An Introduction to Point - Set - Topology (Part II)
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Lecture No. 46
Principle of Transfinite Induction

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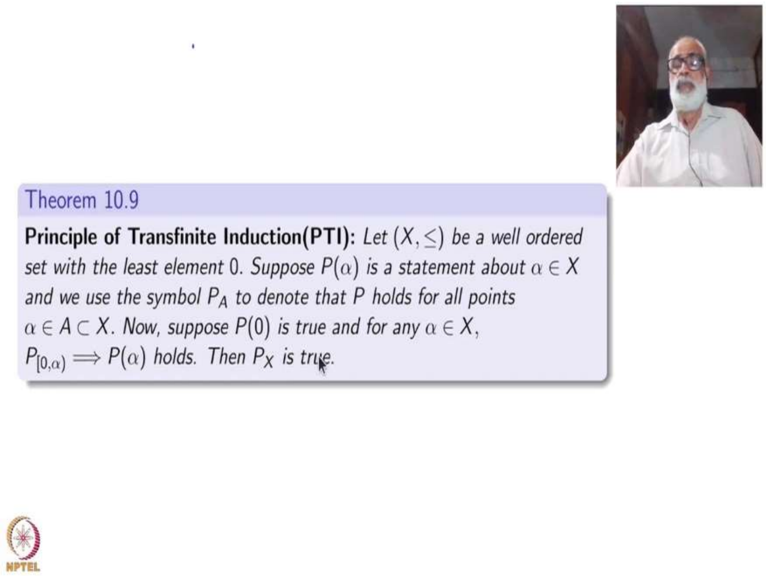

Hello. Welcome to NPTEL NOC an introductory course on Point-Set-Topology, Part II. Today, we shall begin with our study of Principle of Transfinite Induction. As an application, we will also employ this one and give a proof of Tychonoff's theorem. Of course, we have proved Tychonoff's theorem elsewhere. So, that is a good way of learning a new tool test it on something which you already know, where it works.

So, the principle of transfinite induction states the following. Start with a well ordered set with the least element, we will denote it by 0. This is for convenience. Suppose $P(\alpha)$ is a statement about α inside X, and we use the symbol P_A to denote the statement that $P(\alpha)$ holds for all α inside A, where A is a subset of X. Now, suppose $P(0)$ is true and for any α inside X, $P_{[0,\alpha)}$ is true implies $P(\alpha)$ is true. ($[0,\alpha)$ is the initial segment in X consisting of all β which are less than α inside X. Remember this means that $P(\beta)$ is true for all β less than α, that is the meaning of $P_{[0,\alpha)}$. If this is the case, then the principle of transfinite induction concludes that P_X is true, that is, for all elements alpha of X, the statement $P(\alpha)$ will be true.

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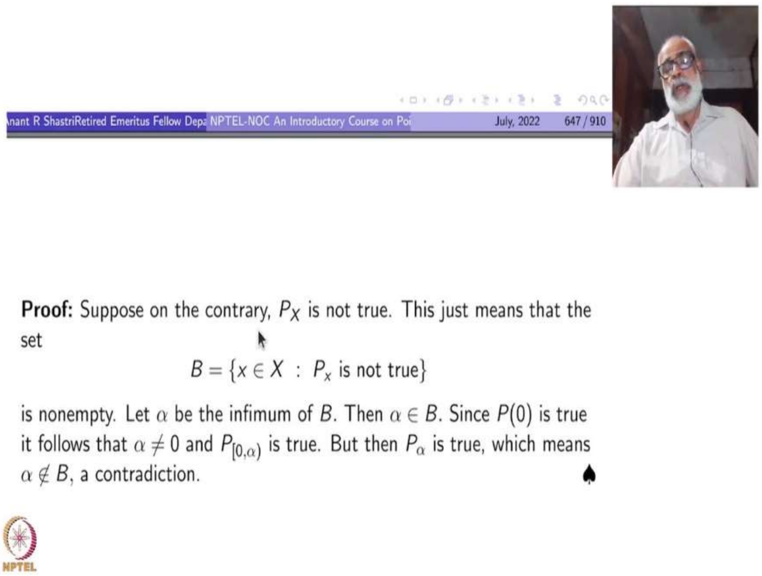

Proof:

Suppose on the contrary, P_x is not true for some $x \in X$. What does that mean? This just means that if you take B to be the set of elements inside X for which the statement is not true, that is a nonempty subset. This B is a nonempty subset. A nonempty subset in a well ordered set has a minimum α and α is inside this B. (That is why we can call it as minimum). Since $P(0)$ is true by hypothesis, that means this α, which is the minimum of B is not equal to 0. Any way 0 precedes α. And for all β which precedes α, $P(\beta)$ is true, otherwise β will be inside B a contradiction. Therefore $P_{[0,\alpha)}$ is true. By hypothesis, this implies $P(\alpha)$ is true which means α is not in B again a contradiction. So B must be empty.

So, that is a proof of principle of transfinite induction, which is, as you see, is an easy consequence of well ordering. So, that is the whole idea. So, we will see, well ordering always holds.

The point is, you are already very, very familiar with the principle of mathematical induction. That is a special case of this, when X is the natural numbers along with the usual order. Of course, that is a well order. So, this principle of transfinite induction is a far, far generalization of mathematical induction. This is true for any X.

As soon as you have a well order there, you can use it like this. So, our next aim is to illustrate the use of principle of transfinite induction with one single example but that example is going to be something very, very important, namely, product of arbitrary nonempty families of compact spaces is compact.

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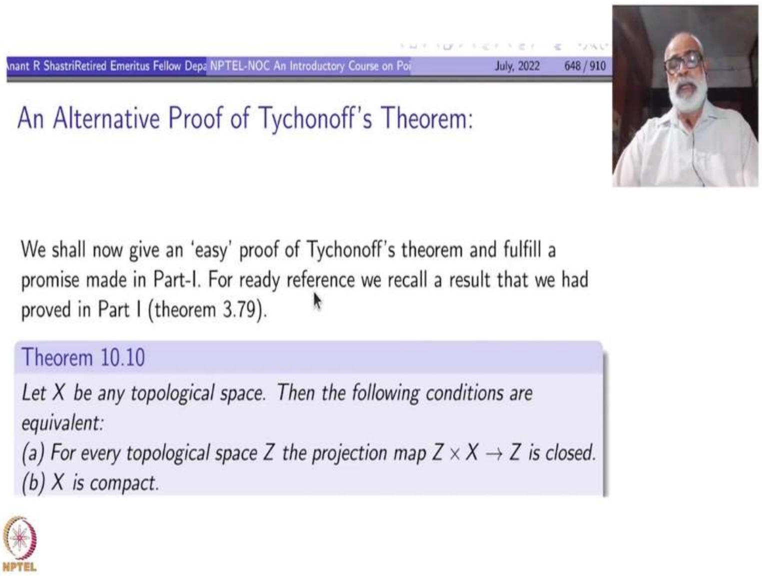

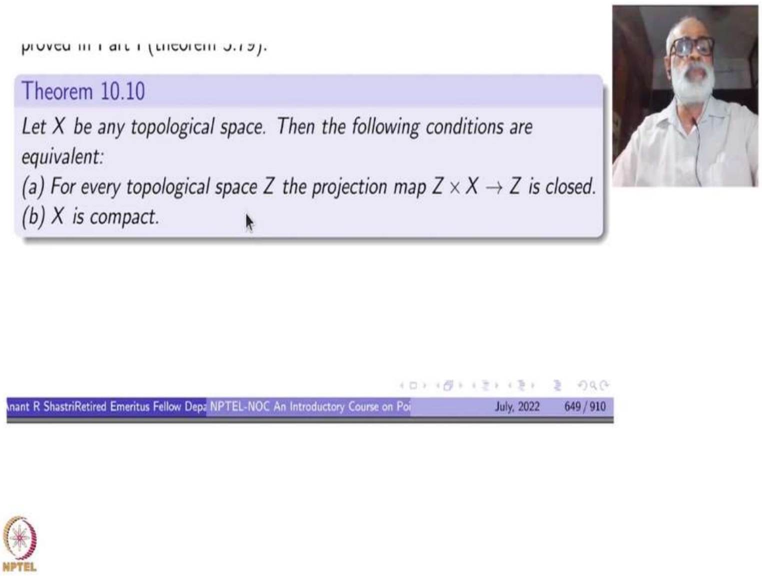

That is the Tychonoff's theorem. We have given several proofs of that. Now, we will give another proof of that one here. So, I am going to use one of the theorems that we proved in part I. I will not have time to do that again here, but I will recall it for ready reference. This is the theorem that we proved in part I.

Let X be any topological space, then the following two conditions are equivalent.

(a) The first condition says that for every topological space Z, the projection map $Z \times X$ to Z is a closed map.

(b) The second condition is X is compact. So, this theorem gives you a characterization of compact spaces. So, what we are going to do is employ this one both ways in proving that

Cartesian product of compact spaces is compact. Of course, you take a nonempty product of non empty spaces. That is important. (statements for products involving empty spaces and empty product etc can be settled easily.)

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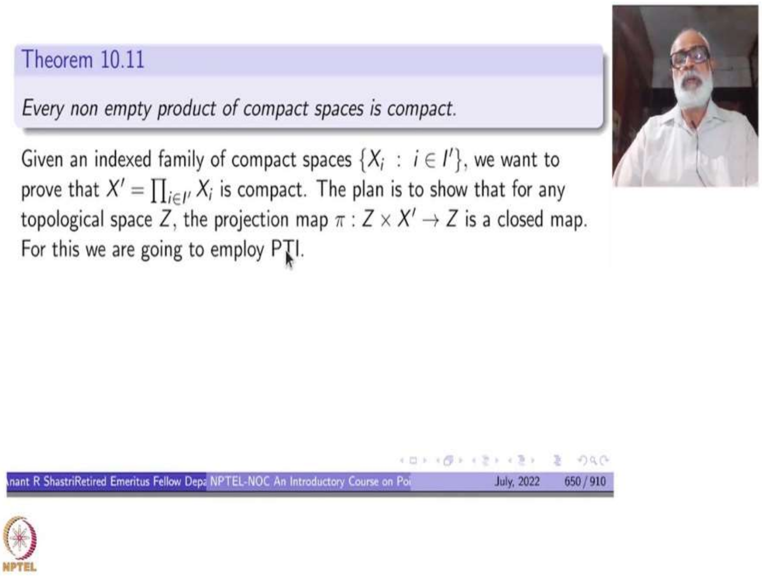

Every nonempty product of non empty compact spaces is compact. So, the proof seems to be just a small trick, by using the principle of transfinite induction correctly, which amounts to more or less selecting appropriate notation here. Otherwise, it is extremely easy proof. So, I will tell you the idea first and then tell you how it works in simple cases. Rest of them is just pure notations and so on.

So, given an indexed family $\{X_i : i \in I'\}$ of compact spaces, we want to prove that X' is the product of all X_i, $i \in I'$ is compact. The plan is to show that for any topological space Z, the projection map $Z \times X'$ to Z, (away from X' to the first factor), is a closed map, and we appeal to the previous theorem. To prove this itself, we will use the principle of transfinite induction.

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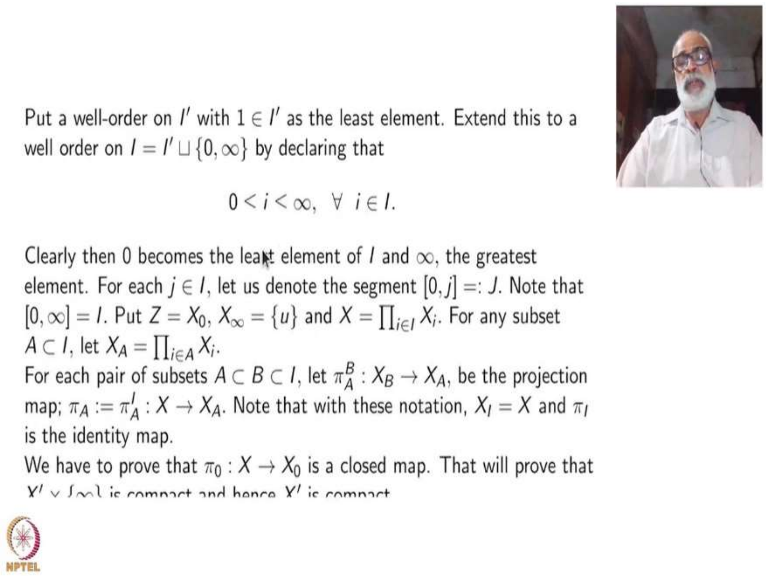

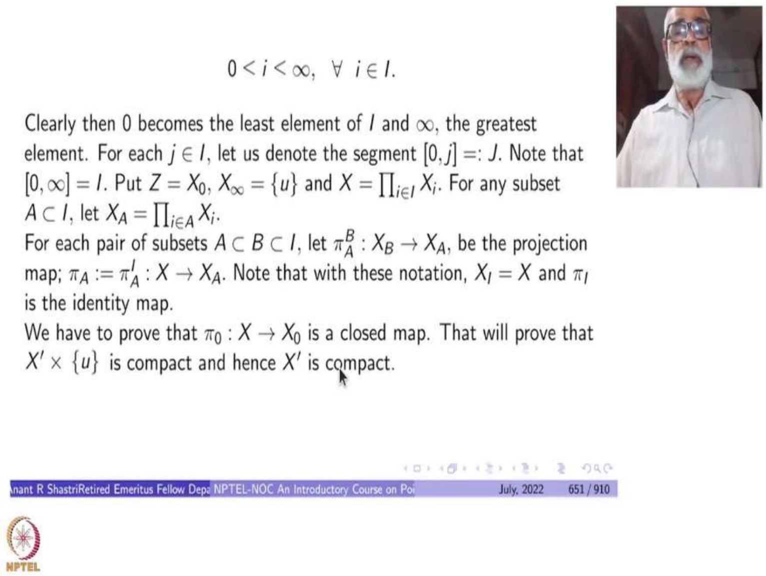

So, here is the first step, which will tell you what kind of tricks we have to do in order to employing principle of transfinite induction. Put a well order on the set, that is, important set, namely, the indexing set I'. Let us denote the least element there by 1. 1 belong to I' is the least element. Take any order on I, no problem. It must be well order, that is all. Extend this to a well order on I which is disjoint union of I' with two extra elements. I am denoting them by 0 and infinity. By declaring that, every element in I' is between 0 and infinity, we are extending the well order on I prime to a well order on I. In other words, now 0 is the least element, and infinity is the greatest element in I. To begin with 1 is the least element of I' but there may not be any greatest element in I'. So, I have put a greatest element also. So, this is the first trick, you can say.

Clearly 0 becomes the least element and infinity becomes the greatest element inside this I, which has two extra elements than I'.

Student: Sir, in the first line, we are using the well ordering principle, right?

Professor: For I'.

Student: And now, on I, you have define an order. It will, it may not remain a well order set.

Professor: It is well order set. Take a subset. If it contains 0, 0 is the least element, fine. If it does not contain 0, it will be inside I', except that it may contain infinity, that does not matter. Infinity is the largest, it cannot be smallest. You throw away infinity also, there will be a smallest element in that set.

Student: And because I' is well order.

Professor: Yeah. Well order, even in the proof of using Zorn's lemma, proof of that one, we have extended this one and so on. So, the extensions of ordering for one element, two element, three element, any finitely many elements is obvious. Wherever there is infinitely many elements to deal with we have to go back to Zorn's lemma, we have to go back to well ordering principle.

In other words, what your objection may be that, suppose I' is well order, and I' is contained inside a larger set, inside J'. Will there be a well order on J', which extends the well order on I'? Even that is true. So, but we have not proved any such thing. Whatever proof we have given for existence of well order, you can modify it to prove such a thing also.

So, that is not of any use for us, I am just, I am going to two elements, I prefer, I want it this way, not an arbitrary order. 0 is the least element and I is the, infinity is the largest element. In fact, we want to work inside I' only, but these two things help us in putting the correct notation instead of getting, instead of buggy notations, that is all. Now, for each $j \in I$, let us denote the segment 0, j which means all elements strictly between 0 and j including 0 and j. So, this, I will denoted by J. Note that 0, infinity is nothing but the whole space I.

So, that is not of any use for us, I am just, I am just adding two elements, I prefer, I want it this way, not an arbitrary order. 0 is the least element and infinity is the largest element. In fact, we want to work inside I' only, but these two things help us in putting the correct notation instead of getting, instead of buggy notations, that is all. Now, for each $j \in I$, let

$J := [0, j]$ denote the set of all elements between 0 and j including 0 and j. So, this, I will denoted by J. Note that $[0, \infty]$ is nothing but the whole space I.

Now, another small trick. I want to prove that for an arbitrary topological space Z, $Z \times X$ prime to X', where X' is a product, that is $Z \times X'$ to Z, sorry, is the projection map, that is a closed map. So, I want to change the notation here. Put Z equal to X_0 now. And for X_∞, choose another space, namely a singleton $\{u\}$, which is a harmless thing, it is again compact. This X_0 is may not be compact, and is not supposed to be compact anyway. Now, to take X to be product of all these $X_i,\ i \in I$.

Remember, if I only take the product over I', then that is X', that is what we want to prove is compact. Taking product with one more factor which is a singleton $\{u\}$ and proving that is compact is the same thing as proving the original thing X' is compact. What we are going to prove now is that $X' \times X_\infty$ is compact.

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So, let F be a closed subset of X and $x^0 \in \overline{\pi_0(F)}$. We have to show that $x^0 \in \pi_0(F)$.
For $j \in I$, let S_j denote the statement that there exists $x^J \in X_J$ such that

$$x^J \in \overline{\pi_J(F)}; \quad \& \quad \pi_{J'}(x^J) = x^{J'}, \ \forall \ j' \in [0, j]. \qquad (34)$$

Our aim is to prove that S_j is true for all $j \in [0, \infty)$. For then PTI would imply that S_∞ is true. This just means that there is $x^\infty \in X$ such that $x^\infty \in \bar{F} = F$, and $\pi_0(x^\infty) = x^0$ which will complete the proof.

Some more notation: For any non empty subset A of I, let X_A denote the product of X_i, where I ranges over A itself. For each pair of subsets A contained inside B contained inside I , (equality allowed),

let π^B_A denote the projection map from X_B to X_A. B is larger, which just means that drop out all the indices which are not in A, that is the projection map. One more simpler notation: let π^I_A be simply denoted by π_A. That is when $B = I$. So, that the projection from the whole space X to X_A. These are all projection maps. Note that with this notation, X_I is X and π_I is the identity map. Nothing is dropped out.

One more notation, π_0 is what? It is just projection of X onto X_0, the 0^{th} coordinate space which Z. We want to show that this is a closed map. Since $X_0 = Z$ is arbitrary, this will prove that $X' \times X_\infty$ is compact. That is the same thing as proving that X' is compact. So, this is what we want to prove.

Now, the next step is, start with a closed subset F of X. You have to prove that the image under π_0 is a closed set in Z. So, take a point in the closure and show that it is inside the original set $\pi_0(F)$. We want to show that a set is closed so we show that it is equal to its closure. Closed. So, I will take a point in the closure and show that it is inside. That is all.

Now suppose, you have just a single compact space X_1. Then you know that the projection map $Z \times X_1$ to X_0 is a closed map. More generally, suppose you have to deal with finitely many compact spaces $X_1, \dots, X_n$. You already know that the projection maps from the product space whee you drop out any number of indices $i = 1, 2, \dots, N$ are all closed maps.

So if you start with a closed subset F of $X_0 \times \cdots \times X_n$ and a point $x_0 \in Z$ which is in the closure of $\pi_0(F)$, you first know that there is $x_i \in X_i$ such that $(x_0, x_1) \in \overline{\pi_{0,1}(F)}$. Next you know that there is $x_2 \in X_2$ such that $(x_0, x_1, x_2) \in \overline{\pi_{0,1,2}(F)}$ and so on..., there is $x_n \in X_n$ such that $(x_0, x_1, \dots, x_n) \in \overline{\pi_{0,1,..,n}(F)}$ which is nothing but $\overline{F}$ which is F. But then $\pi_0(x_0, x_1, x_n) = x_0$ and hence x_0 is in $\pi_0(F)$ itself.

So, this would have been another proof in the case of when we have taken product over a finite family. We need not prove this case anyway.

So this is only to get an idea. However, the above simplified notation are not available for us when dealing with an arbitrary product and that is why we have introduced the above elaborate notation and then use transfinite induction. I hope this makes the idea involved in the in the proof, a bit more clear.

So, start with a point x^0 (use top suffix here because I would like to use the lower suffixes for the coordinates. This is some point in $\overline{\pi_0(F)}$, bar denoting the closure in X_0. So, for each $j \in I$, I will make a statement here S_j: S_j denotes the statement that there exists some x^J belonging to X_J such that this x^J is actually inside $\overline{\pi_J(F)}$. (You have started with F which is a closed subset of the product space X, project it to this X_J, and take its closure. So, this element must be in this closure.) Moreover, it must also satisfy that the J''s projection of this element agrees with $x^{J'}$ for all $j' \in [0, j]$. Note that your $x^{J'}$ belongs to $\overline{\pi_{J'}(F)}$. That is the inductive statement. One single statement S_j means all this.

Our aim is to get X^J such that p_0 of X^J is x^0. But while doing so, we keep track of earlier $X^{J'}$. Namely for all $j' \in [0, j]$, we must have already chosen $x^{J'}$ such that $\pi_{J'}(x^{J'}) = x_0$. We want X^J to satisfy the condition that its projection onto $X_{J'}$ coincides with $X_{J'}$. That may be termed as compatible way of lifting the element x_0 from X_0 to X_J's.

So, this is what I did in the illustration case when dealing with a finite product. First you choose x_0 and then (x_0, x_1) and then (x_0, x_1, x_2) etc. That kind of notation is not possible, in he general case. That is why we have chosen this elaborate notion and the statement S_j. This statement makes sense, even when j' is 0. There is no problem that x_0 has been already chosen. So, we know this statement is true for $j' = 0$. Otherwise, it is just a statement now.

Why am I making the statement? Examine the statement S_∞. What is the meaning of the statement S_∞? That means, there is a point x^∞ inside X, with the property that x^∞ is inside $\overline{\pi_{[0,\infty}(F)} = \overline{F} = F$. But $\overline{F}$ is F. So, x^∞ is inside F. And this one means that in particular, $\pi_0(x^\infty)$ is x^0.

So, what does that mean? That means, this point is in the closure, but it is actually inside F. So, we have shown that this x^0 is inside $\pi_0(F)$. So, that will complete the proof. So, just the idea of making the statement. This statement S_j for j equal to infinity is the one which we want, finally. What is the starting point? The starting point will be S_1 or for example, S_0 is true, obviously. Then just for getting the feeling what to do we shall prove S_1. Then we have to prove that S_j for all $j < k$ implies S_k is true. That step is the hardest step here.

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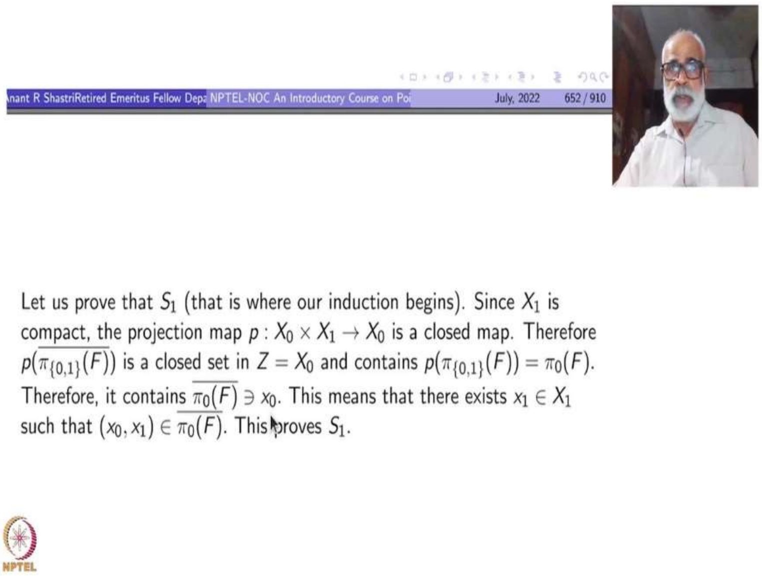

So, let us prove S_1. This is where the induction begins. What is X_1? X_1 is a compact space. The projection map from $X_0 \times X_1$ to X_0 is a closed map because X_1 is compact. Look at π_{01} from X to $X_0 \times X_1$. This I could have written as, in the square bracket also here, no problem, viz. $p_[0, 1]$, both of them are the same (Do not confuse $[0, 1]$ with the closed interval in $\mathbb{R}$.) There is nothing else between 0 and 1 in the set I by the way. So, take the closure of $\pi_{0,1}(F)$. Then take the first projection p from $X_0 \times X_1$ to X_0. That will be a closed map because X_1 is compact. So, $p(\overline{\pi_{0,1}(F)})$ will be closed subset of $Z = X_0$ and contains this subset $p(\pi_{0,1}(F)) = p_0(F)$. Therefore, it contains $\overline{\pi_0(F)}$ because it is a closed subset. Hence it contains x^0. This means that you have an x_1 inside X_1 such that this (x_0, x_1) belongs to $\overline{\pi_0(F)}$. This proves S_1. So, this is why I have explained it already before. Now, I am using this new notation, I have explained it. So, this is the way from x_1 to x_2, x_3 and so on, we can go on, keep going, very easily. But now, we want to use induction, directly.

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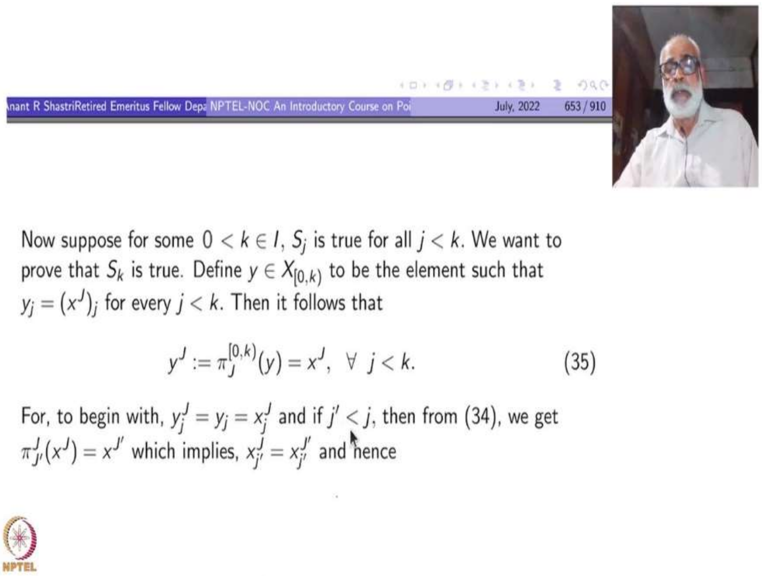

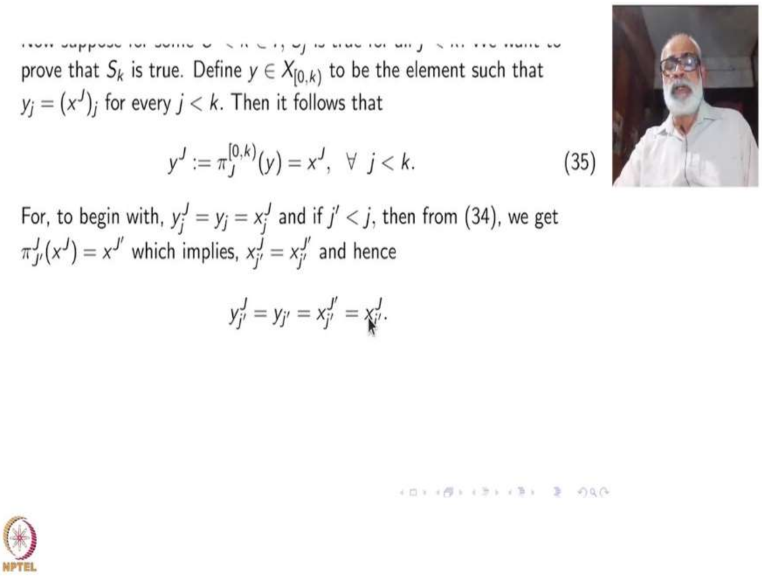

Now, next thing to prove is that suppose for some $k \in I$, $k \neq 0$, this should be 0, S_j is true for all $j < k$. We want to prove that S_k is true. Why? Once we prove this, the condition for transfinite induction is over. That will prove that S_∞ is true. So, all that we have to do is to show that S_k is true.

Define y belongs to $X_{[0,k)}$, the product X_i, i running inside $[0,k)$, such that the j^{th} coordinate of y is the equal to the j^{th} coordinate of x^J for every $j < k$.

Then it follows that this y^J which is nothing but the projection $\pi^{[0,k)}_{J(y)}$ is x^J, for each $j < k$. All that you have to verify is to take the i^{th} coordinate on both sides for $i \leq j$. Then, these are same because I have just projection maps there. So that is what I do now.

For, to begin with fix $j < k$ then the j^{th} coordinate of y^J is y_j, that is same thing as the j^{th} coordinate of x^J. Now if $j' < j$, then from this property (34), what we get? $\pi^J_{J'}$ of x^J is $x^{J'}$. So, this implies that the j'-th coordinate of x^J is equal to j'-th coordinate $X^{J'}$ also, So, that means that j'-th coordinate of y itself is equal to the j'-th coordinate of x^J.

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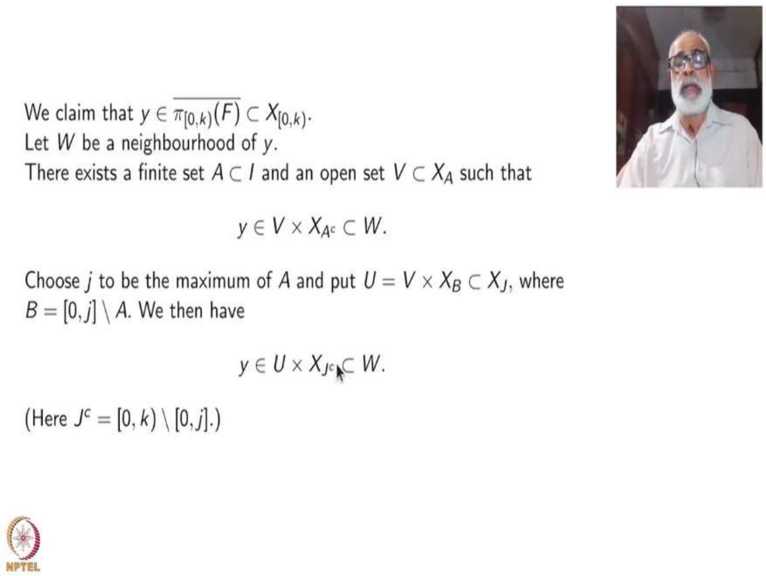

So, I have defined what y should be. This y will be in $\pi_{[0,k)}$ operating upon F, the closure of that. So, all this is happening inside $X_{[0,k)}$ that is the claim.

We have got y which projects correctly. But why it is in the closure of this set? So, this is where something has to be used from the product topology. What is that? The definition of product topology, that is what we are going to use.

Let W be a neighborhood of y in the product topology. What does that mean? There exists a finite set A contained inside I and an open subset V contained inside X_A, such that y is inside V cross this X_{A^c}, X_{A^c} is the product of all X_i where i ranges over the complement of A. So, that is a basic neighborhood. There will be some such neighbourhood contained inside W.

Choose j to be the maximum of A. A is a finite subset of I. That is why a maximum will exists. And put U equal to $V \times X_B$ contained inside X_J, where B is $[0, j] \setminus A$. I have taken the full spaces X_i where i is not inside A but in $[0, j]$, of course that is X_B.

See V is an open subset in the finite product X_A. So, I am taking $[0, j]$, a larger set containing A. Then I take this open subset inside X_J.

Then we automatically get y belonging to the open set $U \times X_{J^c}$ which is contained in W. (Here complement of J is taken in $[0, k)$.) So, only the i^{th} coordinate of points here are restricted for $i \in A$, all the rest of the coordinates are freely in X_i. These are basic neighborhoods.

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Therefore,

$$\pi_J^{[0,k)}(W) \supset \pi_J^{[0,k)}(U \times X_{J^c}) = U \ni y^J = x^J$$

from (35). From (34), we know that $x^J \in \overline{\pi_J(F)}$, and U is open in X_J, it follows that

$$U \cap \pi_J(F) \neq \emptyset.$$

This implies that

$$\emptyset \neq U \times X_{J^c} \cap \pi_{[0,k)}(F) \subset W \cap \pi_{[0,k)}(F)$$

Thus we have shown that $y \in \overline{\pi_{[0,k)}(F)}$.

Therefore, what happens, what is the meaning of this one? If you take the J^{th} projection of W, (so W is a subset of $X_{[0,k)}$ and it contains $U \times X_{J^c}$ and hence the J^{th} projection of W will contain U which contains y^J equal to x^J.

First we choose only finitely many coordinates properly, afterwards, we allowed the rest of the coordinates to be free, so that y will come inside that. So, this y is now inside U. So, that is the whole idea here. But its j^{th} projection is nothing but x^J, from (35), for all $j < k$. So, from (34), we know that this x^J is inside $\overline{\pi_J(F)}$ and U is open in X_J. It follows that $U \cap \pi_J(F)$ is nonempty.

This implies $(U \times X_{J^c}) \cap \pi_{[0,k)}(F)$ is non empty and that is contained in W intersection the projection $\pi_{[0,k)}(F)$. So, each neighborhood of y, intersects this one. Therefore, y must be the closure of $\pi_{[0,k)}(F)$.

So, these notations may be new to you, but this kind of thing, you have proved while proving product is connected etc. So, similar argument, this is not new to you.

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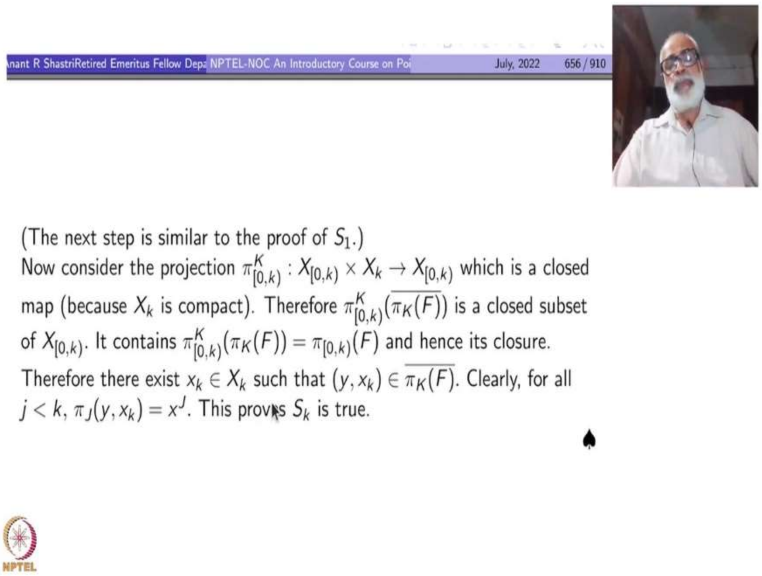

The next step is similar to S_1. What is that? We have yet to prove that this implies the next one S_k.

Now, consider the projection $\pi^K_{[0,k)}$ from X_K to $X_{[0,k)}$, namely, from the full product which has one more factor, K the closed segment $[0,k]$, and omit the last coordinate to come to $X_{[0,k)}$. Because X_k is compact, this projection is closed. (So, you have to use each X_k is compact, after all.

Therefore the image under $\pi^K_{[0,k)}$ of $\overline{\pi_K(F)}$, is a closed subset of $X_{[0,k)}$. But it contains $\pi^K_{[0,k)}(F) = \pi_{[0,k)}(F)$. Therefore, it contains the closure of $\pi_{[0,k)}(F)$. Therefore, there exists x_k belong to X_k, (I can now lift y into the product of one more factor, that is the whole idea), such that (y, x_k) (what is the meaning of this notation now? there is a $y' \in X_K$ such that its projection onto $X_{[0,k)}$ is y. And the k^{th} projection is x_k), and this element is in $\overline{\pi_K(F)}$.

So here, I am using that X_k is compact. One at a time, that step, please note.

So, clearly for all $j < k$, if you take the projection of this (y, x_k) on X_J, that will be x^J, being same as the projection of y.

So, this proves the statement S_k.

So we have proved assuming that S_j is true for every $j < k$, S_k is true. That closes the transfinite induction. Over, the proof is over.

So, this proof is worth repeating quite often, you read it carefully.

Next thing is, we want to go ahead with this order topology and so on. So, how far we can imitate the model, namely, the real line with its order.

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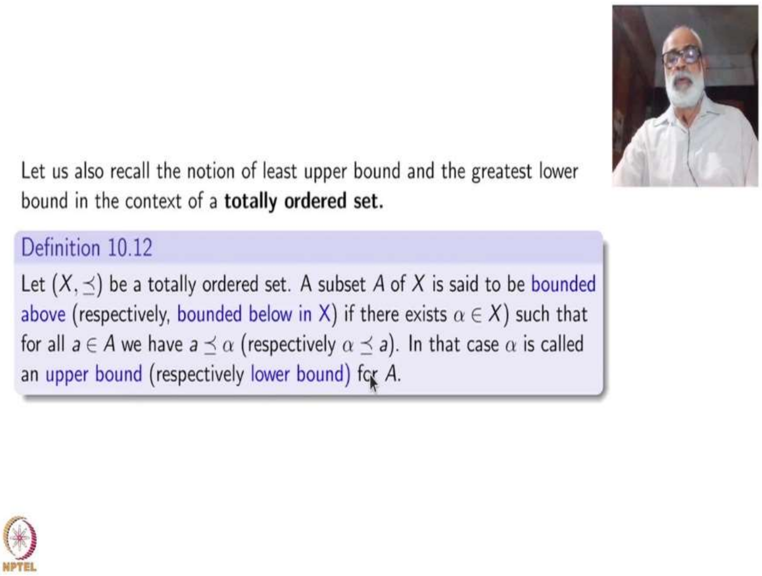

So, we go back to that and introduce a few more terminologies, all based on what we know already, what we are familiar with a real line. So, let us recall the notation of least upper bound and greatest lower bound in the context of a totally ordered set. We are recalling so that everything is devoid of all algebraic properties of real line. Only the order properties are being used. So, that is why we are recalling, emphasis on that.

So, start with any totally ordered set X. A subset A is said to bounded above (respectively to bounded below) if there exist alpha belonging to X such that for all $a \in A$, we have $a \leq \alpha$ (respectively, the other way around, $\alpha \leq a$). Bounded above and bounded below. In that case, this α will be an upper for A (respectively, lower bound for A). Like A be bounded above, and α is an upper bound for it.

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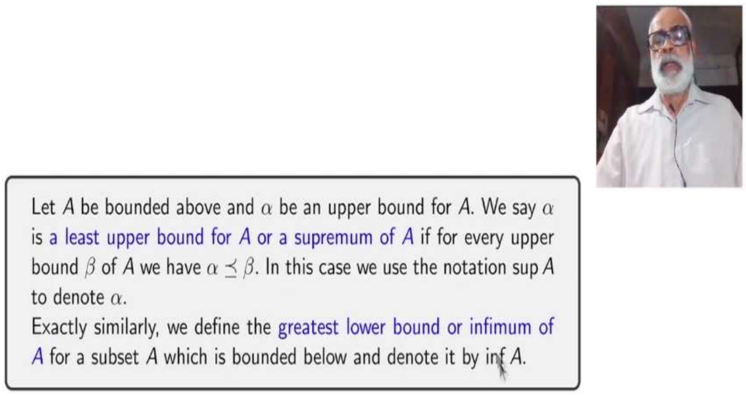

We say α is the least upper bound (another name is supremum) of A if for every upper bound β of A, we have $\alpha \leq \beta$. That means, first of all, the set of upper bounds must be nonempty, which just means that is bounded above. Then, you take alpha to be the least element of the set of upper bounds. For all upper bounds are bigger than α. So, then it is called the least upper bound. It is a very descriptive name. Supremum is another terminology. Exactly similarly, you can define greatest lower bound and infimum also. The notation will be $\inf\ A$ and $\sup\ A$.

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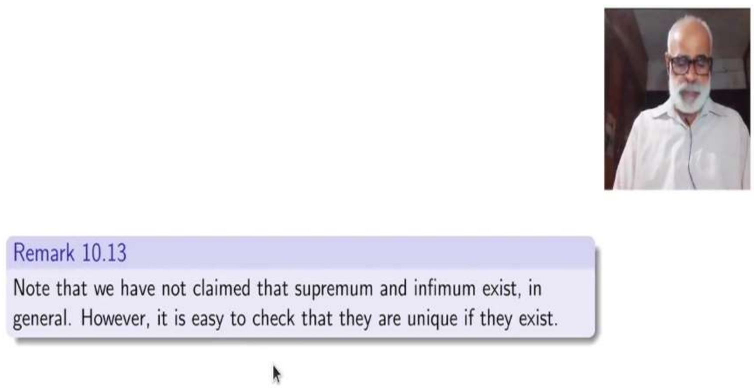

Note that we have not claimed that supremum and infimum exist. However, it is easy to check that they are unique if they exist. And that is where you have to use the anti-symmetry property of the total order. That is all.

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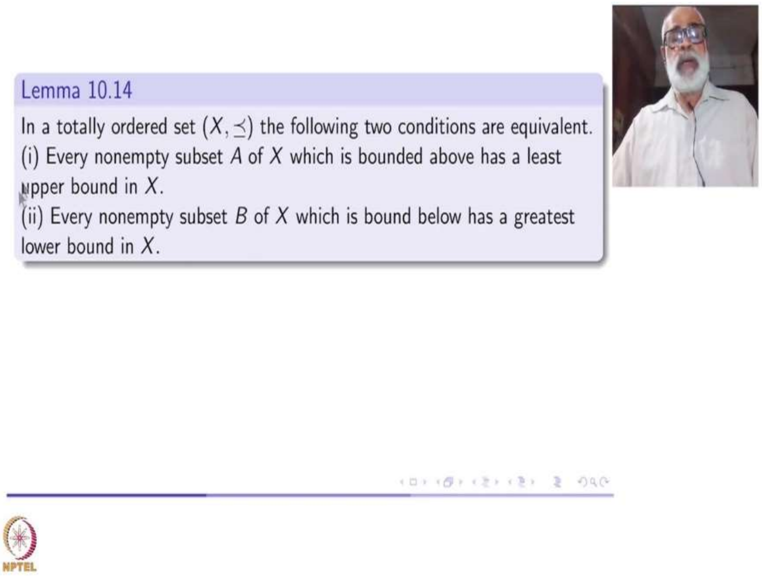

In a totally ordered set X, the following two conditions are equivalent. What is that?

(i) Every nonempty subset A of X which is bounded above has a least upper bound in X. (That upper bound may not be an element of A in X.)

(ii) Every nonempty subset B of X, which is bounded below has a greatest lower bound.

This is just the dual of the statement (i). But these two are equivalent. So, let us prove (i) implies (ii). The proof of (ii) implies (i) is just the other way round. Just that you have to reverse the inequalities everywhere. So, they are similar.

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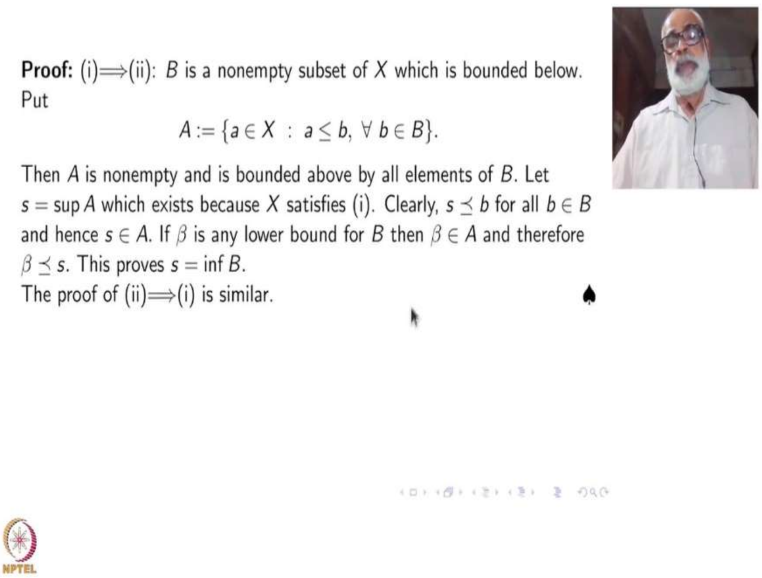

So proof of (i) implies (ii): Start with a nonempty subset B of X, which is bounded below. Look at all the elements a in X such that $a \leq b$ for every $b \in B$, which just means looking at all the lower bounds of B. A is nonempty, follows because we have assumed that B is bounded below. And because B is nonempty to begin with, we now have A is nonempty and it is bounded above. Every element of B is an upper bound for A.

So, (i) implies that $s = \sup A$ exists. Every bounded above set has the least upper bound. So, this is the hypothesis in (i). Now, we have to conclude (ii).

Let B' is the set of all $b \in X$ such that $b \leq a$ for all $a \in A$. Then by definition, s is the least element in B'. Clearly B contains B and hence s is lower bound for B. This means s is in A. If beta is any lower bound for B then by definition beta is inside A. Therefore, $\beta \leq s$. Therefore, s is the greatest lower bound, this infimum. So, by upper bound exist and then the supremum exists if the lower bounded below, lower bound exists is what we have, infimum exists is what we have.

The proof (ii) implies (i) as I have told you, is similar.

Added by reviewer: Note that in case of $\mathbb{R}$, the same thing is usually proved by considering $-B$, which we cannot do here.

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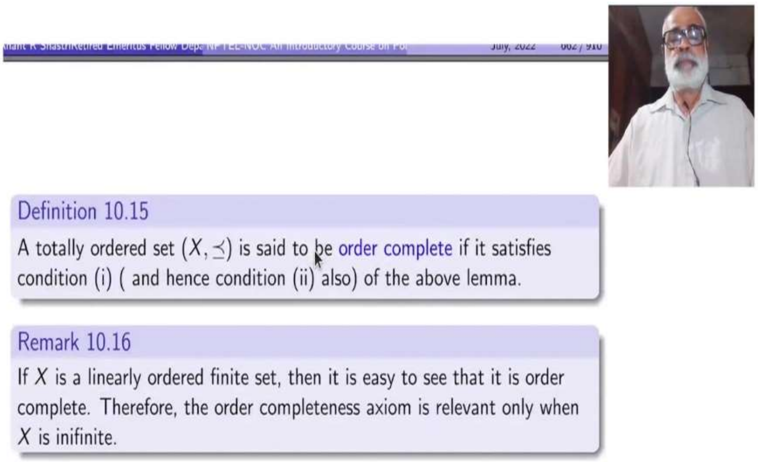

Few more terminologies. A totally ordered set is said to be order complete if it satisfies condition (i) and hence condition (ii) (because they are equivalent to each other by the above lemma). I repeat what is the meaning of order complete? Every bounded above set must have least upper bound which is same thing as saying every bounded below set has a greatest lower bound.

Note that (X, ∂) must be, first of all totally ordered set. In addition, if the above condition is satisfied that we say it is order complete. (So, you may be already knowing that this property, we have been extensively using for the real numbers.) So, we have made the definition here.

If X is a linearly ordered finite set, then it is easy to see that it is order complete. Therefore, order completeness axiom is relevant only when X is infinite.

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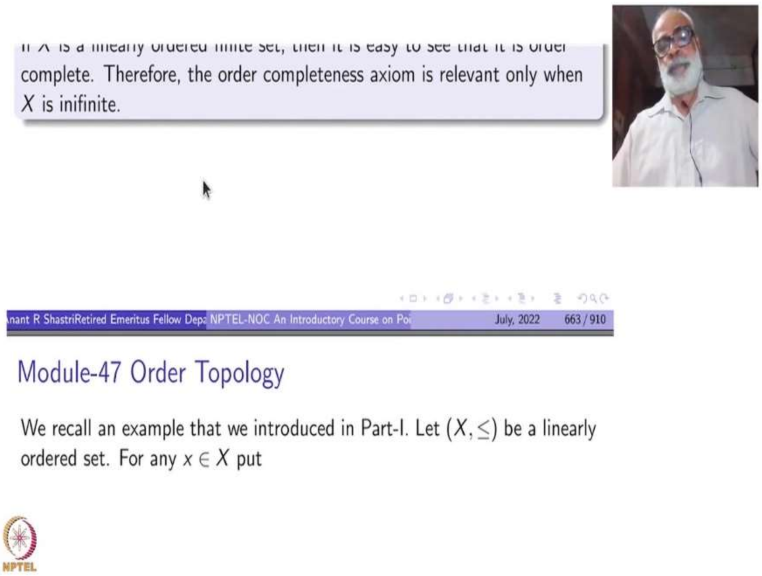

Next time, we shall use all these and we will study order topology. So far, we are not mentioning any topology at all. All the time, we were talking about the combinatorics of this partial ordering, total ordering, well ordering and so on. Now, topology will enter. Thank you.