An Introduction to Point-Set-Topology (Part II)
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Lecture 48
Ordinals

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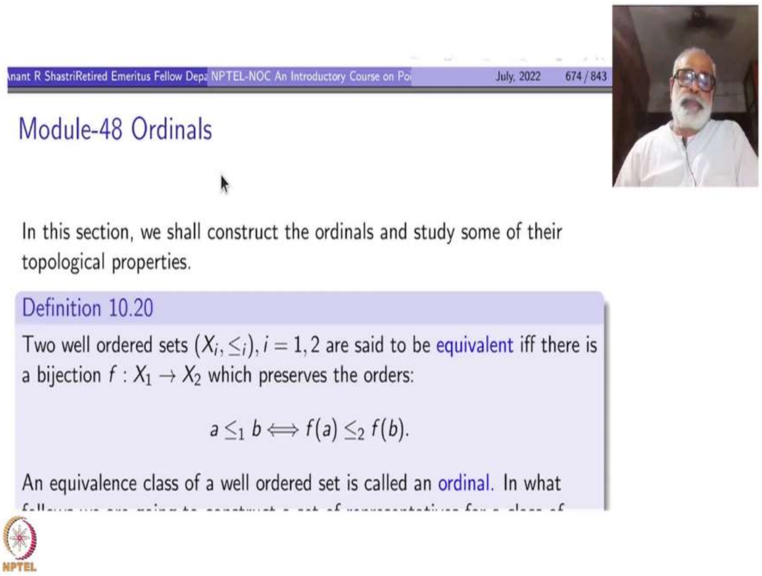

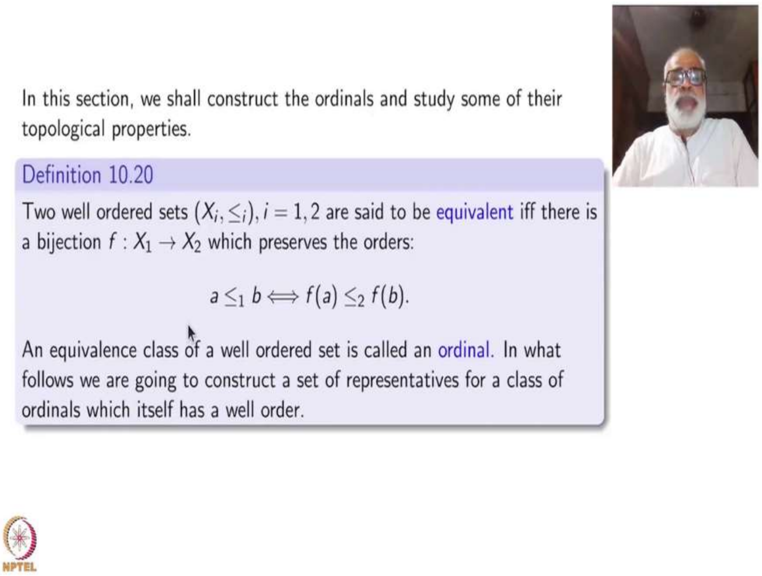

Hello welcome to NPTEL-NOC, an introductory course on Points at topology part 2. So, as I told you yesterday, today we shall construct the ordinals. in this section we shall construct the ordinals and study some of its topological properties.

First of all we have to make a definition. Two well-ordered sets $(X_i, \leq_i)$, for $i = 1$ and 2, are said to be equivalent if there is a bijection from X_1 to X_2, a set theoretic bijective function which preserves the orders. What is the meanings of that?

a less than or equal to b in X_1, first relation, should imply $f(a)$ is less than equal to $f(b)$, in the second relation. There must be an order preserving bijection f from X_1 to X_2.

Obviously, the inverse of f will we also ordered preserving. An equivalence class of well-ordered sets is called an ordinal. You see, you take the collection of all well-ordered sets that is not a set. However, we can define a binary relation on on it as above. Nobody stops you and then you can verify that it is reflexive, symmetric and transitive.

So, you can look at equivalence classes. This word class is very important here. Here neither the entire collection of wel ordered sets on the individual equivalence classes are sets. The equivalence classes are cared ordinals. So, in what follows you are going to construct a set of representatives for a class of ordinals, not all ordinals but a large class of ordinals.

So, we will construct that which itself is a set with a well-order: one set of representatives. So, this is the whole idea, now that will be a sufficiently large class of ordinals for our purpose.

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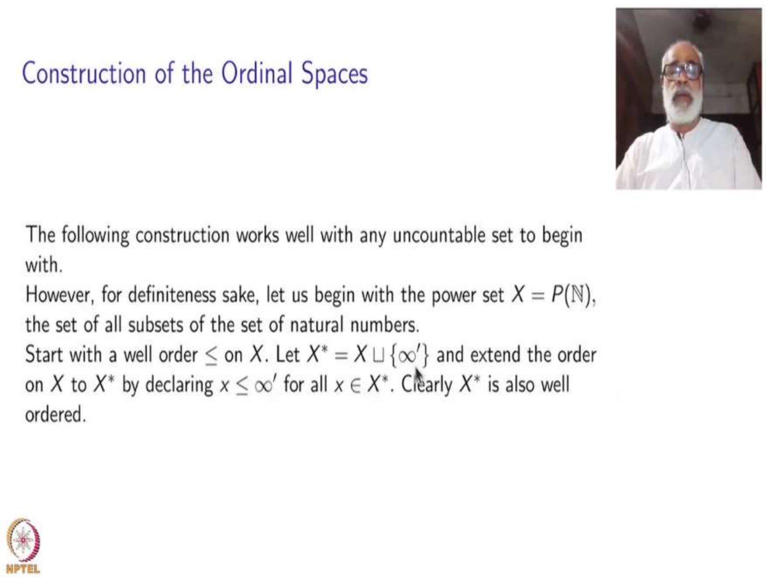

The construction of this Ordinal Space: of course, when you say when there is a well-order on a set, then we can take the topology also and then we can say use the word ordinal space.

The following construction works well with any uncountable set. Now, you must know that uncountable sets exist. So, that part is elementary set theory or whatever. You know, set theory of cardinals, I am assuming that you know it. Or I will pretend that you know it.

So, I am not going to teach you the set theory of cardinals here. So, be sure of that. So, indeed, this is not the correct thing to do in the logical sequence of teaching perhaps. But since you are familiar with natural numbers in whatever quote unquote way you know, you can take the power set of the natural numbers and call it X, that will do our job of understanding what are ordinals to begin with.

Strictly speaking, we should not assume that we know the natural numbers. They will be also constructed out of what we are going to do. So, all that you have to know is that somehow that there is an uncountable set. Now, so, fix one uncountable set X, start with a well order on it. The theorem that we have proved, every set can be a well ordered.

Start with a well order on X and then ad one extra point.

So, once again I denote it by X^* equal to $X \sqcup \{\infty'\}$ and extend the order on X to X^* by declaring that all the elements of X are less than, strictly less than infinity prime. That will be automatically a well order in which X will be an initial segment. In fact, we have used this process earlier, extending a well order above as well below also.

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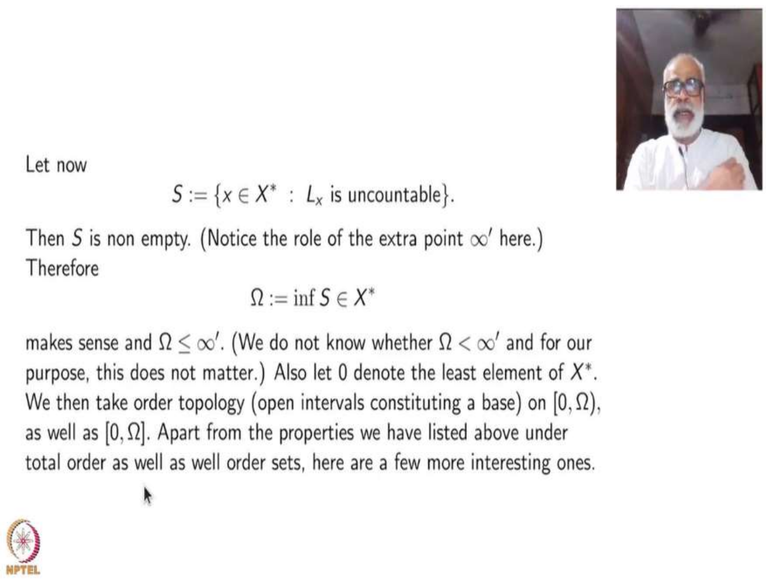

Now, let us put S equal to all points of X^* such that L_x is uncountable. Remember what is L_x? It is the left open ray: take S to be all those x in X^* for which L_x is uncountable. Then S is non empty. How do you prove that S is non-empty? Well, you can take x to be infinity prime here. Then what is L_x? L_x is precisely X and we have started with the assumption that X is uncountable.

So, therefore this S contains infinity prime and hence non-empty. That is the role of infinity prime here. If I do not put this one, I will be hard put to prove why this S is not empty. In fact, it may not be true also. That is the trick. infinity prime is there just for this purpose. One extra point and you have got a non-empty set you.

Once it is non-empty, you can take infimum of S belonging to X^* because X^* is well ordered, take the least element of S and call it Ω. So, this makes sense because S is not empty. That is all.

So, this Ω will be automatically a member of X^*. So, it is less than equal to infinity prime, we do not know whether this Ω is less than infinity prime or for that matter is equal to infinity prime.

So, this depends upon the set X itself, but we are not bothered about it. We started with any uncountable set. Therefore, we do not know much about it. If you take natural numbers and power set then maybe you can say that this has to be actually infinity that is a different aspect. So, we do not bother. This is an element of this X^* that is all we know.

Also, now let 0 denote the least element of X^*. This time I am bold enough to use the symbol 0 itself. Perhaps, I better use $\hat{0}$ or $0'$ as before. But that is too much of work. Whenever it is needed later on, I will do that. Right now let us just have this symbol 0, for the least element of X^*. The least element exists anyway because this is a well ordered set.

We then take the order topology (open intervals and half open intervals constituting a base) on both subsets $[0, \Omega)$ and $[0, \Omega]$ of X^*. We have seen earlier that the restricted order makes them a well ordered set.

So, you can take the order topology on both of them. Clearly this Ω will be in the closure of $[0, \Omega)$. That much is clear. Apart from the properties we have listed above under total order, (except for one property which is for a well ordered set), now, we will have more interesting properties of this spaces $[0, \Omega)$ and $[0, \Omega]$.

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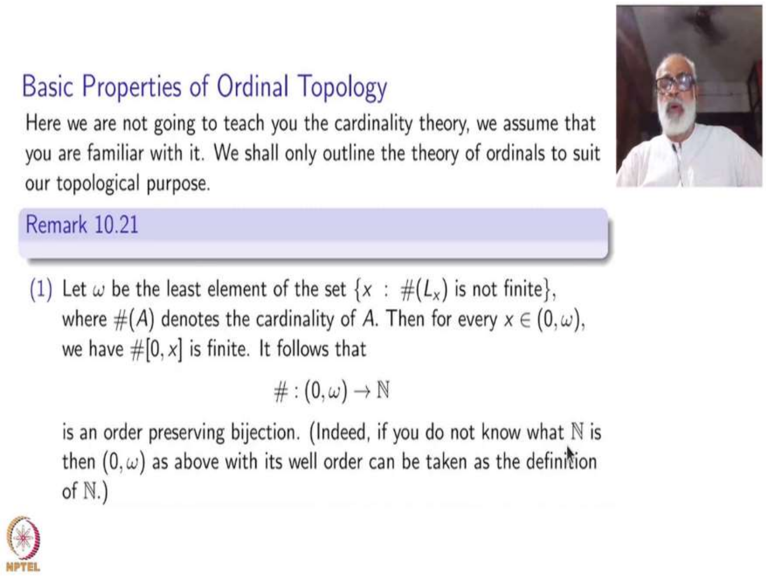

So, basic properties of this ordinal topology. (I told you I am not interested in teaching you cardinality theory, I hope you know it. We assume that you are familiar with this.)

So, start with ω (little omega) equal to the least element of the set of all $x \in X^*$ such that, the cardinality of this left ray L_x is not finite.

For example, if I take $x = 0$, L_x is empty, if x is the immediate successor of 0, then L_x will have only one element 0 and so on. (So, that is where actually the Piano construction of natural numbers would start, if you have do it by yourself. Otherwise, you may pretend that you know that stuff). So, you take all x such that the cardinality of L_x is not finite and take the least element of that set. Because the least element is also an element of this set, it follows that L_ω is infinite.

So, here this $\#(A)$ denotes the cardinality of A.

Now for every $x \in (0, \omega)$, (that means what? 0 is less than x and is less than ω) the cardinality of $\overline{L_x}$ is also finite. L_x is finite, we add one more point to it, its cardinality is also finite. It follows that if you take this check as a function from $(0, \omega)$, this interval to the natural numbers, (namely cardinality is now a finite, a finite number, a natural number, not equal to 0) it will be an order preserving bijection. See the domain, whatever we have started with is an ordered set and the codomain we pretend we know it, is the set of natural numbers with the usual order. Both of them are well-ordered sets also. This check is an order preserving bijection. So, in our definition, they belong to the same equivalence class and represent an ordinal and that ordinal will be denoted ω itself. That is the whole idea here. Indeed, if you do not know what $\mathbb{N}$ is, then $[0,\omega)$ as above, you can think of that as the definition of the set of natural numbers. And then construct the algebra out of that by using the successor theory. I am not going to do that here.

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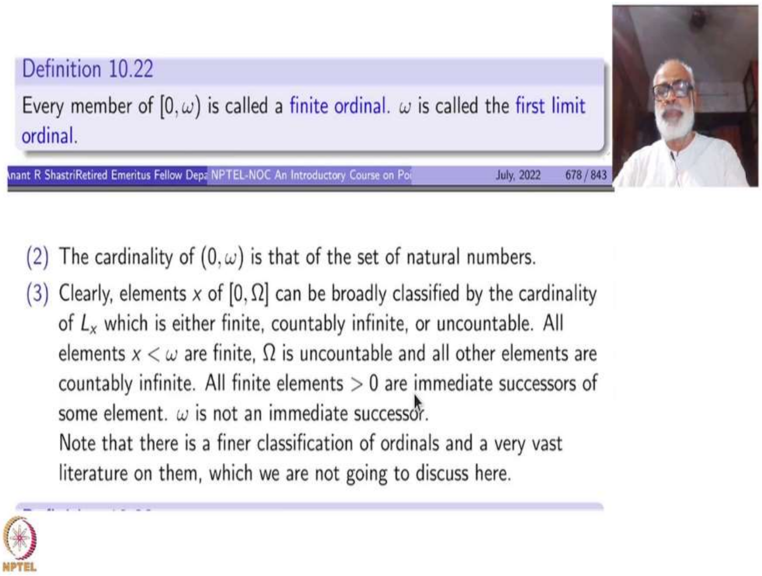

So, here are a few terms I want to recall: Every member of $[0,\omega)$ is called a finite ordinal. ω is called the first limit ordinal. There are other limit ordinals, obviously, there are many more. This is some name you can say there are many elements which are not finite. Among them omega is the least one. That is why you call it first limit ordinal. Now, cardinality of $[0,\omega)$ is the same as that of the set of natural numbers because there is a bijection like that. Clearly elements x of $[0,\omega]$ can be broadly classified by the cardinality of L_x. How many elements are there before x? look at the cardinality of that. So, it may be finite, it is countably infinite when $x=\omega$. Beyond that, we cannot be sure that the cardinality classifies the ordinal. I do not want to go on further classifying these what kind of countability is there etc. that is purely for logic, which we are not interested in it right now. We are only interested in some broad classification of members of X^*.

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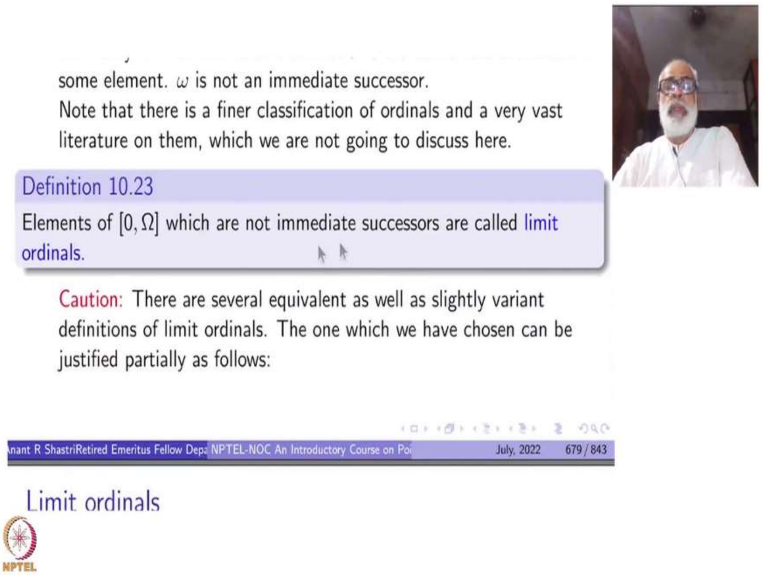

So, elements of $[0, \Omega]$ which are not immediate successors are called limit ordinals. I am just repeating this one here. We only defined first limit ordinal but all of them which are not immediate successor are called limit ordinals. Of course every element in $[0, \Omega]$ has an immediate successor, whereas not all elements are susscessors. So ω is not an immediate successor, $\omega + 1$ is an immediate successor, $(\omega + 1) + 1$ etc are all immediate successors.

So, again there will be another one which is not an immediate successor and then again after that only immediate successor will be there and so on. This is a wonderful space never ending! That is why we have put this Ω here. So, that is the end this is the maximum element amongst all of them.

There are several equivalent as well as slightly variant definitions of limit ordinals, different names are also, one of which we have chosen. This name can be justified partially as follows: why limit ordinal? Just ordinary name. So why so I am going to tell you that.

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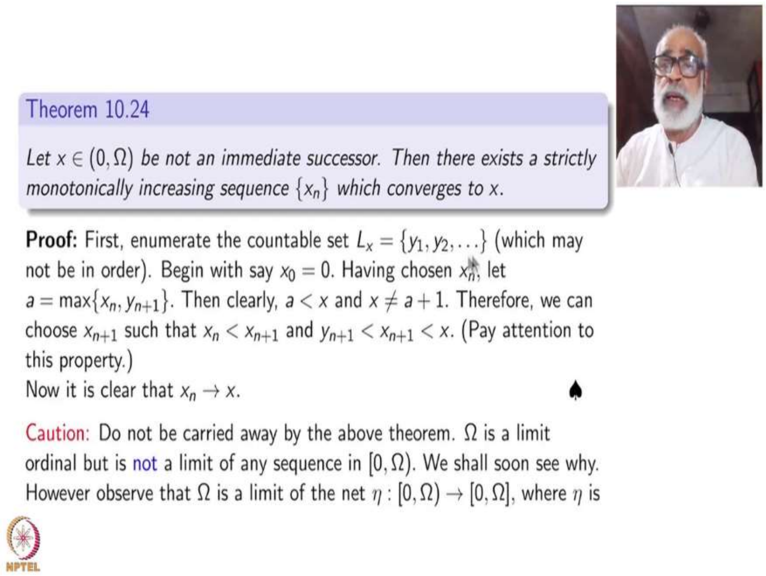

Take any element in the open interval $(0, \Omega)$. Suppose it is not a successor. (0 is not a successor anyway, we do not know what Ω is. So, we have omitted them.) Then there exists a strictly monotonically increasing sequence $\{x_n\}$, (strictly monotonically increasing, very important) which converges to x.

Since every element which is not an immediate successor has this property, viz., it is a limit in the above sense we are calling it a limit ordinal, that is the justification for the name. Soon we will see that that is not a very good justification either.)

Proof: First enumerate the countable set tell $L_x = \{y_1, y_2, \ldots\}$, L_x is countable set means it can be put in one-to-one correspondence with set of natural numbers. However, these $y_1 y_2, \ldots$ are not in the order in which L_x is ordered. L_x comes with a well order.

That does not matter just take an enumeration to start with.

Now, I am constructing the sequence $\{x_n\}$ as required. Start at x_0 equal to 0 or any element in L_x, no problem. Having chosen x_n, I want to define x_{n+1} inductively. So, how do I do that? Once x_n is chosen, look at the maximum of $\{x_n, y_{n+1}\}$, call it a. Clearly this is still less than x because both x_n and y_{n+1} are less than x.

Since x is not equal to $a + 1$, because x is not an immediate successor, so, $a + 1$ is also is in L_x. So we can choose $x_{n+1} = a + 1$. Then it follows that x_n is less than x_{n+1} and y_{n+1} is less than x_{n+1}.

So, $\{x_n\}$ is strictly monotonically increasing. Given any $s < x$, say $s = y_k$ for some k. Then it follows that for all $n > k,\ s = y_k < x_n$. That just means x_n converges to x.

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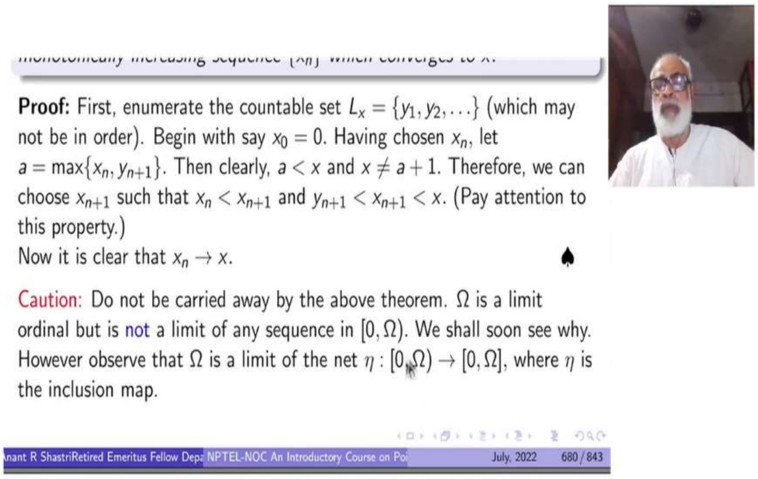

So, here is a caution: do not be carried away by the above theorem. Ω is a limit ordinal but it is not a limit of any sequence in $[0, \Omega)$. Ω is not a successor yet there is no sequence in $[0, \Omega)$. (Of course, if include Ω also then you can construct a sequence converging to Ω.)

So, we shall soon see why this is true. Observe that Ω is a limit of the net viz., you think of $[0, \Omega)$ as the domain of a net because it is a directed set. this is totally ordered set anyway. So, $[0, \Omega)$ as the domain of a net, what is the net? Just inclusion map. Take this net inside $[0, \Omega]$ closed.

Then this Ω will be obviously a limit point. So, in this one single paragraph I have both justified as well as cautioned you. So I have given you another justification for the name limit ordinal. This time by taking, not a sequence but a net.

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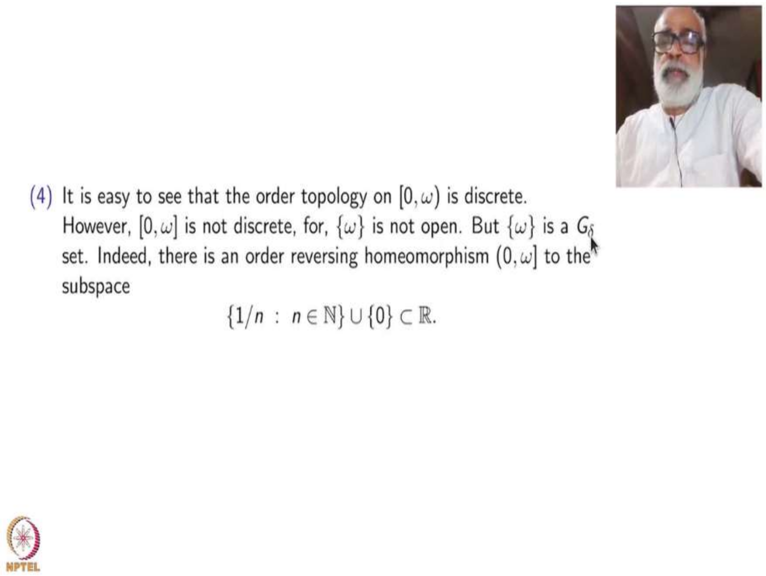

It is easy to see that the order topology on $[0, \omega)$ is discrete, just like natural numbers, for which we have already seen that. So, ordered topology, that is just discrete. However, if you include ω, it is not discrete. Something nice can be seen here, viz., singleton ω, $\{\omega\}$ is not open $[0, \omega]$. Discrete means all singletons are open.

So, to see singleton ω is not open, suppose it were. Then we should have two elements x, y such that $(x, y) \in [0, \Omega) \cap [0, \omega]$ should be $\{\omega\}$. If $x + 1 = y$ then (x, y) is empty. Therefore $x + 1 < y$ and belongs to this intersection whereas $x + 1$ is not equal to ω.

But $\{\omega\}$ is a G_δ set. So, it is countable intersection of open sets. What are they? start with any sequence $\{x_n\}$ as above which converges to ω and look at the intersection of all $(x_n, \omega + 1)$.

Indeed, there is another way of looking at is $[0, \omega)$. There is an order reversing homeomorphism from $[0, \omega)$ to the space of all $1/n$, n ranging over natural numbers along with 0. So, send each x to $1/n$ where is cardinality of L_x, then this omega itself send it to 0.

So, that will be a continuous function that will be a bijection. it is order reversing homeomorphism as you can take this also has an ordered sets, an ordered topology, then we will have this, 0 is a limit point. So, that is the way you have to think of this one. This ω is a limit point of $[0, \omega)$.

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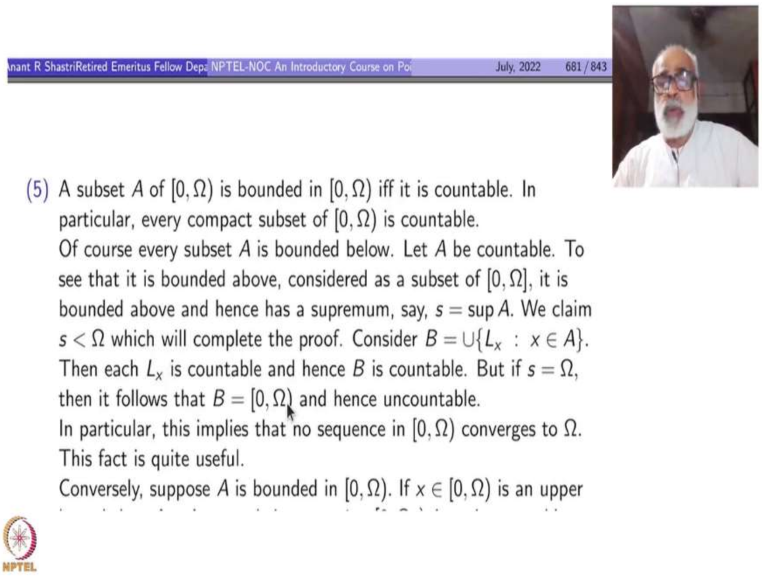

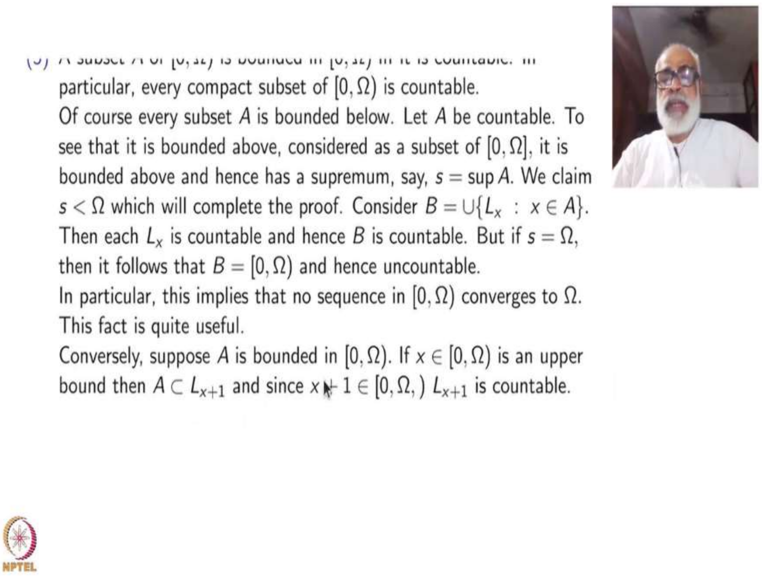

A subset A of $[0,\Omega)$ is bounded in $[0,\Omega)$ (bounded in $[0,\Omega)$ open is very important here because everything is bounded in $[0,\Omega]$ closed, i.e., if you include Ω) if and only if it is countable. Note that all subsets are bounded below obviously by 0. So, we are only interested in bounded means bounded above everything is bounded below in a well-ordered set.

So, if and only if it is countable, is the statement. Only countable subsets and all countable subsets are bounded inside $[0,\Omega)$ itself. So, this is the crucial thing I said. Wait a minute. Slowly we are building it up.

So, why so, in particular, if this is the case then it happens at every compact subset of $[0,\Omega]$ is countable. Because compact subsets are bounded, that we have seen last time.

So, bounded subsets are countable is what we have to prove of course, every subset A is bounded below. So, we are only interested in whether it is bounded above or not.

So, let A be a countable subset of $[0, \Omega)$. To see that it is bounded above, considered as subset of $[0, \Omega)$ what we do: take the smallest bound namely least element among all bounds of A in $[0, \Omega]$, and show that it is inside $[0, \Omega)$. That is what we plan.

You know that every subset of $[0, \Omega)$ is bounded above by Ω itself. Hence has a supremum A makes sense, Put $s = \sup A$. We claim that this s is less than Ω, s is not equal to Ω. That is all. So, how do we do that? Look at the set B which is the union of all the left rays L_x, where x ranges over A. So, A is some scattered countable set.

So, for each point x in A, you fill up all the things missing which are below x. So, it is like saturating the set A. everything below them comes inside, so, that is B. So, this is a very, very big set now. So, for all x inside A, you take L_x and take the union. Each L_x is countable by the very definition of Ω. But A itself is countable and a countable union of countable sets is countable. Therefore, B is countable.

But now it follows that s is equal to $\sup B$. Therefore, if $s = \Omega$, then it follows that B must be the entire of $[0, \Omega)$. But $[0, \Omega)$ is uncountable. (We started with a set X which is uncountable then construct this Ω, to be the least element such that $[0, \Omega)$ is uncountable.)

In particular, this implies that no sequence in $[0, \Omega)$ converges to Ω. See, because the supremum s of the set of images of sequence will be inside $[0, \Omega)$. A sequence is what? Countable set first of all right? There supremum s will be less than Ω and then $(s, \Omega]$ is an open neighbourhood of Ω which does not contain any member of the sequence. So, the sequence cannot converge to this Ω. So, this fact is useful.

So, this is an extra thing I am telling already. I have proved that if A is countable, then it is bounded, bounded inside $[0, \Omega)$. Now, let us prove the converse. Suppose A is bounded in $[0, \Omega)$. If x is in $[0, \Omega)$, is an upper bound for A, that means this A is contained inside L_{x+1}, because every thing is less than or equal to x. Since $x + 1$ must be inside $[0, \Omega)$. But L_{x+1} is a countable set. So, A is countable. Over.

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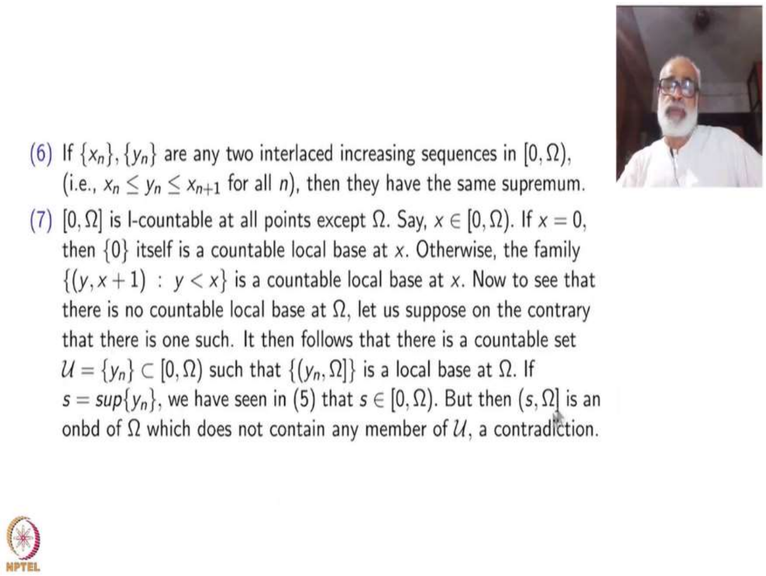

So, now we come to a little more serious business here. If $\{x_n\}$ and $\{y_n\}$ are any two interlaced, increasing sequences in $[0, \Omega)$, (what is the meaning of interlaced? You take x_1 that will be less than or equal to y_1, x_2 will be bigger than y_1 and then x_2 will be less than y_2 and so on. Interlacing can occur in slightly different ways also. If you change the labels there, you will get the same thing that is all. So, I can define like this also). Then they have the same supremum. See increasing sequences in $[0, \Omega)$, they are countable. So, they are bounded. This much we have seen already. So the supremum exist. Conclusion here is that they are the same. So, this is similar to what happens inside real numbers. So, I will leave it to you as an exercise.

$[0, \Omega)$ is I-countable. Now, we are talking about topology now. $[0, \Omega]$ is I-countable at all points except Ω. So, how do we show that? Say x belongs to $[0, \Omega)$? If x is 0, then singleton $\{0\}$ itself is open, therefore, that itself is a base countable base at 0. Over. Otherwise, look at the open interval $(y, x+1)$, where y is less than x. Look at all these where x is fixed and y is varying. They are neighbourhood so x. They form a countable family, because there are only countably many points less than x. It is also a local base at x. Every neighbour of x must contain an open interval around it. First of all that interval must contain $x+1$. It may contain $((x+1)+1)+1$ etc, $x+1$ has to be there and the lower limit of the interval has to be some $y < x$.

Now, to see that there is no countable local base at Ω : I said accepted at that point right? So, why there is no countable local base at Ω. Let us suppose there is one such. Then it follows that there is a countable set $\mathcal{U}$ which I may enumerate as $\{y_n\}$, such that the family $\{(y_n, \Omega]\}$ is a local base at Ω. Half closed intervals, because there is nothing bigger than Ω.

Now, you take $s = \sup\{y_n\}$, because a countable set is bounded in $[0, \Omega)$, s exists and belongs to $[0, \Omega)$. If you look at the neighbourhood $(s, \Omega]$ of Ω, no element of the family $\{(y_n, \Omega]\}$ is contained in it, because, s is bigger than all y_n.

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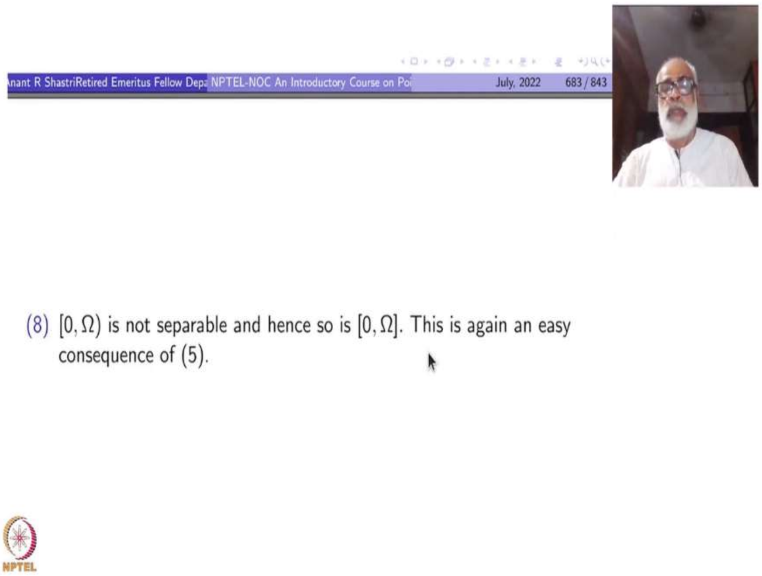

Next $[0, \Omega)$ as well as $[0, \Omega]$ are not separable. We have seen the I-countability. We will now see that separability is not possible. Enough to prove that $[0, \Omega)$ is not separable, because separability is hereditary. So, we shall prove this one is not separable automatically it follows that $[0, \Omega]$ is not separable.

So, why this is true? Take any countable subset A. What happens, you will have its supremum s. If you take the supremum plus 1 that will still be inside $[0, \Omega)$. But then the non empty open interval $(s + 1, \Omega)$ will not contain any element of A. Therefore, A cannot be dense.

So, this is also an easy consequence of (5). viz., every countable set is bounded. Well, let us stop here. Next time we will see some more serious topology of this ordinals. Thank you.