An Introduction to Point-Set-Topology (Part II)
Professor Anant R. Shastri
Department of Mathematics
Indian Institute of Technology, Bombay
Lecture 44
Local Separation to Global Separation

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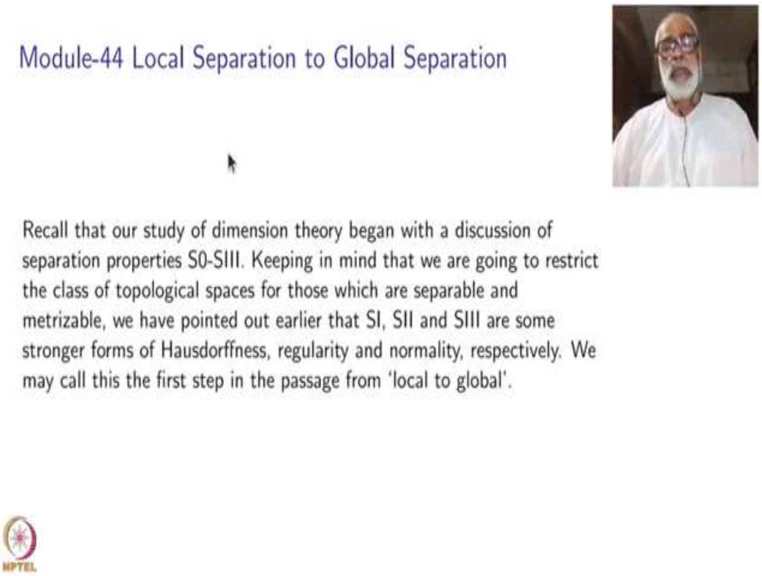

Hello. Welcome to NPTEL NOC, an introductory course on Point-Set-Topology, Part II. Today, we will have the last section on the dimension theory, module 44, Local Separation to Global Separation. Recall that our study of dimension theory began, actually, in the previous chapter with a discussion of separation properties, which we have named S0 to SIII.

Keeping in mind that we are going to restrict the class of topological spaces for those which are separable and metrizable, and in particular T_1, we have pointed out earlier that SI, SII and SIII are some stronger forms of Hausdorffness, regularity and normality, respectively. We may call this itself the first step in the passage from local to global.

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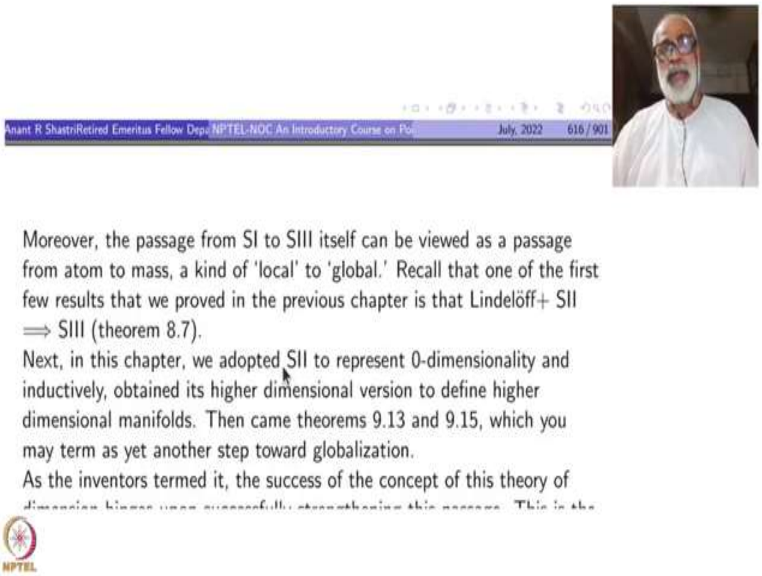

Moreover, the passage from SI to SIII, (there is one SII in between), itself can be viewed as a passage from atom to mass, atomic to aggregate or whatever you want to say, mini scale to larger-scale and so on, a kind of local to global. It is also a passage from local to global. Recall that one of the first few results that we proved in the previous chapter is that Lindelof plus SII implies SIII. So, that made SII our central object of study.

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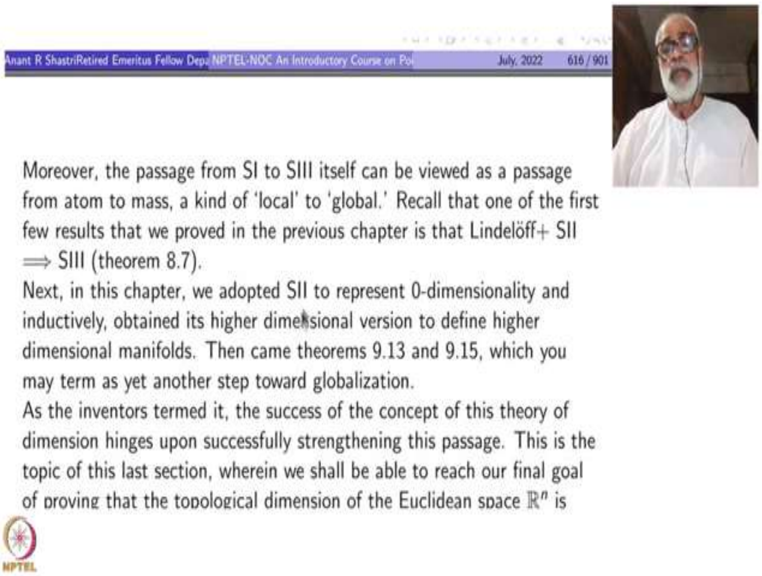

Next, in this chapter, we adopted SII to represent 0-dimensionality and then inductively, obtained its higher dimensional versions to define higher dimensional manifolds. Then came the theorem 9.13 and 9.15, which you may term as another step toward globalization. Let me just show you these steps.

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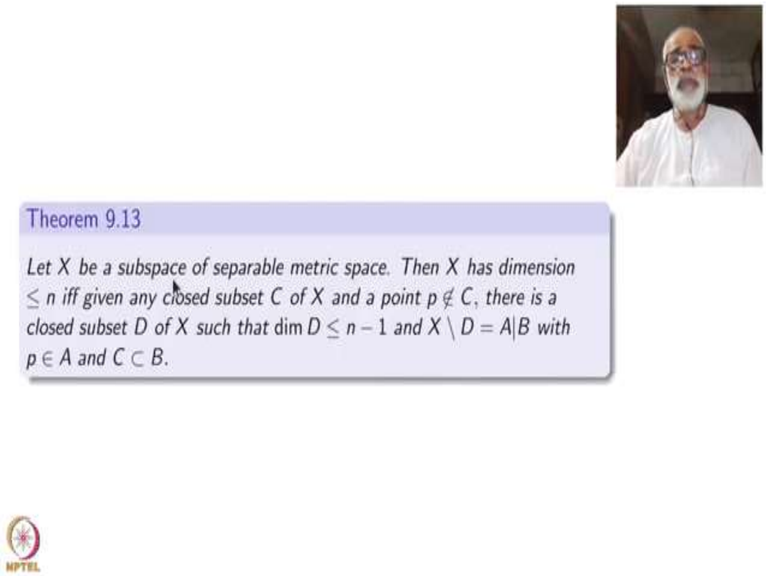

This was the theorem. If we have a subspace of a separate metrizable space then X has dimension less than or equal to n if and only if, given any closed subset C of X and a point outside, there is a closed subset D of X such that dimension of D is less than or equal to $n-1$ and $X \setminus D$ is equal to $A|B$, A and B are both open and closed subset disjoint, with p inside A and C inside B. So, we have pointed out that in the case of n equal to 0, dimension of D is -1 means D is empty set. This is precisely the condition SII.

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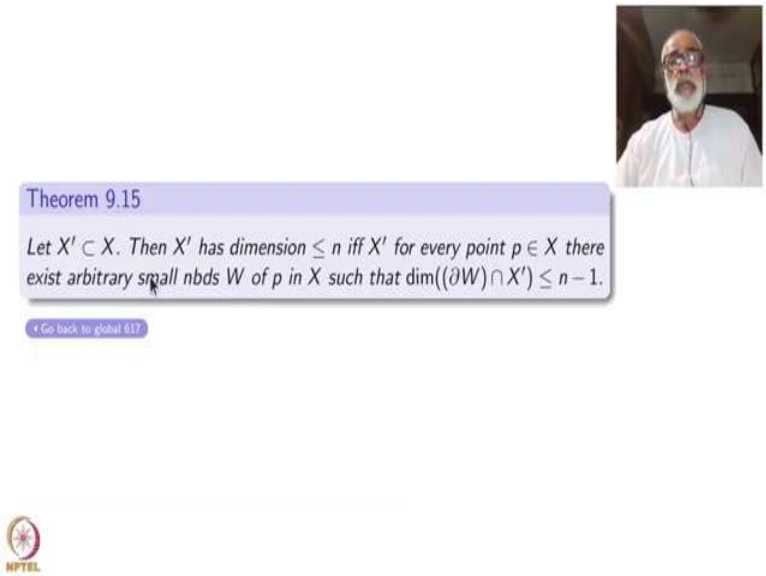

So, similarly, theorem 9.15, you can also do similar interpretation. I just wanted to show you one of them here. If X is metrizable space and X' is a subspace of that, dimension of X' is less than or equal to n if and only if, for every point be inside X, something happens. Now I am putting conditions on points of X all together. You see, I am getting condition like that, condition for dimension of X' to be less than or equal to n. What is the relation between points inside X and the subspace having dimension less than or equal to n. So that step is again another step towards globalization. This is what we meant. So, let us go back to what we were doing today.

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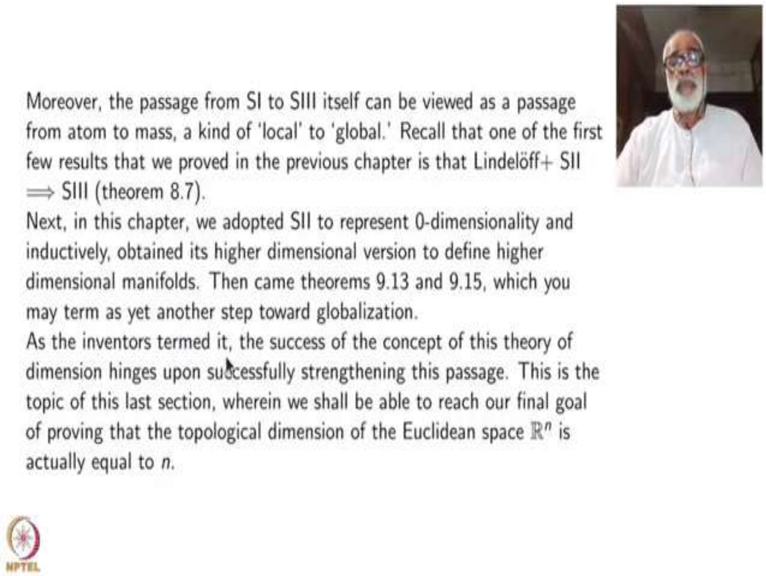
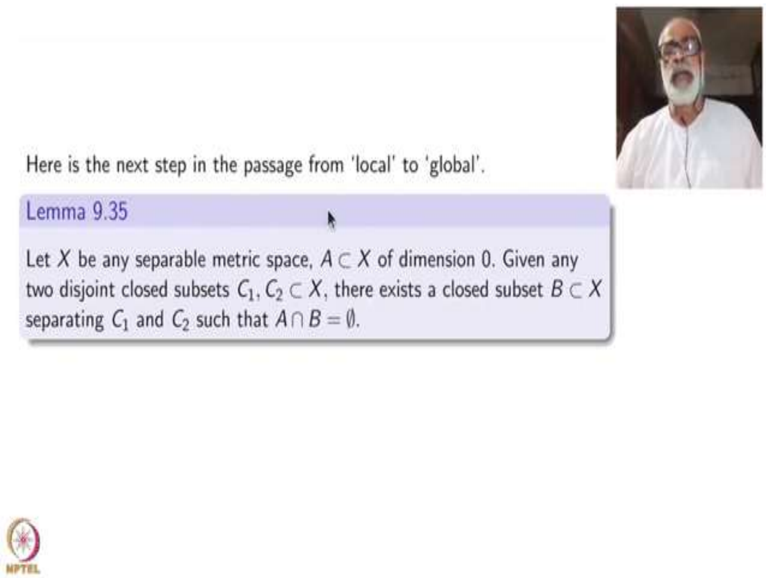


So, as the inventors have termed it, the success of the concept of this theory of dimension, (why I am calling this a theory, because there are other theories also), things upon successfully strengthening the passage from local to global. This is the topic of this last section, wherein we shall be able to reach our goal, the final goal of proving that the topological dimension of the Euclidean space $\mathbb{R}^n$ is actually equal to n. So, that I call it as a success of the theory. (Refer Slide Time: 05:38)



Here is the next step in the passage from local to global. So, we have to prove all these things now.

Let X be any separable metric space, and A be a subset of X of dimension 0. Given any two disjoint closed subsets C_1 and C_2, there exists a closed subset B of X separating C_1 and C_2 such that $A \cap B$ is empty.

The subsets C_1 and C_2 are disjoint closed subsets. By normality, you can separate them by open sets. That is a different aspect. Of course, that will be the starting point in the proof of this. What we are going to do is there is a separation by a closed subset B which does not intersect A at all. So, this you can call it really the crunching fact. Of course, we have to improve on this also of the separation properties being globalized, for globalization property.

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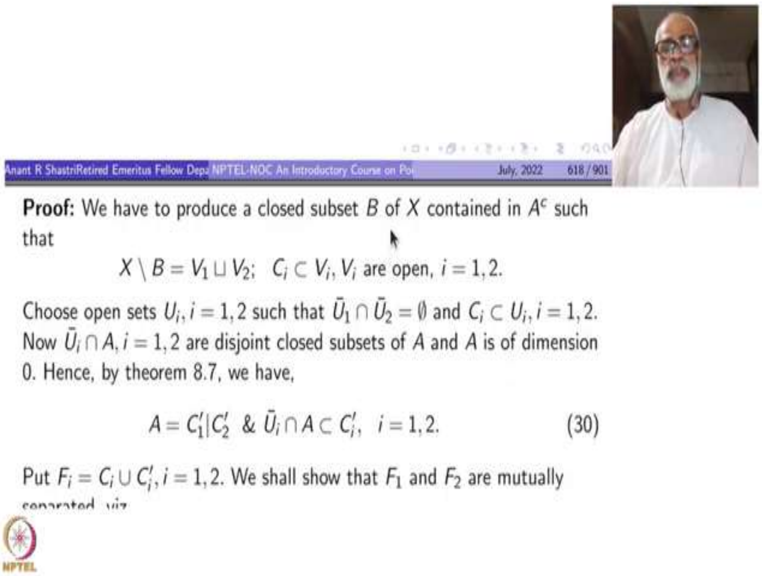

So, let us start the proof of this. We have to produce a closed subset B of A^c, i.e., contained in the complement of A, such that when you throw away this B, viz., $X \setminus B$, can be written as a disjoint union of open sets V_1 and V_2, with C_1 contained inside V_1, and C_2 contained inside V_2. That is the meaning of separation of B separates C_1 and C_2.

First, choose open sets $U_i,\ i = 1$ and 2, such that $\overline{U_1} \cap \overline{U_2}$ is empty and C_i are inside U_i. So, this step just uses the metric space property here, normal, normal property or metric space. Once you have these U_1 and U_2, look at their intersection with A. In fact take $\overline{U_i} \cap A$. These will be disjoint subsets of A and A is of dimension 0. So, apply the SII property there. We have A equal to $C'_1|C'_2$ here, $\overline{U_i} \cap A$ is inside C'_i for $i = 1, 2$. So, that is the property for a 0-dimensional set.

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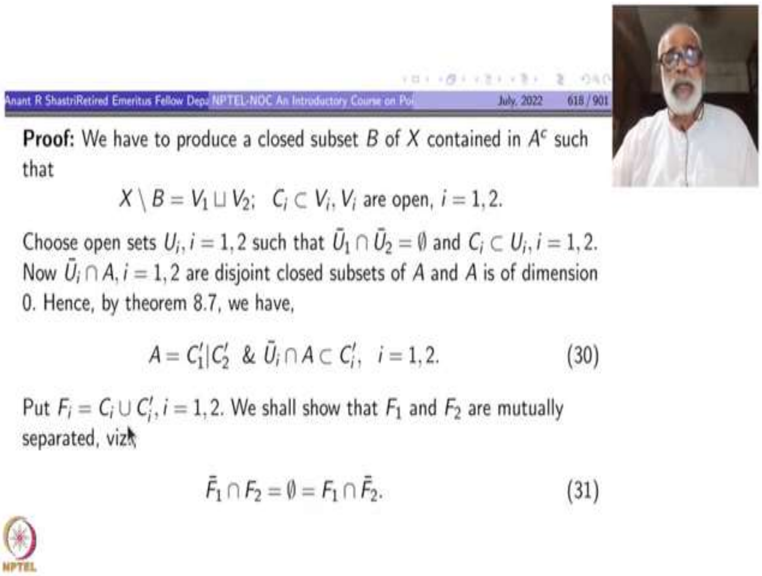

Now, you enlarge this C'_1 and C'_2, along with C_i, put $F_i = C_i \cup C'_i$, i equal to 1 and 2. C'_i are subsets of A, C_1, C_2 are subsets of the larger space X. C'_i is closed inside A, but A itself is not closed inside X. So, there is some problem here. Otherwise, F_i would have been easily, closed subsets.

So, we should show that F_1 and F_2 are mutually separated sets (next best to saying disjoint closed subsets, but enough for our purpose). So, I mean they are not closed subset but $\overline{F_1}$ does not intersect F_2 and $\overline{F_2}$ does not intersect F_1. So, mutually separated subsets. So, which is slightly weaker hypothesis than having disjoint closed subset, disjoint closed subset are easy to separate, but this is only a little more difficult. So, for here, we will need more than normality, namely complete normality, which is there since we are working in a metric space. Wait a minute, so let us solve this one now, Namely $\overline{F_1} \cap F_2$ is empty? Indeed, once you prove this one, the other one is symmetric. All these conditions are symmetric in $i = 1, 2$. So the proof of $\overline{F_2} \cap F_1$ is empty would be similar.

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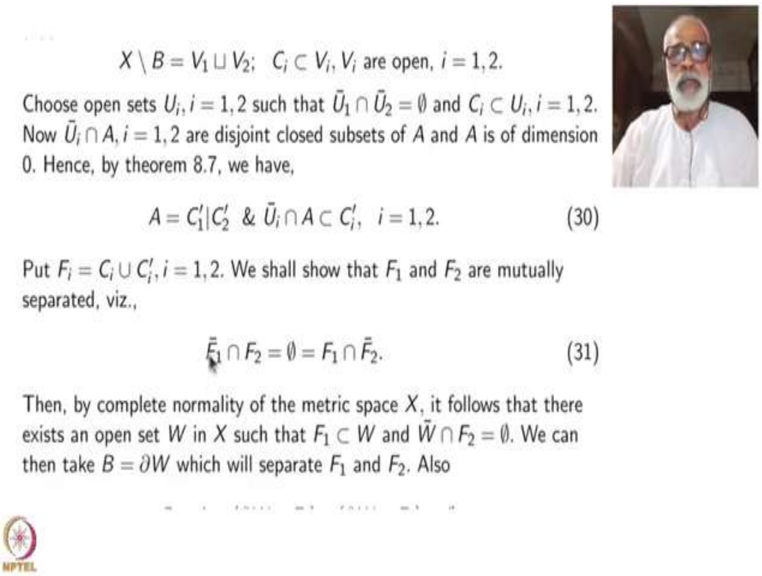

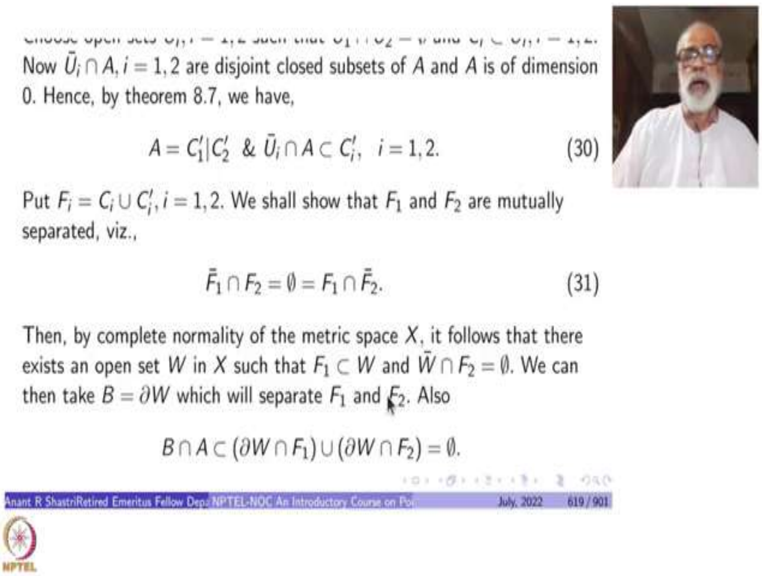

So, once you can use complete normality of the metric space, it follows that we get open subset W in X such that at F_1 is inside W, and closure of W does not intersect F_2, which is same thing as having disjoint closed subset and so on. This way, it is easier to state: F_1 contained in W and closure of W does not intersect F_2.

You can then take B as the boundary of W. Automatically boundary of W is closed. So, this B is a closed subset, which will separate F_1 and F_2. Because F_1 will be contained inside W and does not intersect B, F_2 does not intersect B because it does not intersect $\overline{W}$ at all. So, F_2 are inside complement of $\overline{W}$.

If you take W union complement of $\overline{W}$ that will be precisely equal to the whole X minus boundary of W i.e., $X \setminus B$.

Moreover, $B \cap A$ is empty, Why? This I have not yet shown. What I have shown F_1 and F_2 are disjoint from boundary of W. We want to ensure that $B \cap A$ is empty. But $B \cap A$ is contained in boundary of $W \cap F_1$ union boundary of $W \cap F_2$, because this entire A being $C_1' \cup C_2'$ is contained in $F_1 \cup F_2$. But both of them are empty. Just now we shown that. So $B \cap A$ is empty.

So, this B will serve the purpose.

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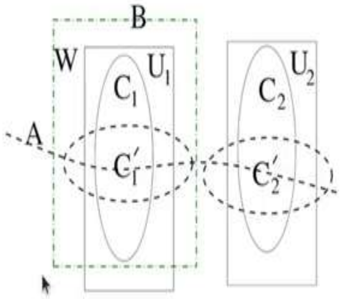

Figure 12: A step toward 'local-to-global'

So, we have yet to prove that F_1 and F_2 are mutually separated. So, here is the schematic picture of what is happening. Started with C_1 and C_2 which are disjoint closed subsets, shown by these ellipses, we enclose them in open subsets U_1 and U_2 where closure of U_1 and closure of U_2 are disjoint. This is my A which is of dimension 0. That is why I have shown it with dot dot dots here.

Intersect A with $\overline{U_i}$ so that, you get two disjoint closed subsets A_i. One is from here to here. Similarly, another from here to here. So, they are disjoint closed subsets of A. So, you can separate them by C_1' and C_2' shown by dotted ellipses. Note that C_1' and C_2' may go out of U_1 and U_2.

After all open subsets of A are nothing but open subsets of X intesected with. So, that is what I have shown here.

Then I put F_1 equal to $C_1 \cup C_1'$ up till here and F_2 equal to $C_2 \cup C_2'$ up till here. I have to show that they are mutually separated. In the picture, it is obvious. You can not use the picture to prove a theorem. You can take the help, but finally everything should be purely logical.

And finally, what we want is this green rectangle thing W, such that it contains F_1 and its closure does not intersect F_2. And if you throw away its boundary from x, you get a separation. So, let us prove that F_1 and F_2 are mutually separated.

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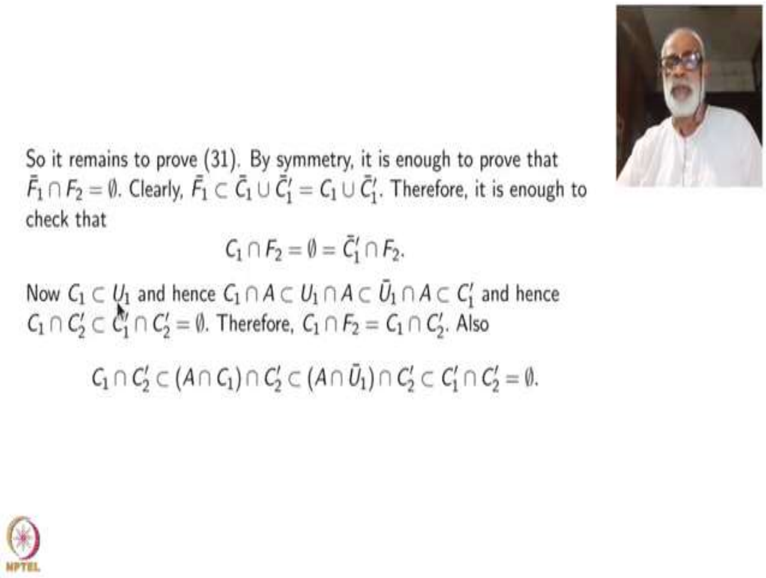

So, let us do that. So, it remains to prove (31). (31) is what? $\overline{F_1} \cap F_2$ is empty. The other one is similar. So, by symmetry it is enough to prove that $\overline{F_1} \cap F_2$ is empty.

First of all, F_1 itself is $C_1 \cup C_1'$, therefore $\overline{F_1}$, (bar denotes the closures in the whole space X), $\overline{F_1}$ is $\overline{C_1} \cup \overline{C_1'}$, because it is just a finite union.

But C_1 is already closed in X. So, it is $C_1 \cup \overline{C_1'}$. Therefore, it is enough to check that $C_1 \cap F_2$ and $C_1' \cap F_2$ are empty. So, this is the first step. I have to show these two things one by one. So, let us see.

Now C_1 is inside U_1, because by the very choice U_1, U_1 is an open subset containing in C_1. Hence $C_1 \cap A$ which is contained in $U_1 \cap A$. But $\overline{U_1} \cap A$ is contained in C_1' by (30). Therefore, $C_1 \cap C_2'$ is contained in $C_1' \cap C_2'$ which is empty.

So, $C_1 \cap F2$ is equal to $(C_1 \cap C2) \cup (C_1 \cap C_2')$, F_2 has two parts, both the parts intersection with C_1 are empty and therefore, we conclude that $C_1 \cap F_2$ is empty.

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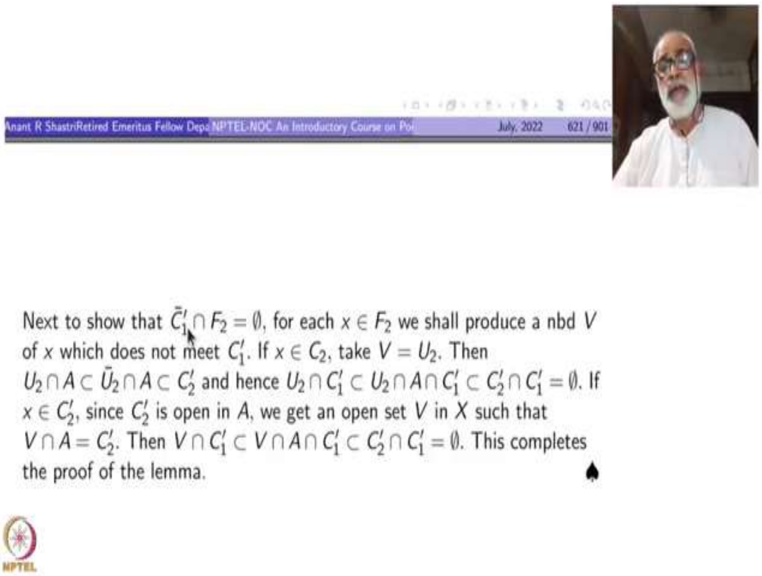

Next to show that $\overline{C_1'} \cap F_2$ is empty. So, how do you show? For each x belonging to F_2, we shall produce a neighborhood V of $x \in X$ which does not meet C_1'. Then it follows that the point is not in the closure of C_1'.

If x is inside C_2, (there are two parts to F_2, one is C_2 and other is C_2', we are taking the two cases separately), you can just take V equal to whole of this U_2. And then $U_2 \cap A$ which is contained inside $\overline{U_2} \cap A$, that is contained inside C_2', and hence $U_2 \cap C_1'$ is $U_2 \cap A \cap C_1'$, which is contained in C_2‘ $\cap C_1'$. That is empty.

If x is in C_2', then what do I do? Note C_2' is an open subset of A, being one part of a separation of A. So, we get an open set V in X such that $V \cap A$ is equal to C_2'. Then what happens? $V \cap C_1'$ is $V \cap A \cap C_1'$ which is equal to $C_2' \cap C_1'$ and that is empty.

And that completes the proof.

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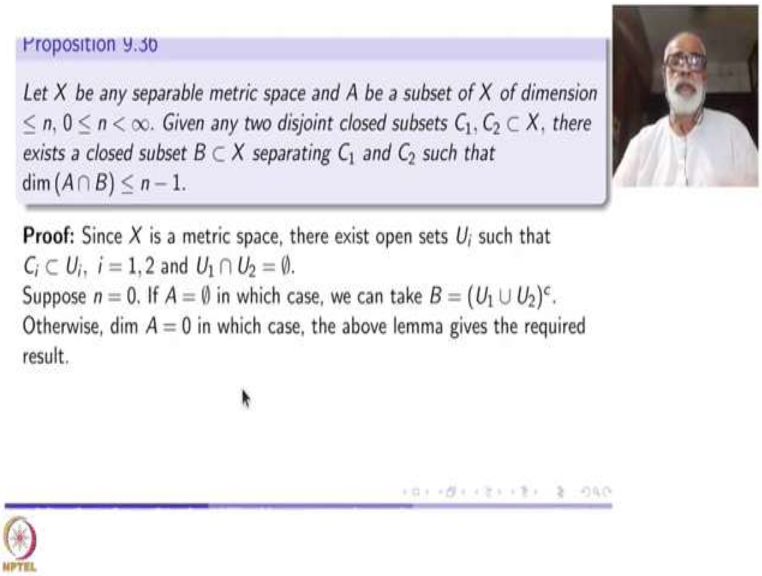

Now, we can state a more pliable statement and easy to remember statement. Start with any separable metric space X, take a subset A of dimension less than or equal to n, where n itself is finite. (Of course, I assume that n bigger than equal to 0 because, if A is empty there is no statement. Those things we have seen already.) So, given any two disjoint closed subsets C_1 and C_2 of X, there exists a closed subset B of X separating C_1 and C_2 such that the subset $A \cap B$ has dimension less than equal to $n-1$.

So, from the case when A is of dimension 0, we have come to arbitrary dimension here now. So, how will you do that. Of course, since X is a metric space, there exists open subsets U_1 and U_2 such that C_i's are inside U_i, i equal to 1 and 2, and intersection U_1 and U_2 is empty. This is from the normality of X because C_1 and C_2 are disjoint closed subsets.

Now, suppose n is 0 (this n is between 0 and infinity). So, n is 0, there are two cases to be handled. If A is empty, in which case you can take B equal to complement of U_1 union U_2. that is a closed subset. And then $X \setminus B$ is just the disjoint union of U_1 and U_2. Over.

If A is non empty, dimension of A is 0, in which case, the earlier lemma which we did just now, that gives the required result.

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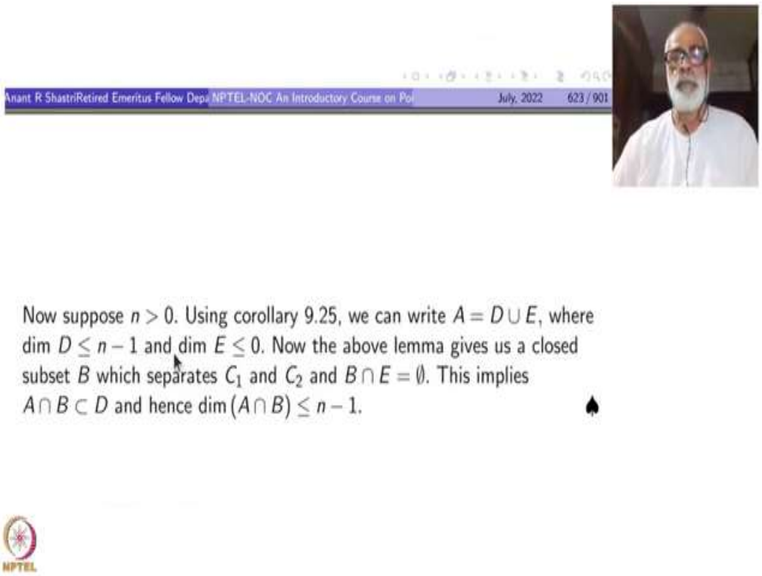

Now we have the inductive hypothesis, where we suppose n is bigger than 0. Using a previous corollary, we can write A as a union of two subsets D and E, where dimension of D is less than or equal to $n-1$ and dimension of E is less than equal to 0. This is one of the theorems that we have derived last time. Now, you use the above Lemma to get a closed subset B, which separates C_1 and C_2 such that $B \cap E$ is empty. I do not know what is happening intersection with D, we will come to that later, but intersection with E part is empty. That is the starting point.

But then look at $A \cap B$. That will be now contained inside D. D is of dimension less than or equal to $n-1$. So, $A \cap B$ is also of dimension less than equal to $n-1$. over.

So, this after the hard work in the above lemma, this comes quite easily. Why? Of course I have to use this crucial thing here, namely anything which is of dimension n can be written as a union of two subsets, one is dimension $n-1$, another one is 0.

Here is a theorem.

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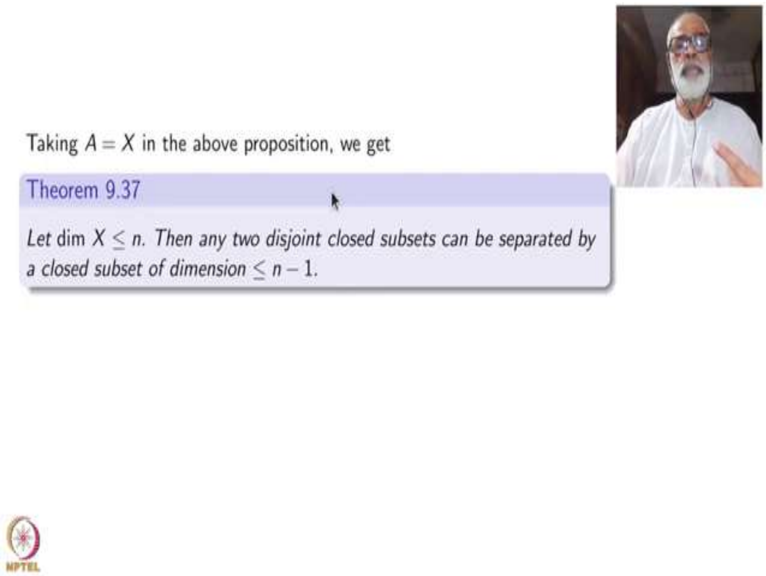

Now in the above theorem (well, this was not a theorem, it is a proposition, does not matter), take A is equal to X. Then what do I get? Let X be of dimension less than equal to n. Then any two disjoint closed subsets can be separated by a closed subset of dimension less than or equal to $n-1$. (There is no question of intersecting with A because A is the whole space X.)

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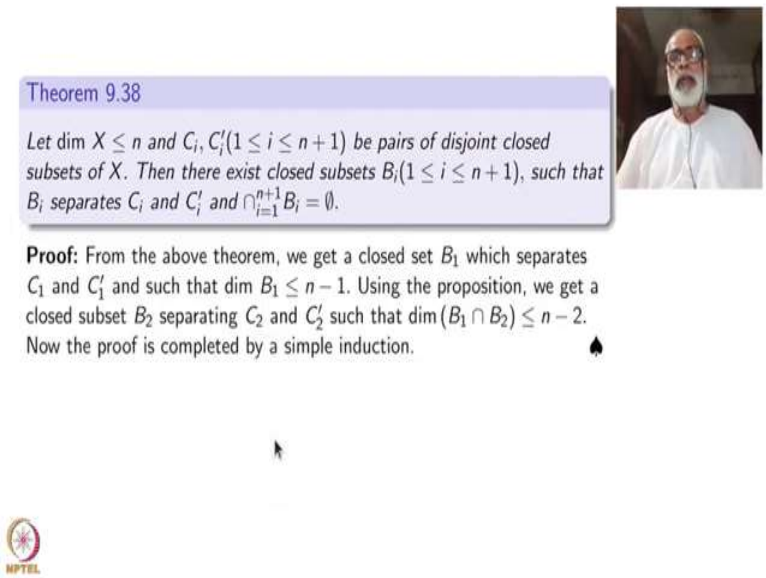

Now, we want to improve upon that one. Let X be of dimension less than or equal to n and $C_1, C_2, \ldots$ and similarly $C_1', C_2', \ldots$ be pairwise disjoint closed subsets, $C_i \cap C_i'$ is empty for each i. How many are there? $n+1$ of them. This then, there exist closed subsets $B_i, i = 1, 2 \ldots, n+1$ such that each B_i separates C_i and C_i', and intersection of all the B_i's, i ranging from 1 to $n+1$ is empty.

So, how do we get this one? This is also easy. From the previous theorem, applied to C_1 and C_1', you get a closed set B_1, which separates C_1 and C_1', such that dimension of B_1 is less than or equal to $n-1$. Now, use the proposition, not the theorem, we get a closed subsets B_2 of X again, separating C_2 and C_2', such that when you intersect B_2 with B_1, its dimension is less than or equal to $n-2$, because you already dimension of B is less than or equal to $n-1$.

Now, you keep on doing this, you repeat this step, get a B_3 such that it separates C_3 and C_3', with dimension of $B_1 \cap B_2 \cap B_3$ is less that or equal to $n-3$. How far you can go till you get $n-1$ that is empty. (So, you have to have dimension less than or equal to n here and there must be $n+1$ then only you will actually handle $n+1$ pairs of disjoint closed sets.)

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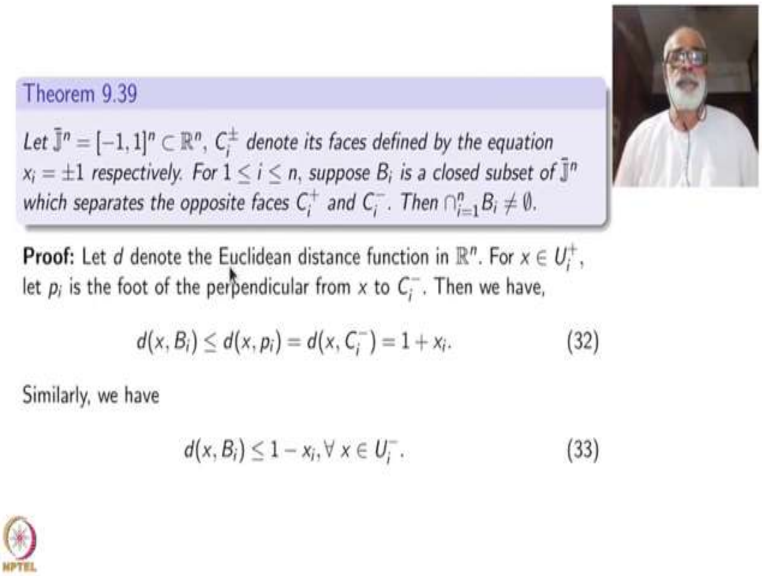

Now, we are very close to the end here. Consider the rectangular box $\overline{\mathbb{J}^n}$ closed interval $[-1,1]^n$ contained inside $\mathbb{R}^n$. Suppose you denote by $C_i^\pm$ the faces of $\overline{\mathbb{J}^n}$, defined by equations x_i equal to ± 1, $i = 1, 2, \ldots, n$. (For $n=1$, this is nothing but $C_+ = \{1\}$ and $C_1^- = \{-1\}$. If $n=2$, there will be four faces, two pairs of opposite faces. So, that is the way you have to take these faces, defined by the equations: the i^{th}-coordinate x_i equal to ± 1. For $1 \le i \le n$, there are n pairs here.)

Suppose B_i is a closed subset of $\overline{\mathbb{J}^n}$ which separates the opposite faces C_i^+ and C_i^-, which just means that when you throw away B_i from $\overline{\mathbb{J}^n}$, you get two disjoint open subsets $U_i^\pm$, each of them containing either C_i^+ or C_i^-, that is a separation. Then we want to show that intersection of all B_i's is non empty.

So, together these two results are going to imply a big theorem for us. However, the proof of this is now based on something different that we did last time, namely, Brouwer's fixed point theorem comes here. Let us see how. Pay attention to the method of proof because that can be used in several other places.

Let d denote the Euclidean distance function in $\mathbb{R}^n$. For each x in U_i^+, (I have already told you what are U_i^+ here, the open subset containing C_i^+, and this U_i^- is containing C_i^-.)

For x belongs to U_i^+, let $p_i = p_i(x)$ be the foot of the perpendicular from x to C_i^-. So, move all the way to C_i^- from the point x in U_i^+. Do not change the i^{th} coordinate, the $p_i(x_0)$ and x have the same i^{th} coordinate. That line is perpendicular to the hyperplane containing C_i ^minus. It is an elementary observation that distance between x and B_i, (B_i is the set that separates $C_i^{\pm}$ is less than or equal to distance between x and p_i and this distance is actually equal to distance between x and C_i^-. Why? Because p_i is the foot of the perpendicular from x. And what is this distance? It is equal to 1 plus the i^{th} coordinate of x. So, what I have proved by this? Distance between x and B_i is less than or equal to $1 - x_i$, for every $x \in U_i^+$. So, very easy to remember. Let me justify this with a small picture here, for $n = 2$.

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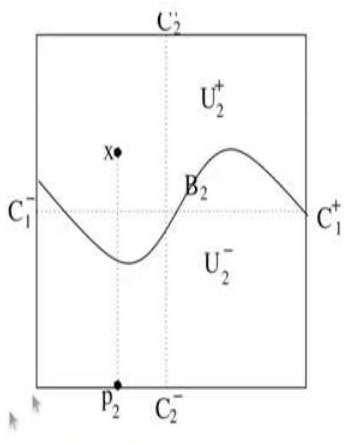

Figure 13: Application of BFT

So, this is $[-1, 1] \times [-1, 1]$. This is square. And this is C_1^+, this is C_1^-, this is C_2^-, and this is C_2^+. This is my B_2 which separates C_2^+ and C_2^-. Take a point x in U_2^+, take its projection onto this plane. So, here it is just a segment parallel to x-axis. What is this? x_1 coordinate will not change.

What is its y-coordinate? i.e., the x_2 coordinate of this point 1 is equal to the distance of x from p_2 and that is the same thing as the distance between x and C_2^-, which is bigger than distance between x and B_2. That is precisely what we are claiming, more generally, that distance between x and B_i is less than $1 + x_i$. Similarly for x in U_i^-, we have $d(x, B_i)$.

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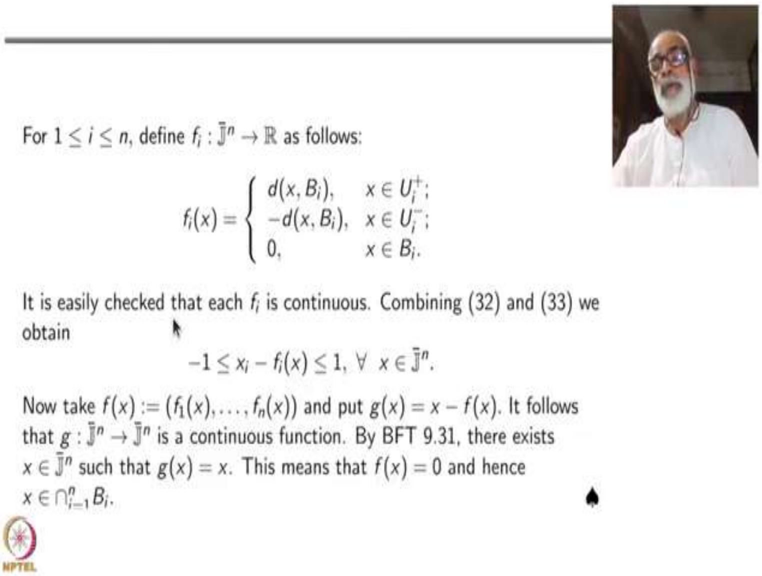

Now, I define a function here for each i between 1 and n, let f_i from $\overline{\mathbb{J}^n}$ to $\mathbb{R}$ be as follows: If x is in U_i^+, just take $f_i(x)$ equal to the distance between x and B_i. This is a continuous function on U_i^+, we know that. What is this? This is the minimum of distance between x and b_i, where b_i runs over B_i.

If x is in U_i^-, you put $f_i(x) = -d(x, B_i)$, a minus sign in this case.

Finally, if x is in B_i, put $f_i(x) = 0$. Look at this one. As x tends to a point in B_i, $f_i(x)$ tends to 0 in both cases, because B_i is the common boundary of $U_i^{\pm}$. It follows that f_i is continuous on the whole of $\overline{\mathbb{J}^n}$.

Now, you combine these two inequations here, inequalities, what you conclude is that -1 is always less than or equal to $x_i - f_i(x)$ is always less than or equal to 1. There are two different cases you may have to work, you have to work on. This is very easy.

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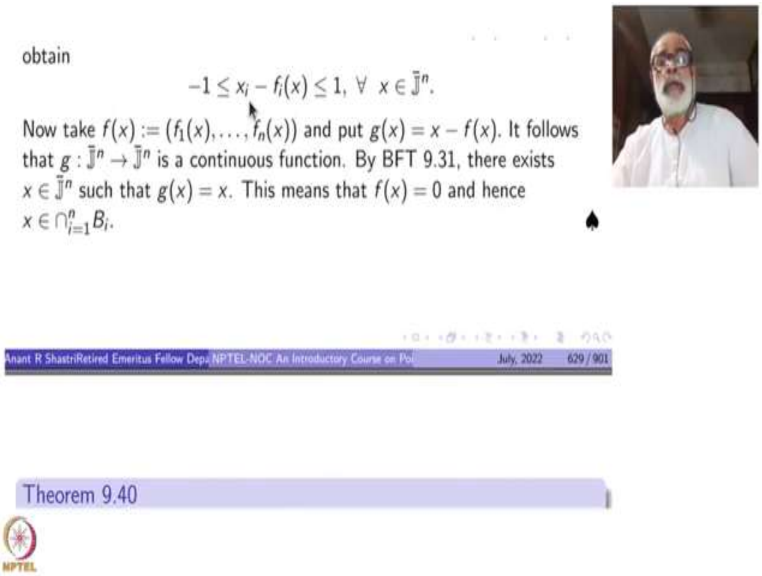

So, $x_i - f_i(x)$ is between -1 and 1. Therefore if I define, $f(x)$ equal to $(f_1(x), f_2(x), \ldots, f_n(x))$ and $g(x)$ equal to $x - f(x)$, then what happens? Both f and g a function taking values inside $\mathbb{R}^n$, but $g(x)$ will be taking function inside $\overline{\mathbb{J}^n}$. It will be always between -1 and 1, all the coordinates. So, obviously, both of them are continuous.

Therefore, you have got a function g from the closed rectangle $\overline{\mathbb{J}^n}$ to $\overline{\mathbb{J}^n}$. We can apply Brouwer's fixed point theorem to g. So, we get a point $x \in \overline{\mathbb{J}^n}$ such that $g(x) = x$. What does that mean? $f(x) = 0$. $f_i(x) = 0$ for all i. What does that mean? x must be inside each of these B_i's. Which means, x is in the intersection of all B_i's.

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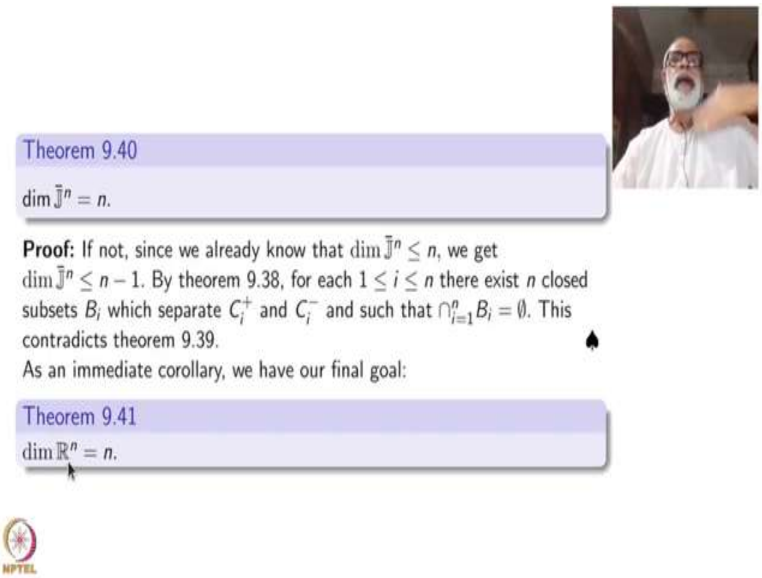

So, this theorem is proved now. As I told you now, you can combine this with the previous theorem, to get a wonderful result now, namely. Dimension of $\overline{\mathbb{J}^n}$ has to be equal to n.

Let me go through this one. It is not so clear. If not what happens? Dimension of $\overline{\mathbb{J}^n}$ is less than or equal to n. We know that, this part we have already proved, dimension for $\mathbb{J}$ is one and by the product theorem, dimension of $\overline{\mathbb{J}^n}$ is less than or equal to n. That is already proved.

So if it is not equal to n, then the dimension must be less than or equal to $n-1$. As soon as you see $n-1$, by our theorem 9.38, whatever we have proved today, means that, for each i, $1 \leq i \leq n$, there exist n closed subsets B_i which separate C_i^+ and C_i^-, such that the intersection of all B_i's is empty.

But Brouwer's fixed point theorem applied just now, in the previous theorem says this is non empty. So that is the contradiction to this theorem 9.39. Therefore, dimension is tight, it has to be equal to n.

$\mathbb{R}^n$ being larger space, containing $\overline{\mathbb{J}^n}$ must also of dimension n.

So, we have proved that main result. Not only that, you can take now any non empty open subset which is homeomorphic to some disc or $\mathbb{J}^n$ and so on, or any subset which contains some such set, all of them will be of dimension n inside $\mathbb{R}^n$.

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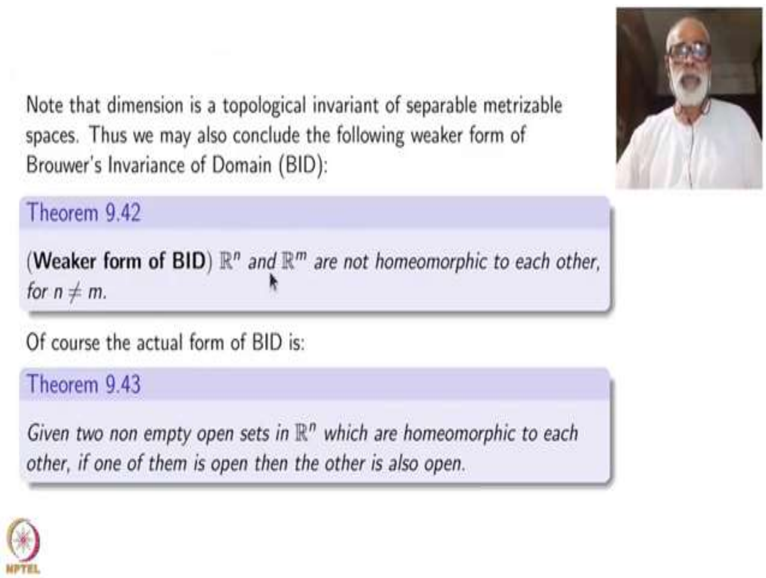

So, here is a remark. Topological dimension, whatever you have defined, is a topological invariant for the class of separable meterizable spaces. We have not defined it for arbitrary spaces. That is one point you have to remember.

Thus, we may also derive the following weaker form of Brouwer's Invariane of domain(BID).

So, I am going, I am going to give that, namely: $\mathbb{R}^n$ and $\mathbb{R}^m$, if $n \neq m$, cannot be homeomorphic.

Because we have just shown that dimensions of $\mathbb{R}^n$ is n, dimension of $\mathbb{R}^m$ is m, but dimension is a homeomorphism invariant. This is a weaker form of Brouwer's invariance of domain.

Of course, the actual Brouwer's invariance of domain is the following: namely, take any two non empty open subsets of $\mathbb{R}^m$. Suppose, they are homeomorphic. Then, if one of them is open, then the other one is also open.

And that is why the name invariance of domain. The word domain was used more often than just open subsets in the older days. So, being open was same thing as being a domain. And that is an invariance. So, that is why it is called invariance of domain here. Unfortunately, we are not able to touch this one. We have come very close, but there is still a big gap here. So, the proof of this will take us much deeper into Dimension theory, which is beyond the scope of this course.

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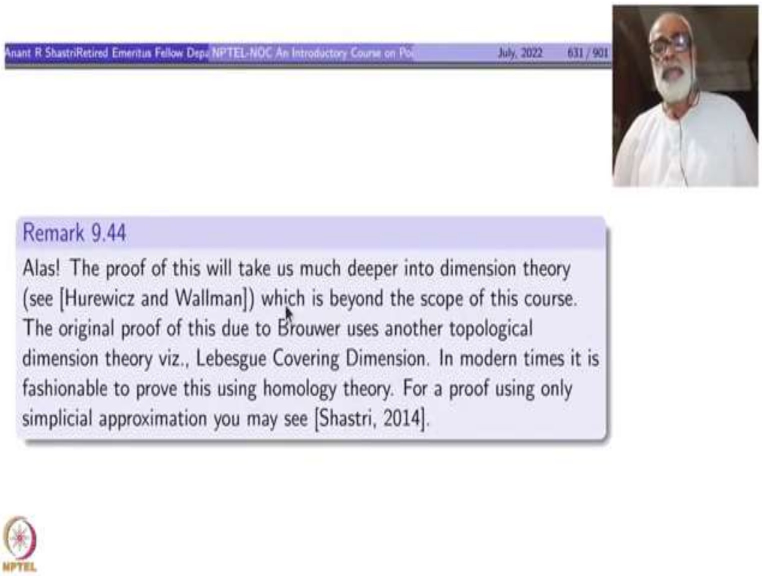

The original proof due to Brouwer uses another topological dimension theory namely Lebesgue covering dimension. In modern times, it is fashionable to prove this using homology theory. So, there are many proofs of this one, this great theorem, but a proof using only simplicial approximation, (part of which is there in the book of Hurewicz and Wallman, implicitly), you may see my book.

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So, here is an easy exercise for you. Deduce 9.42 from 9.43. Namely this, I said, is a stronger form I said. Why? So, you assume this a and prove this one. We have proved it using all our result in two different chapters which we developed so carefully. But assuming this, you prove this one. That is your exercise. So, next time, we will start a new topic. Thank you.