set $\langle K_0, K_0 \rangle$ does not change this property.

Even after taking the call intersection it is not contained is the RHS above. Even the smaller set is not contained in. Of course, larger set also will not be contained in. So that is why I can take this intersection with this extra open set also. So this is a special one I have taken but these are general things, so that will show you that $\langle C_0, C_0 \rangle$ is not a neighborhood of identity.

So you have to construct a special function $f \in H(X)$ here belonging to LHS but not to RHS The idea is to construct a homeomorphis f from C to C itself, by shuffling suitable Cantor subsets such that 0 goes to 0 so that if you throw away 0, it is still a homeomorphism from X to X, and of course, this f is on the LHS here but not on the RHS.

Now the construction starts. Look at these finitely many $K_1, K_2, \ldots, K_r$ which are compact subsets of X. 0 is not a point of X. It is inside the Cantor set C.

So this distance $d = d(0, \cup K_i)$ is positive. This is the usual distance in $\mathbb{R}$ nothing more than that. It is positive. Choose n such that $1/3^n < d$, so that $[0, 1/3^n]$ that interval does not intersect any of these K_i's that is all. A small closed interval around 0 is taken of course 0 will be thrown out afterwards. That interval is not intersecting any of these K_i's. That is the important step here.

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Note that K_0 is a Cantor set. By the above remark (iv), with $A = K_0 \subset \cup_{i=0}^r K_i$, there exists a clopen subset B which is a copy of the Cantor set such that

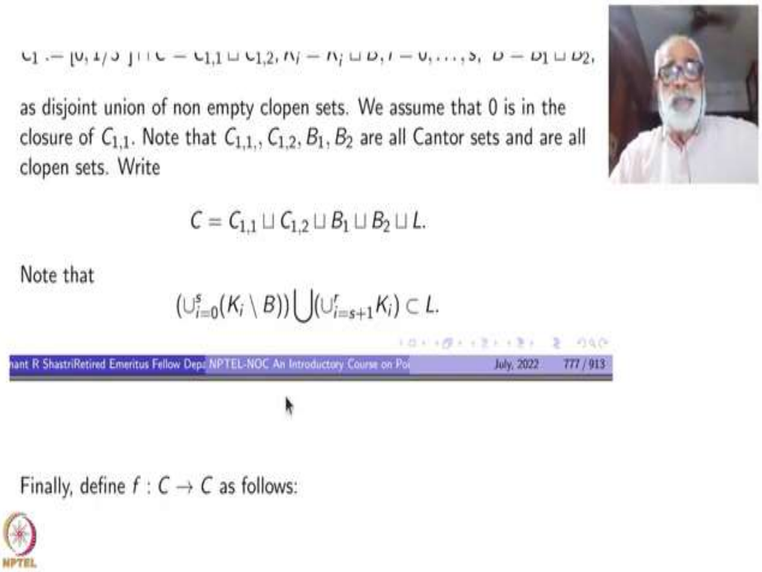
$$B \subset K_i, i = 0, 1, \ldots, s \text{ and } B \cap K_i = \emptyset, i = s+1, \ldots r; s \geq 0,$$

after re-indexing the sets $K_1, \ldots, K_r$.
Write

$$C_1 := [0, 1/3^n] \cap C = C_{1,1} \sqcup C_{1,2}; K_i = K_i' \sqcup B, i = 0, \ldots, s; \; B = B_1 \sqcup B_2;$$

as disjoint union of non empty clopen sets. We assume that 0 is in the closure of $C_{1,1}$. Note that $C_{1,1}, C_{1,2}, B_1, B_2$ are all Cantor sets and are all clopen sets. Write

$$C = C_{1,1} \sqcup C_{1,2} \sqcup B_1 \sqcup B_2 \sqcup L.$$



Note that this K_0 whatever we have chosen, is one of the Cantor subsets. Put A equal to K_0. Then A is contained in the union of $K_i, i = 0, 1, \ldots, r$. So, by remark (iv) above, there exists a clopen subset B which is a copy of the Cantor set such that this B is contained inside K_i, i range from $1, 2, ..., \ldots s$ and the rest of the K_i do not intersect B. I have to possibly re-index these K_i's for this.

For example suppose K_i's were all disjoint from K_0. Then I can take this B equal K_0 itself and there is no need to re-index. Note that K_0 is a special one, all other K_i are arbitrary, you can label them whichever way you like. You do not know in which order they are occurring. This s may be just zero also as in the above case or it may be r iteself. It does not matter, the rest of them do not intersect K_i at all. So, s has to be chosen that way.

Now, let us do some more splittings. Whole idea is that a Cantor subsets can be written as a disjoint union of as many copies of itself. So that is the whole idea. (This fractal property of the Cantor set is used here very nicely.)

Take $C_1 = [0, 1/3^n] \cap C$. C_1 does not intersect any of the K_i's. The number n has been chosen that way.

Write C_1 as a disjoint union of two Cantor subsets $C_{1,1}$ disjpoint union $C_{1,2}$. Recall how C_1 is constructed from the interval $[0, 1/3^n]$, you remove the open middle one third first, so you get two disjoint intervals, intersected with A they will give you $C_{1,1}$ and $C_{1,2}$. You can always do this, whichever way you like, it does not matter.

Now you write each K_i equal to K_i' disjoint union with B for $i = 1, \ldots, s$, possible because these K_i's contain B.

What is B? B is a Cantor set. Therefore it is clopen, so K_i' will be also clopen in K_i. But what happens to other K_i's? Do not have to worry because B does not intersect that part. Finally you also write B itself as disjoint union of clopen subsets B_1 and B_2. So all these $B_1, B_2, B, C_1, C_{1,1}, C_{1,2}$ they are all Cantor subsets, the disjoint union of non empty clopen sets.

Now we assume that 0 is in $C_{1,1}$. Remember the whole space C_1 contains 0 and so 0 is either in $C_{1,1}$ or $C_{1,2}$. By changing the indexing you may therefore assume that 0 is in $C_{1,1}$.

Now look at the union of all these subsets that is a clopen set inside C, so the entire C can be written as a disjoint union of C_{11}, C_{12}, B_1, B_2 and finally another clopen set L which is jst the complement of the union of earlier ones in C. Now the major work is over. Now I can define the homeomorphism by merely shuffling these subsets.

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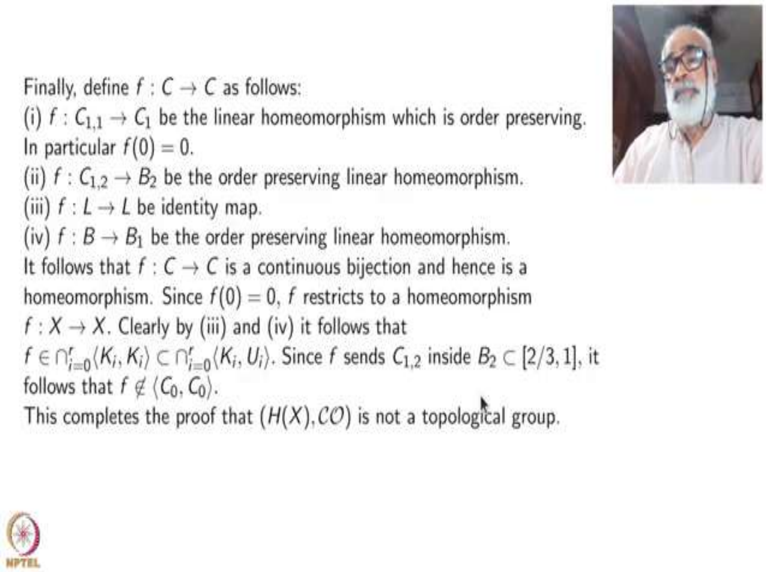

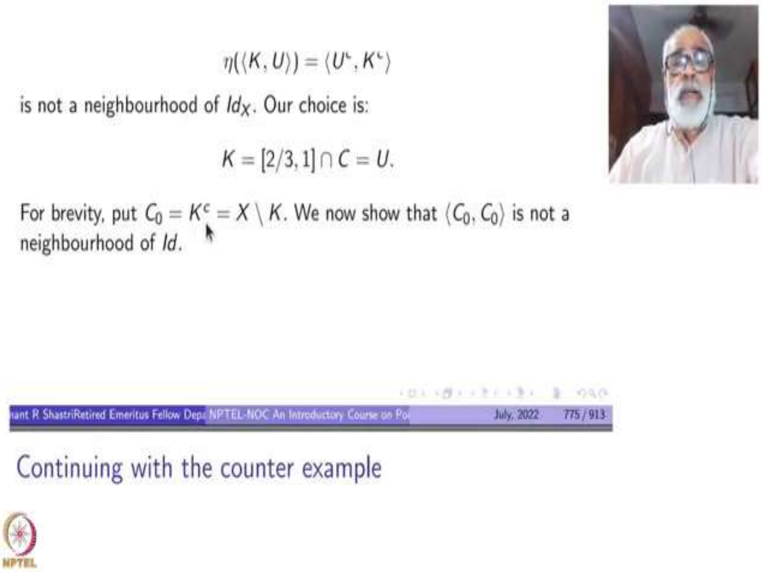

So one thing you have to notice is this L contains the complement of B inside K_i' and all those K_i's which B does not intersect, namely, K_i for i ranging from $s+1$ to r. L may contain other things also. So now define f from C to C as follows.

(i) First f from C_{11} to C_1; C_{11} is a subset of C_1, but both of them are Cantor subsets. So take a linear homeomorphism which is order preserving. Note both of them contain 0 as the least element so automatically $f(0) = 0$. Order preserving will automatically implies that $f(0) = 0$

(ii) So let f from C_{12} to B_2 be the order preserving linear homeomorphism again, both of them are Cantor sets there is such a homeomorphism. (So they are all disjoint, so I am totally independent in defining these homeomorphisms, so f on this one, f on this one and so on.)

(iii) Then f from L to L be the identity map.

(iv) Lastly let f from B to B_1, again the order preserving affine linear map. Earlier, I had covered the B_2 part. Now I am covering B_1 part.

So, since I have covered the entire of C because they are all disjoint subsets, and they are going to disjoint subsets, the domain and image are covered completely. So all this partial homeomorphism patch up to define one single f which is is a homeomorphism from C to C.

Since $f(0)$ is 0, if you throw away 0, then f restricted to X is a homeomorphism from X to X. Now look at (iii). f is identity on L. And (iv) says B is mapped to B_1, so these two are important. It follows that this f takes K_i to K_i for $i = s_1$ onward, because where are they are inside L and f is identity on L. What about these K_i's, $i = 1, \ldots, s$?

For them K_i' are again inside L and so no problem, only the (ii) part you have to check. But B is going inside B_1, which is again inside each K_i. So this K_i's going inside K_i. I am not claiming that f restricted to K_i are homeomorphisms onto K_i. Just that f takes K_i into K_i for each i.

So f is here. So that was the first thing we want to show.

Now finally I have to see that f is not in $\langle C_0, C_0\rangle$, since f sends $C_{1,2}$ which is inside B_2 to all the way inside $[2/3, 1]$. Remember this B was a subset of $[2/3, 1]$, C_{12} is somewhere away. What is C_0? Remember C_0 is the complement of K. Let me just show you what C_0 is to begin with. C_0 is a short notation for complement of K in X. and my K itself is $[2/3, 1] \cap C$. So it has gone out of the complement so it is not going inside C_0 at all. Therefore, this f does not take C_0 inside C_0.

So the proof is over, namely, the group of homeomorphisms of X, the punctured Cantor set is not a topological group.

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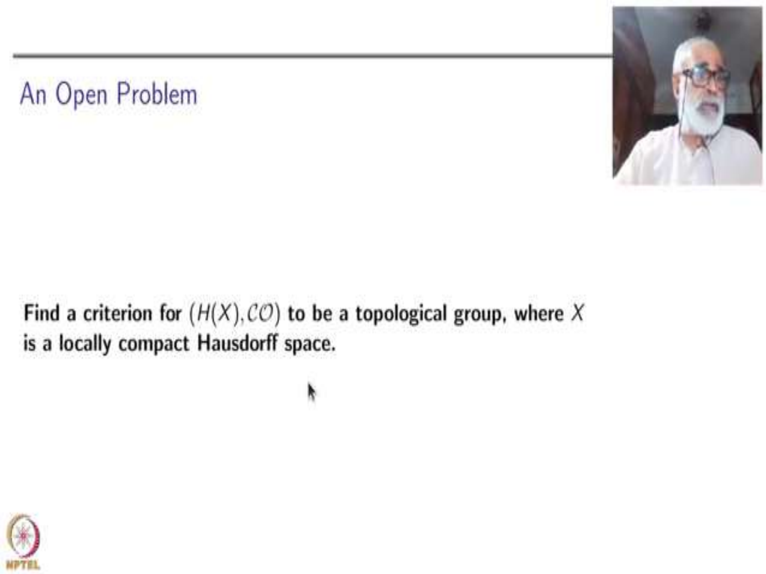

I thank Shameek Paul, Allen Hatcher, and Dennis Sullivan for some useful conversations. Special thanks to Parameswaran Sankaran, who painfully went through initial versions of this write-up and gave me some warnings.

So there are a couple of questions that automatically arise in this context, namely, find a criterion for $H(X)$ under compact open topology to be a topological group where you try to put some condition other than X to be locally compact Hausdorff space. The criterion means what? if and only if statement.

Equivalently you can take one point compatification, try to solve the subspace problem, if the subspace topology on $H(X)$ coming from $H(X^*)$, whether that is the same as compact open topology. Or you can try to solve this problem in the case of sine curve and so on Interesting cases like that so there are quite a few problems left here.

I told you that these things were done very recently about 3 years back, so during a workshop in characteristic classes in which Shameek Paul attended it and we were sharing a room for some time and that time I was discussing this with him. Then I am thankful for Allen Hatcher also. I asked him whether he knows anything about this. He says he does not know but maybe he will have a look at what I did. Then I asked Dennis Sullivan and so on.

The most important of all, my initial attempts were discussed with Parameswaran, so he said look here you have to be careful. And he painfully went through the initial versions with so many typos and so many miscalculations there. But he saw through the whole thing and made a remark where I should be careful and so on, some warnings.

So I should thank all of them. Of course I thank you for listening to this one also it is a good opportunity of presenting this one, so thank you. We will meet next time.