**An Introduction to Point - Set - Topology (Part II)**
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**Lecture No. 51**
**Motivation and Definition**

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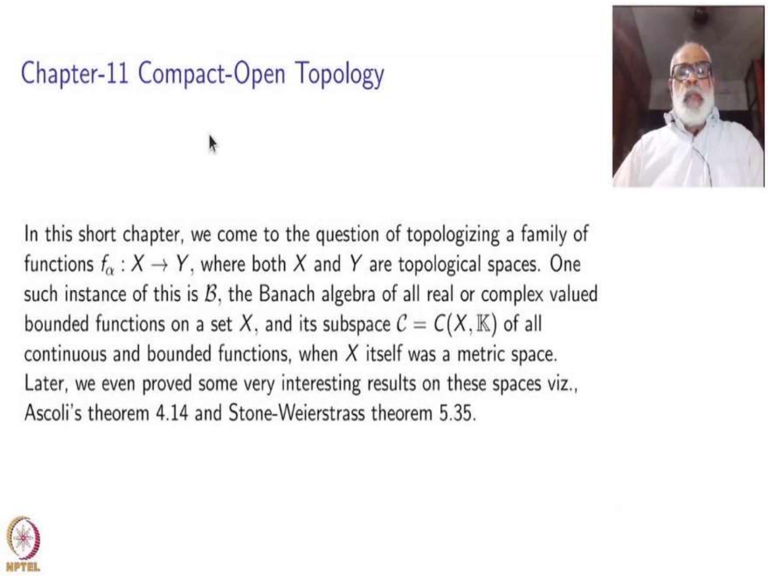


Welcome to NPTEL-NOC an introductory course on Point Set Topology Part 2. So, today we shall begin chapter 11 a short chapter on Compact Open Topology. Here we will discuss the question of topologizing a family of functions from one set to another set but with some specific topologies on both sides.

For example, we have discussed the Banach space, the Banach algebra of all real or complex functions from one set to another set which are bounded and then one of its subspace, where $X$ is some topological space, you take just continuous functions and bounded functions.

Later on, we will study some specific properties of the spaces such as Stone-Weierstrass theorem, Ascoli's theorem and so on. So, function spaces have a lot of importance in mathematics after all.

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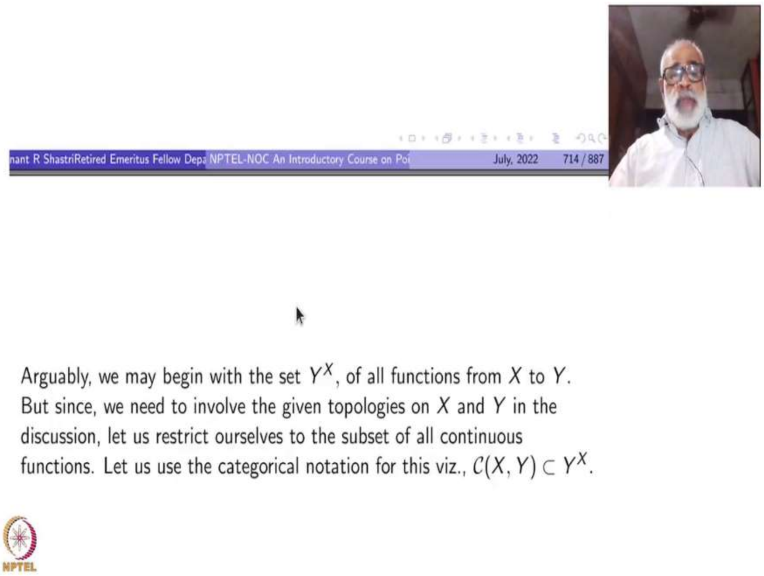


Arguably, we may just begin with the set $Y^X$. This notation is just for the set of all functions from $X$ to $Y$. It also is equivalent to another description, namely, product of copies of $Y$ as many as there are elements of $X$. You can think of each copy of $Y$ indexed by $\alpha$ where $\alpha$'s runs over $X$ and then take the product. So, that is the same thing as the set of all theoretic functions from $X$ to $Y$. Usually it is taken with the product topology. But, that will involve only the topology on $Y$. Product topology has nothing to do with the indexing set $X$, but we want to bring the topology of $X$ also in the picture. So, let us restrict ourselves to subsets of all continuous functions from $X$ to $Y$. So, that is clearly a subset of set of all functions from $X$ to $Y$, $Y^X$.

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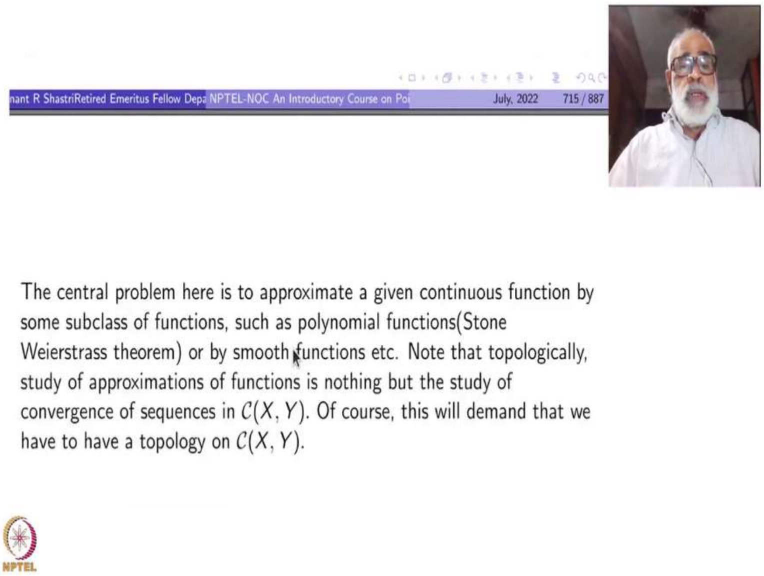

Now, the central problem here is to approximate a given continuous function with some special properties maybe or totally arbitrary continuous function by some subclass of functions such as polynomial functions. So, that is Stone-Weierstrass theorem or may say just Weierstrass theorem in the case of functions defined on the closed interval or something like that or it could be smooth functions or it could be embeddings and various things, or you may want to approximate by just what are called piecewise linear functions and so on.

So, what is the meaning of approximation? Approximations of functions is nothing but the study of convergence properties in the ambient space, namely, the space of all continuous function from $X$ to $Y$. Of course, this will demand that we have topologies on $X$ and $Y$, so that continuous functions make sense. Another way would be just take the product topology on $Y^X$ and restrict it to subspaces that you can take the product topology and restrict it.

Of course, that will not involve the topology of $X$ itself only the choice of subset you have got this one. What about other subsets, what are the open sets? There also we want to involve the topology of $X$ in some way or the other. That is what we want to concentrate upon.

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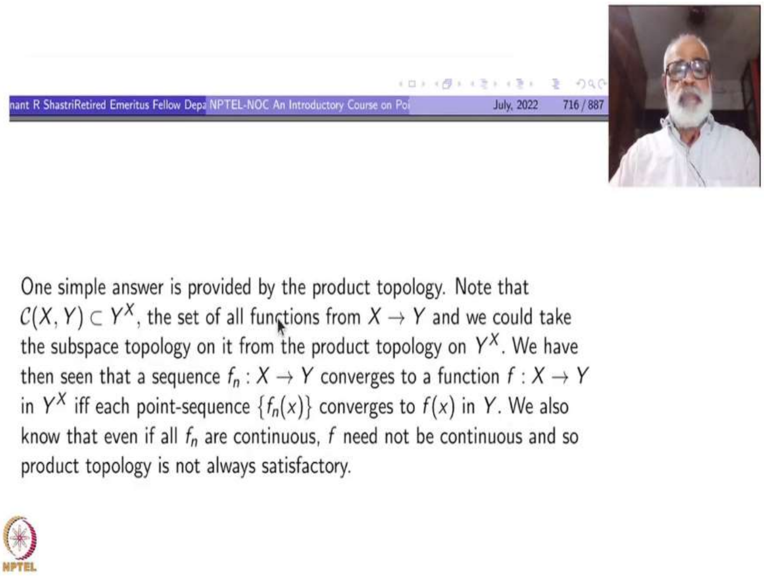


So, we need to examine even the simplest answer provided by taking the product topology on $Y^X$, because $\mathcal{C}(X,Y)$ is after all a subset of $Y^X$. The subspace topology has some decent properties but for example what happens if you have a sequence $f_n$ of continuous functions from $X$ to $Y$, which converges to a function from $X$ to $Y$ in the product topology? That just means that each coordinate sequence $\{f_n(x)\}$ converges to $f(x)$. This is known as pointwise convergence in analysis.

We have seen that point wise convergence of continuous functions need not imply that the limit is continuous. So, we will be going out of the subspace $\mathcal{C}(X,Y)$, which we may not accept. With this particular point of view, the product topology is not quite satisfactory.

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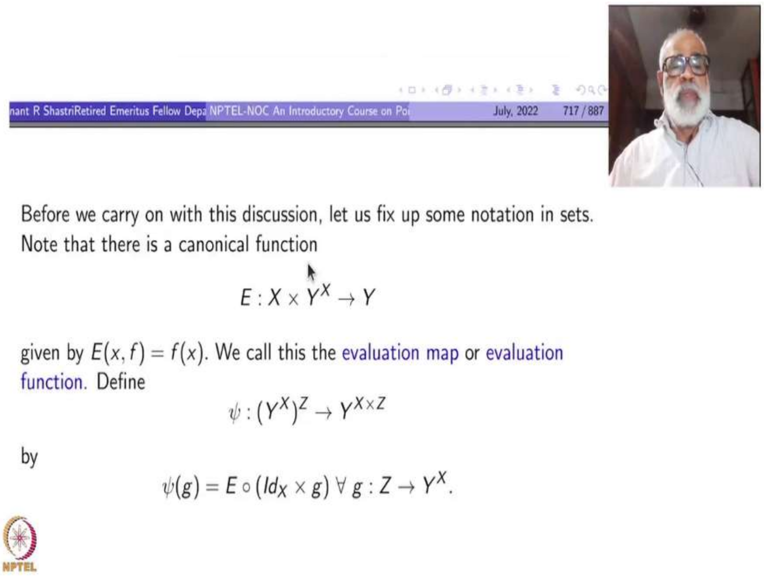


Before we carry on with this discussion, let us have some notation, so that we can discuss it more carefully. Note that there is a canonical function, which we will denote by $E$, $E$ representing the evaluation map from $X \times Y^X$ to $Y$; take a point $x$ here and take a function $f$ from $X$ to $Y$, you evaluate that function at this point, that is $E(x,f) = f(x)$.

Next, we can define a little more complicated function here. By the way, this is just the exponential law as far as sets are considered. I am just taking the set of all functions here, all functions from a set $Z$ to the set of all functions from $X$ to $Y$. So, this is $Y^X$, whole power $Z$.

So, I am going to define a function on this, namely, take a function $g$ from $Z$ to $Y^X$, take the identity map of $X$ and take the product of these two and then follow it by the evaluation map $E$ to get a function from $X \times Z$ to $Y$. So you get a map from $(Y^X)^Z$ to $Y^{X\times Z}$. That is denoted by $\psi$, then you can compose it with $E$. So, that composition we will denote by $\psi(g)$.

This map is called the exponential correspondence. It is easy to see that it is a bijection. Namely, whenever you have a function $f$ from $X \times Z$ to $Y$, you can restrict it to $X \times \{z\}$

they are called sections of $f$, and are denoted by $f_z$ from $X$ to $Y$, viz., $f_z(x) = f(x, z)$. So, for each $f_z$ is a function from $X$ to $Y$. Thus we get a function $\phi$ from $Y^{X \times Z}$ to $(Y^X)^Z$. You can easily check that $\phi$ is the inverse of $\psi$.

So, if you denote cardinality of a set $X$ by $x$, then this bijection says that $(y^x)^z$ is equal to $y^{xz}$.

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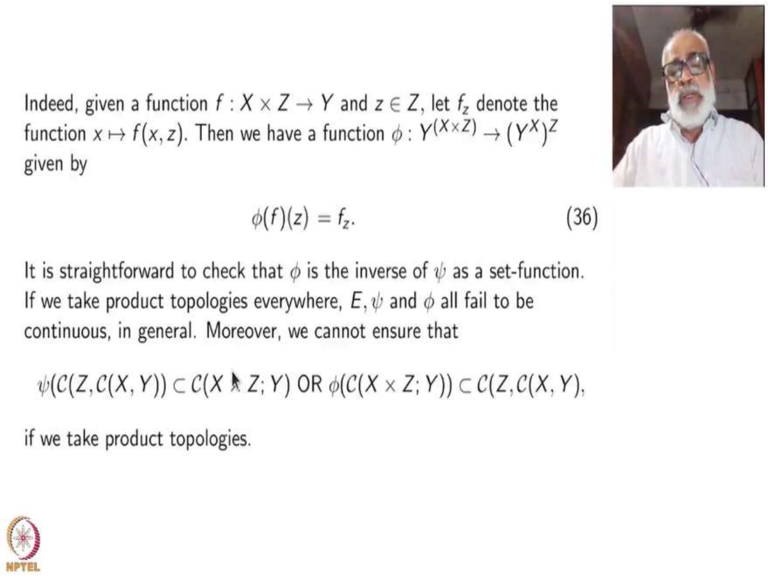


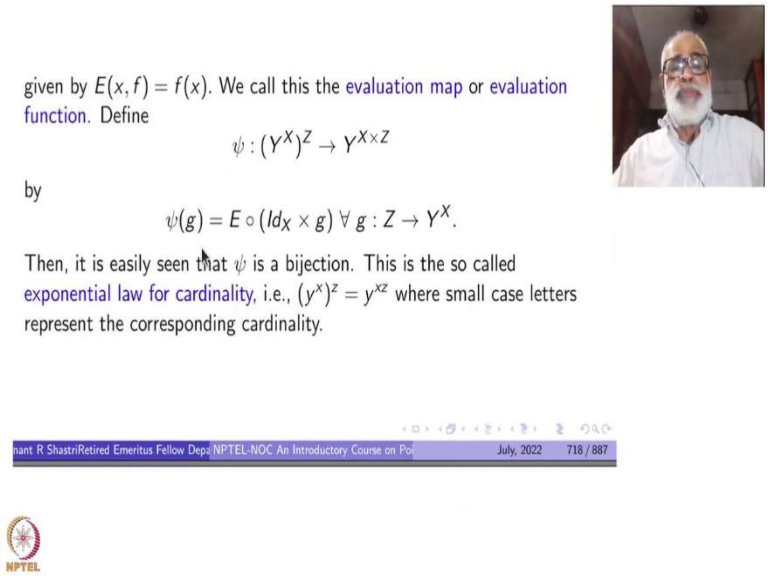


Now, when we take product topology everywhere, viz., starting with some topological spaces $X, Y$ and $Z$, and take product topology everywhere, on $X \times Z$, on $Y^{(}X \times Z)$, on $(Y^X)^Z$ etc., we can ask whether $\psi$ and $\phi$ preserve continuous functions.

They may not, one way or the other way. We cannot ensure that. So, even for this reason, product topology is not satisfactory.

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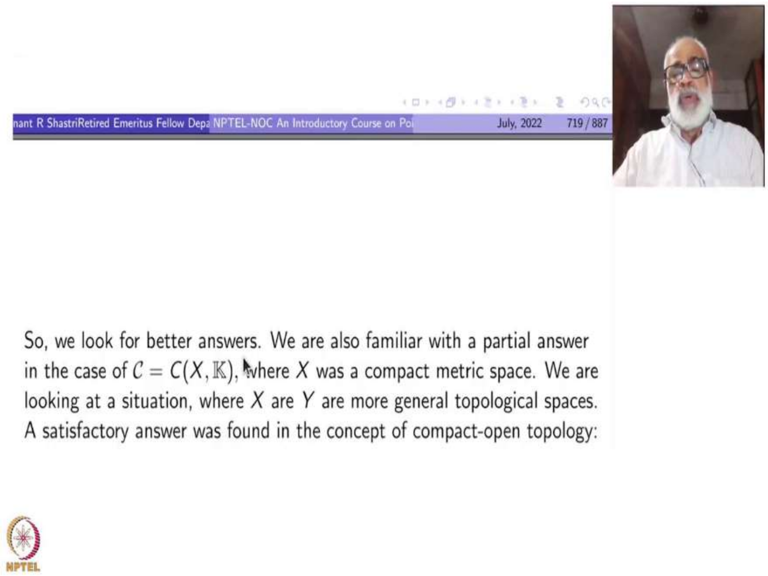


So, we have look for better answers. We are also familiar with a partial answer in the case when you have to deal with real or complex valued functions on a topological space $X$. Usually $X$ itself was also a compact metric space and then automatically all continuous functions are bounded. This way they formed a Banach algebra on which we have studied many other results also.

Now, we do not want to restrict $X$ to be a metric space. That is all let alone a compact one. Namely, take any topological space $X$ there is one concept which will imitate the sup norm topology, i.e., which is the same as the topology of uniform convergence. This will imitate, to a large extent, a very good imitation. And that is called the compact-open- topology.

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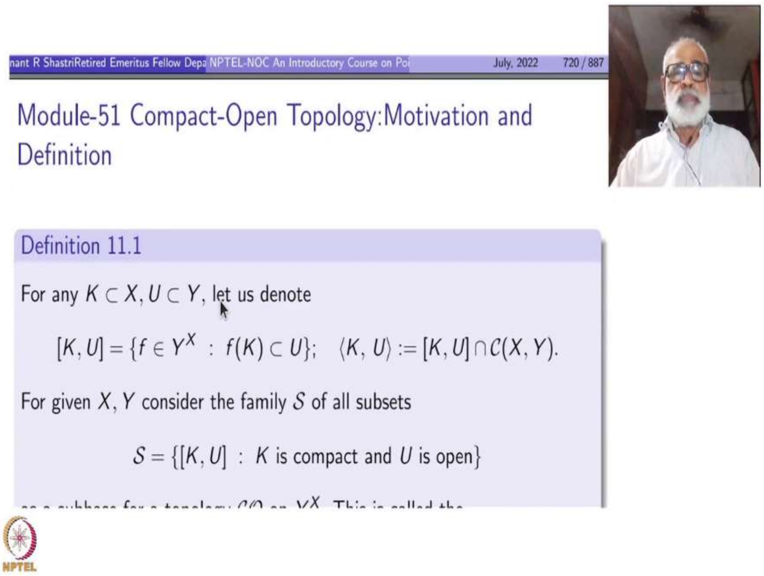


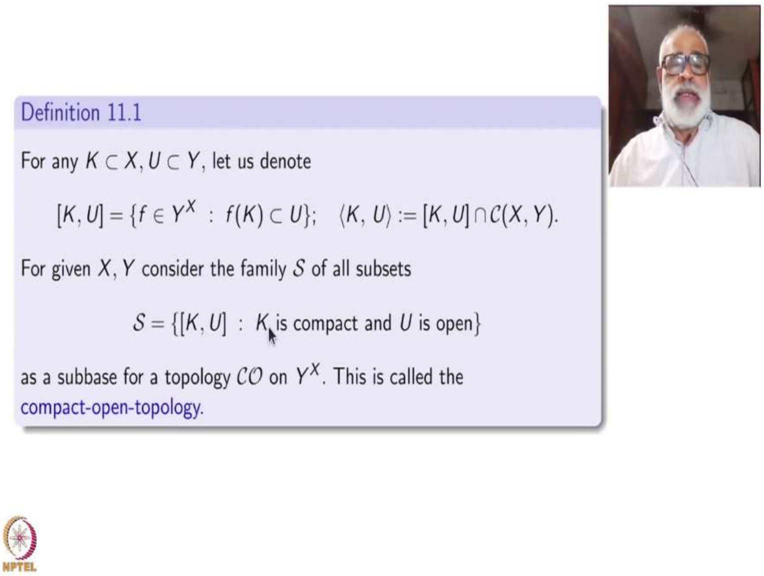


So, that much for motivation for today's topic. Let us start the study Module 51. The earnest study of compact-open topology. There are many other topologies on function spaces. The study of various topologies itself is a very interesting and thriving business in function analysis. Here are some more notation here.

For any subset $K$ of $X$ and $U$ of $Y$, let us denote this square bracket $[K, U]$ to be the set of all functions from $X$ to $Y$ such that $f(K)$ is contained inside $U$.

So, this is where we are imposing some restrictions on functions we take. The function should take our given $K$ inside $U$ to be inside this subset.

Another notation, namely these angled brackets (looking somewhat like inner product, but nothing to do with that concept). So, I will call this also $\langle K, U\rangle$ is all those $[K, U] \cap \mathcal{C}(X, Y)$. Condition is the same, but only take continuous functions that is it.

Now, given $X$ and $Y$ consider family $\mathcal{S}$ to be the set of all subsets $[K, U]$, where $K$ ranges over all compact subsets of $X$ and $U$ ranges over all open subsets of $Y$. Now, both $X$ and $Y$ are topological spaces, of course. $\mathcal{S}$ equal to all $[K, U]$, $K$ is compact and $U$ is open. So, that is suggestive. That is why I put $K$ here in a subset of $X$ and subsets of $Y$ are taken $U$. $K$ is compact. $U$ is open. The symbol represent the same things only now qualification is that $K$ must be compact and $U$ must be open.

Take this collection $\mathcal{S}$. Declare it as a subbase for a topology on $Y^X$. Let us denote it by $\mathcal{CO}$, and call it the compact-open-topology. Remember this square brackets denote the set of all functions such that... So, $\mathcal{CO}$ will be perhaps different from the product topology, let us see. Finally, we take the subspace topology on the subset $\mathcal{C}(X, Y)$ and denote it also by $\mathcal{CO}$.

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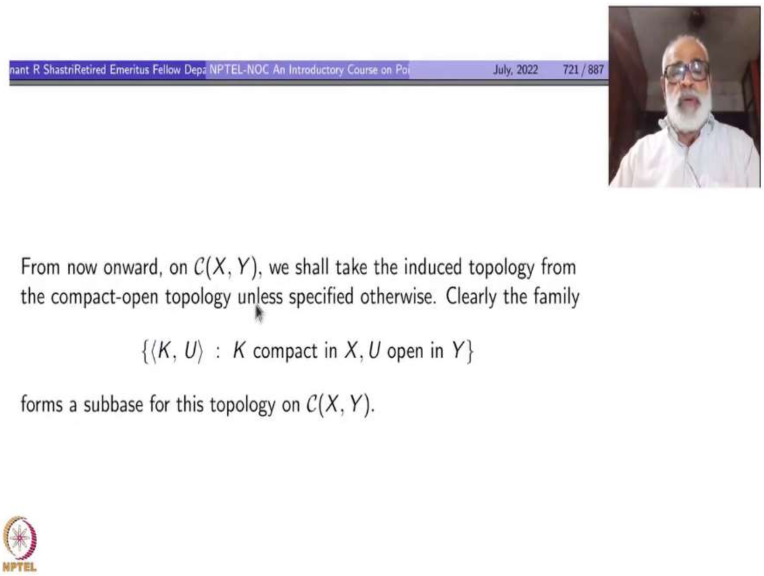


From now onward,  we should take the induced topology on $\mathcal{C}(X, Y)$ from the compact open topology unless specified otherwise. There are so many different topologies on $\mathcal{C}(X, Y)$ but right now we are going to take compact-open-topology only.

Clearly, the family of subsets $\langle K, U\rangle$ as $K$ ranges over all compact sets of $X$ and $U$ ranges over all open subsets of $Y$ forms a subbase for $\mathcal{CO}$ on $\mathcal{C}(X, Y)$. Family of subsets got by intersecting with a give subset the members of a subbase for the larger space will form a subbase for the subspace topology on the subset. This is a general fact.

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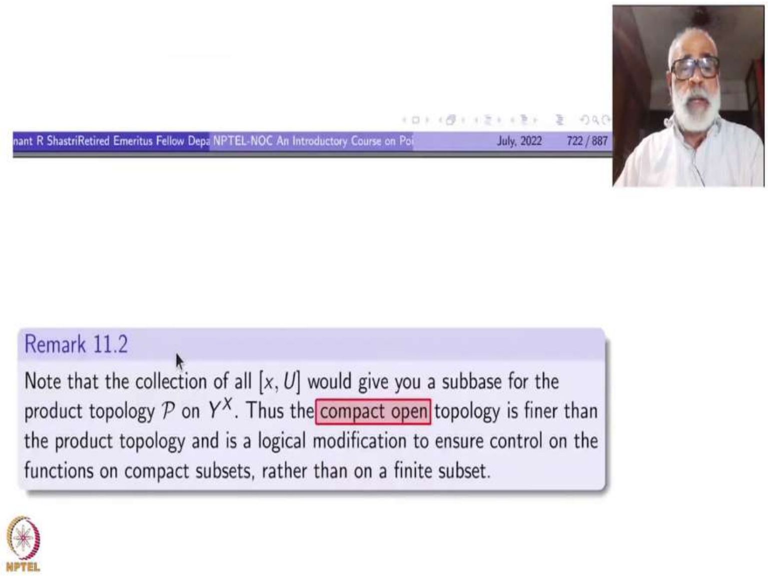


Note that the collection of all $[x, U]$, (now I am specifying $x$ instead of $K$ in the first slot, justified because singletons are compact) forms the standard subbase for the product topology on $Y^X$. What is $[x, U]$? All functions whose $x^{th}$ coordinate is inside $U$ which is the same as $\pi_{x^{-1}}(U)$ where $\pi_x$ from $Y^X$ to $Y$ is the $x^{th}$ coordinate projection.

Therefore, you see that at least the definition this compact-open-topology is some kind of a generalization of the product topology. Clearly, $\mathcal{CO}$ is finer than the product topology, because subbase for $\mathcal{CO}$ contains the standard subbase for the product topology. Therefore every open set in the product topology is also open in $\mathcal{CO}$.

Therefore, convergence become more stringent. For example, immediately you can see that convergence with respect to this compact open topology immediately implies convergence with respect to the product topology, which is nothing but pointwise convergence.

So, what is the meaning of this one? $[x, U]$? control only at points, whereas $[K, U]$ are controlling over compact subsets. Functions have been controlled over the compact subsets. It is control over the functions not over the compact sets.

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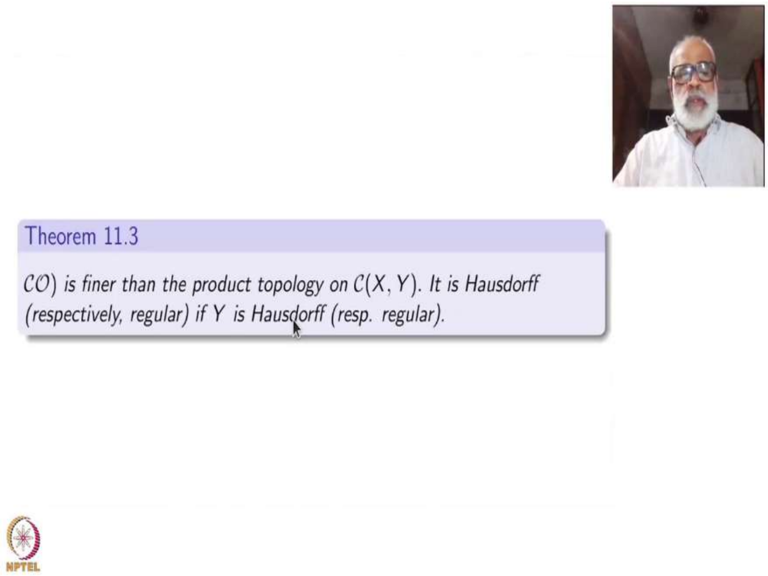


So, here is a theorem. The $\mathcal{CO}$ this topology is finer than the product topology on $\mathcal{C}(X, Y)$. It is Hausdorff (respectively regular) if $Y$ is Hausdorff (respectively regular). So, these two are more or less consequences of the first observation which we have already seen.

The topology is $\mathcal{CO}$ is finer than the product topology because it contains the subbase itself contains all the subbase for the product topology. As soon as it is finer than a Hausdorff topology, it will be also Hausdorff. If $Y$ is Hausdorff, the product topology on $Y^X$ is Hausdorff. That is old game for us. Therefore, it is Hausdorff. Regularity is not that quick.

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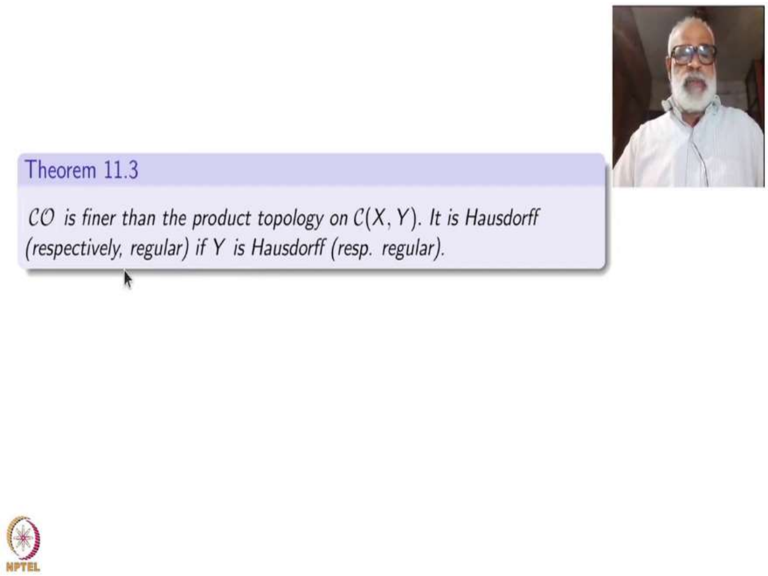


So, let us see why regularity is also true. So, assume $Y$ is regular. It suffices to consider $f$ belonging to $\langle K, U\rangle$. (These are subbasic open sets. Instead of doing it for all open sets, you can just do it for some subbasic open sets and then you can take finite intersections then that will give you the proof for all open sets.)

Now, $f$ belongs to $\langle K, U\rangle$ implies $f(K)$ is contained inside $U$. Since $K$ is compact $f(K)$ is compact. So, if you have a compact subset of an open set inside a regular space what happens? This is all an old game for us, we know that.

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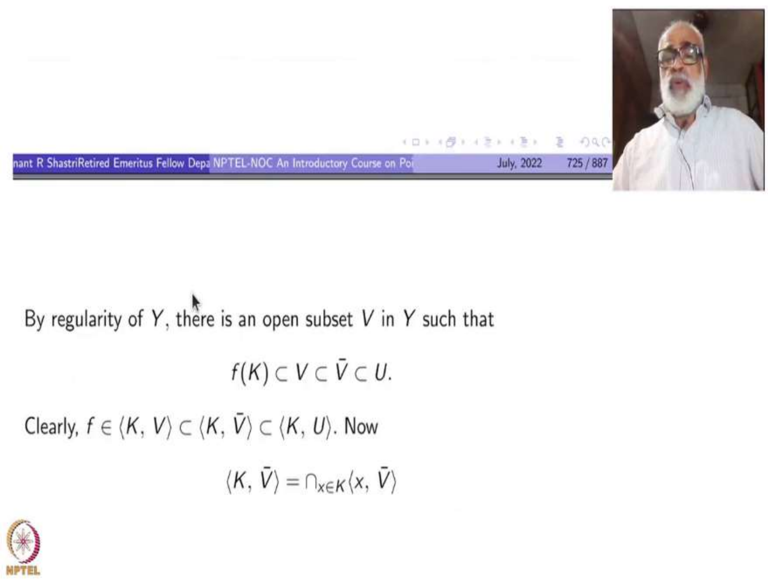

By regularity there is an open subset $V$ of $Y$ such that $f(K)$ is contained inside $V$ contained inside $\overline{V}$ contained inside $U$. Clearly $f$ belongs to this $\langle K, V\rangle$ now. But $\langle K, V\rangle$ is contained in $\overline{\langle K, V\rangle}, \overline{V}$ being larger than $V$ and that is contained in $\langle K, V\rangle$.

Now $\overline{\langle K, V\rangle}$ is nothing but intersection of all $\overline{\langle x, V\rangle}$ where $x$ ranges over $K$.

Remember $\overline{\langle x, V\rangle}$ means all (continuous) functions which take the point $x$ inside $\overline{V}$. If all the points given a function if all the points of $K$ are inside $\overline{V}$, $f(K)$ is inside $\overline{V}$ and conversely. So, this is the intersection of all these $\overline{\langle x, V\rangle}$. Now, what are $\overline{\langle x, V\rangle}$? They are $\overline{\pi_x^{-1}(V)} \cap \mathcal{C}(X, Y)$ and hence are closed subsets of $\mathcal{C}(X, Y)$, in the product topology and therefore in $\mathcal{CO}$ also.

So, $\overline{\langle K, V\rangle}$ is a closed subset of $\mathcal{C}(X, Y)$. These is the closed subset. So, what we have found for each $f$ inside this open set contained in its closure contained in the given open set. So, that is the regularity for the $\mathcal{CO}$ topology. Over.

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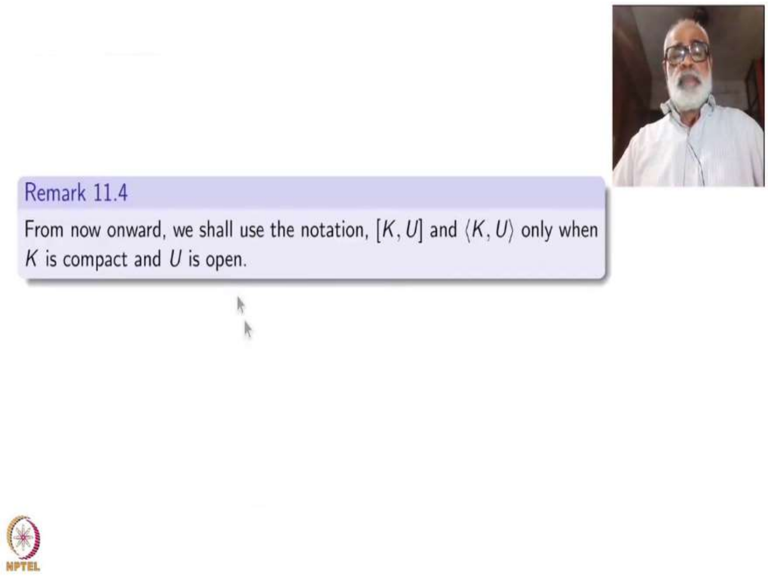


From now onwards, we shall use the notation $\langle K, V\rangle$ or the ordinary square bracket $[K, V]$ only when $K$ is compact and $U$ is open. This is just a lazy way of saying but sometimes I may forget it that is why. We do not write this kind of notation unless $K$ is compact and $U$ is open.

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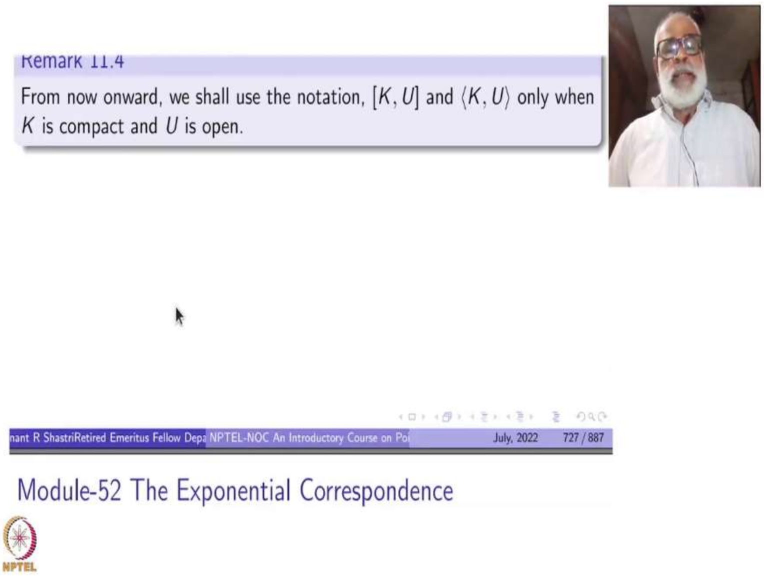


So, let us take a break here. Next time we shall study carefully the exponential correspondence which we have introduced today for only sets. Now we will study them for continuous functions with $\mathcal{CO}$ topology. Thank you.