An Introduction to Point-Set-Topology (Part II)
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Lecture 43
Analytic Proof of Brouwer's Fixed Point Theorem

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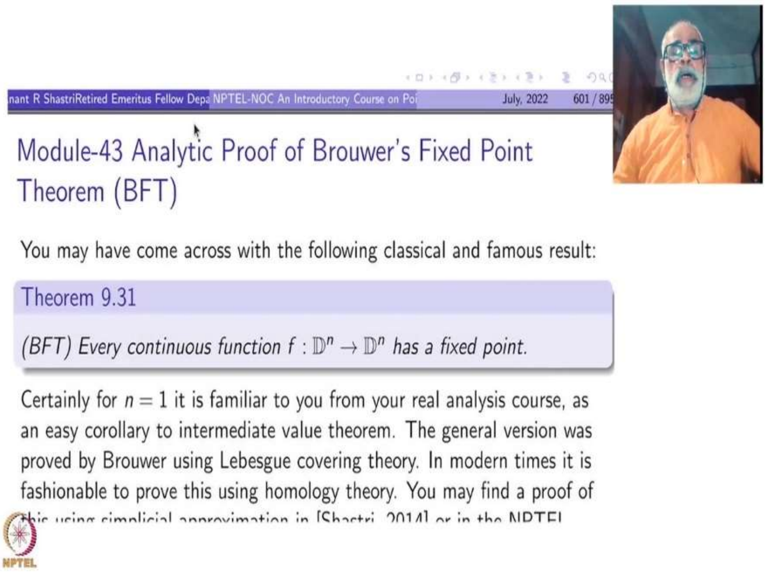

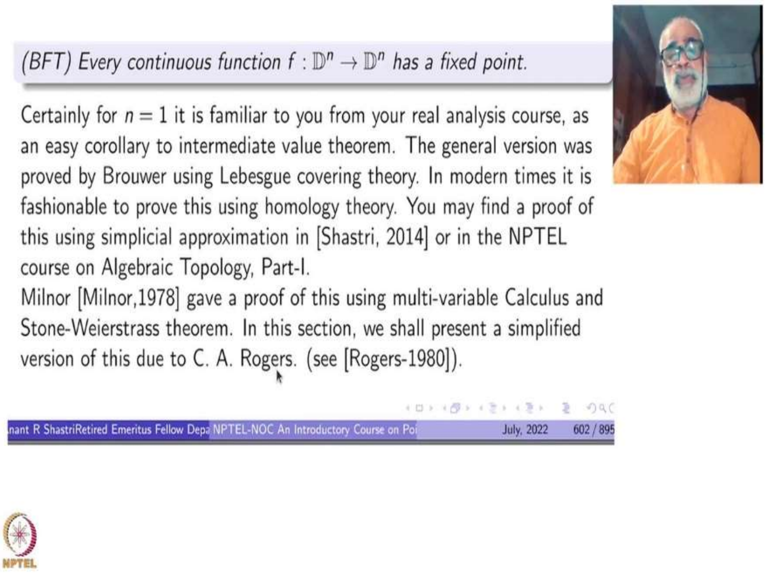

Hello, welcome to NPTEL NOC introductory course on Point-Set-Topology Part II, module 43. Today we will present a proof of Brouwer's fixed point theorem which you may term as an analytic proof. There are quite a few different proofs of this famous theorem. And this proof is due to Milnor. You may have come across with the following classical and famous result, which we call Brouwer's fixed point theorem.

Every continuous function from a closed disk $\mathbb{D}^n$ to $\mathbb{D}^n$ has a fixed point.

Certainly, for $n = 1$, viz., if you take the closed interval $[-1, 1]$, it is a familiar result to you from the real analysis course. Indeed, it is an easy corollary to the intermediate value theorem. Maybe you have done it. Any closed interval $[a, b]$ to $[a, b]$, if you have a continuous function, then it has a fixed point.

The general version was proved by Brouwer using Lebesgue covering theory. In modern times it is fashionable to prove this using homology theory. You may find a proof of this using simplicial approximation, in my book on Algebraic Topology, for example, or in the NPTEL course on Algebraic Topology Part I.

Milnor gave a proof of this using just multi-variable Calculus and Stone-Weierstrass theorem. In fact, Stone-Weierstrass theorem for functions defined on a closed and bounded subset of $\mathbb{R}^n$ is much easier than the generalized Stone-Weierstrass theorem that we have proved. So, in this section, right now, we will present a proof which is a simplified version of Milnor's proof, and that is due to C. A. Rogers.

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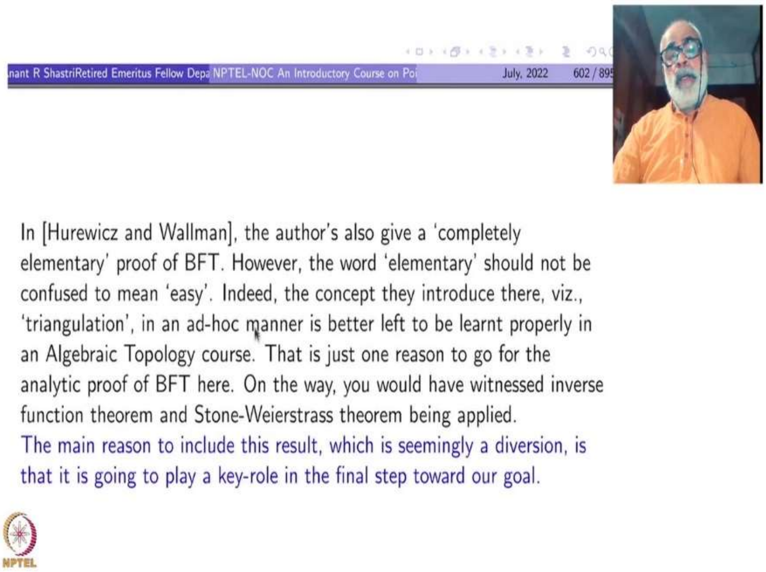

In Hurewicz and Wallman book, which I am refer to quite often, the authors also give a completely elementary proof, quote unquote, of the Brouwer's fixed point theorem. However, the word elementary should not be confused to mean easy. Indeed, the concept they introduced in order to prove this theorem, namely triangulation, (they do not use the word triangulation at all, but that is what they are doing), in an ad-hoc manner, is better left to to be learnt properly in an algebraic topology course. That is one reason which we are not very keen to present that proof here.

On the way, while learning Milnor's proof, you would have witnessed inverse function theorem and Stone-Weierstrass theorem being applied. So, that is also another motivation. So, both of these things, we have studied in this course itself. So, the main reason to include this result, which is seemingly a diversion, is that it is going to play a key-role in the final step toward our goal in dimension theory. So, that is why it is here.

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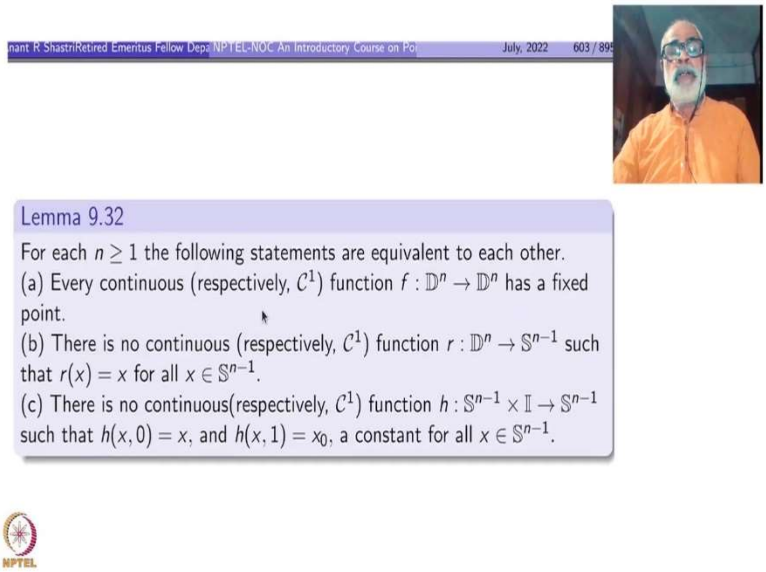

So, let us begin with earnest, this lemma, which gives you three equivalent conditions. All of them are equivalent to the statement of (not the conclusion) Brouwer's theorem.

(a) Every continuous (respectively, C^1) function f from $\mathbb{D}^n$ into $\mathbb{D}^n$ has a fixed point.

(b) There is no continuous (respectively C^1) function r from $\mathbb{D}^n$ into $\mathbb{S}^{n-1}$ (note that the codomain of the function has changed here, pay attention to that) such that $r(x) = x$ for all $x \in \mathbb{S}^{n-1}$. (Note that such a map r is called a retraction of the $\mathbb{D}^n$ to $\mathbb{S}^{n-1}$. We do not need that word. But just for your information, I am telling you that.)

(c) The third condition is that there is no continuous (respectively, C^1) function, h from $\mathbb{S}^{n-1} \times \mathbb{I}$ to $\mathbb{S}^{n-1}$ such that $h(x,0)$ is x and $h(x,1)$ is x_0, a constant for all $x \in \mathbb{S}^{n-1}$.

(The third condition just says that the identity map of $\mathbb{S}^{n-1}$ is not homotopic (respectively smoothly homotopic) to the constant function.

So, in each of these statements, I have put in the bracket C^1, C^1, C^1. So, in fact, this way we get one version in which you have taken C^1 functions everywhere. Another version, in which you have are taking just continuous functions. That is the meaning of this. These three are equivalent conditions. Notice that (b) and (c) are negative statements, whereas (a) is an assertive one. So, that is the beauty of this approach here. So, what we will do is we will prove that these three conditions are equivalent. Then, in order to prove (a) we can either prove (b) or (c). So, that is the whole idea.

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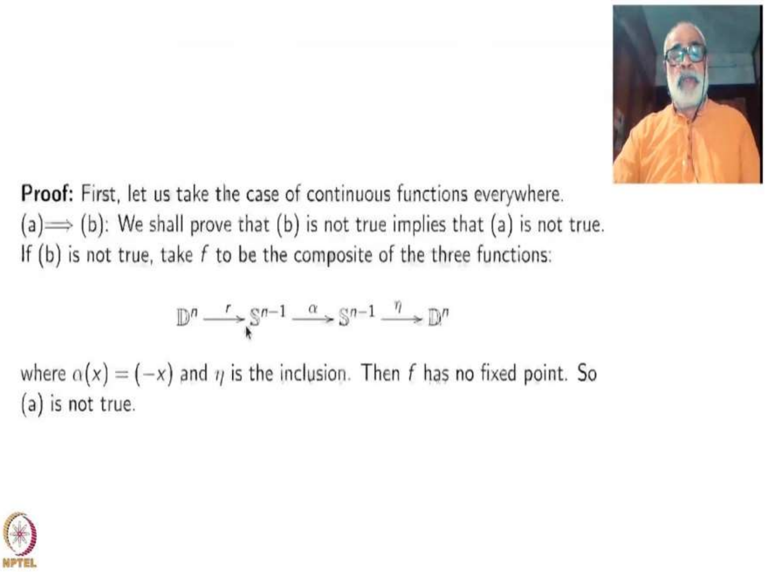

So, let us prove that (a) implies (b). Everywhere first, I am now going to take only continuous functions. Then I will indicate what to do when you have a C^1 functions.

So, assume that every continuous function from $\mathbb{D}^n$ to $\mathbb{D}^n$ has a fixed point. Now, in order to prove (b) we will assume (b) is not true and show that (a) is not true.

If (b) is not true, ((b) itself is in a negation), that means we have a retraction, namely, a continuous function r from $\mathbb{D}^n$ into $\mathbb{S}^{n-1}$, which is identity on the boundary. Take that map, compose it with this α which sends x to $-x$, anti-podal map. So, you are inside $\mathbb{S}^{n-1}$. Now you go back to $\mathbb{D}^n$ by the inclusion map, that is all. What happens? This function will never have any fixed point at all.

So, let $f = i \circ \alpha \circ r$, where i is from $\mathbb{S}^{n-1}$ to $\mathbb{D}^n$ is the inclusion map. If $f(x) = x$, first of all means that the point x, has to be on the boundary because $f(x) = (i \circ \alpha \circ r)(x)$. But for a boundary point r is identity, i.e, $r(x) = x$. But then it is followed by α which takes x going to $-x$. So, identity x goes to $-x$ here. So, $f(x)$ is $-x$, as well x that cannot happen. So, that is all.

So there are no fixed points for this composite function. So, that is a contradiction to (a). So that means not (b) implies not (a), which means that (a) implies (b).

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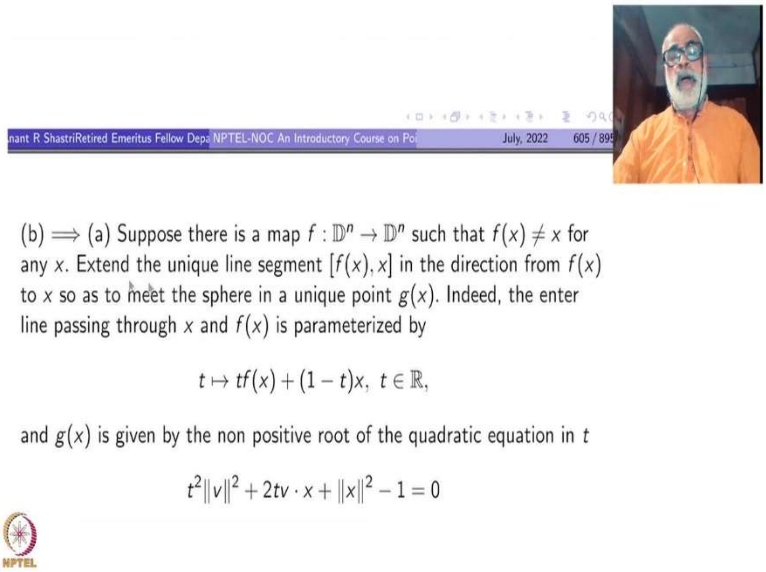

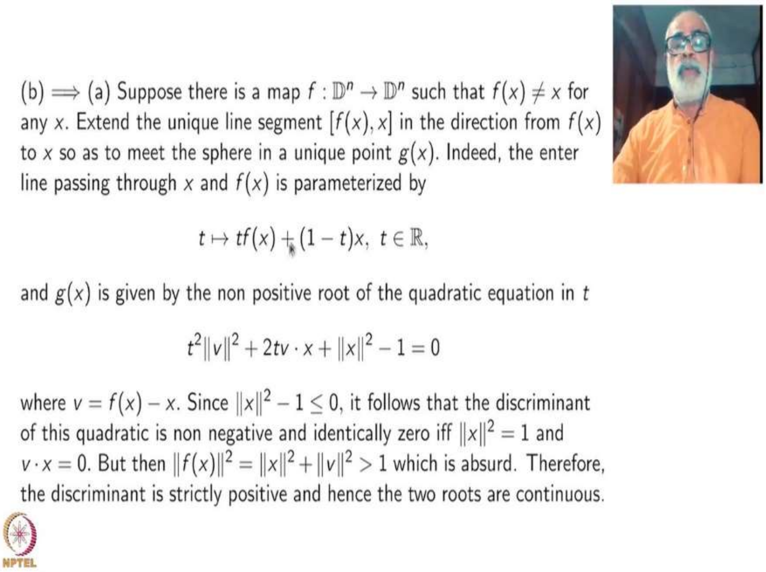

Now, I want to prove (b) implies (a). Suppose there is a map f from $\mathbb{D}^n$ into $\mathbb{D}^n$ such that $f(x)$ is never equal to x. Consider the unique line segment $f(x)$ to x. See, these are vectors inside $\mathbb{R}^n$. So, the line segment between two distinct point makes sense. But you take this line segment, extend it in the direction from $f(x)$ to x till you hit the sphere $\mathbb{S}^{n-1}$.

So, you will get a unique point on $\mathbb{S}^{n-1}$. call it $g(x)$. So, why this makes sense? Because the entire line segment is contained in the closed unit disk $\mathbb{D}^n$. Maybe it is strictly inside the unique disk. So, you will have to extend it. As soon as we extend it, because the entire line is unbounded, it will have to hit the boundary somewhere and the boundary is the sphere. So, this is just a geometric way of getting this function g.

In olden days, that was enough for people to understand that g is continuous. Even today, I can leave it as an exercise, but here because you may be seeing such things for the first time,

I will give you a full proof of why g is continuous. Just because, this f is continuous. So, how do we do that? Look at the entire line, passing through x and $f(x)$. So, what is the parameterization? It is $tf(x) + (1 - t)x$, as t varies over the real numbers $\mathbb{R}$.

So, this right-hand side is, they are all points inside this line. Now, we want a point on the intersection of the sphere and the line and the should be beyond the line segment on the side of the point x. Maybe it is x itself, or beyond x. Therefore, I have to take the non-positive root of the quadratic equation in t to arrive at this point. Why quadratic equation? Norm of the right-hand side must be equal to 1. Norm squared equal to 1 will give you a quadratic equation.

So, norm square, when you write, I am rewriting it this way. $t^2\|v\|^2 + 2tv + \|x\|^2 - 1 = 0$. Here v is a short form for $f(x) - x$, the vector $f(x) - x$. You can check that. Here t is the variable, x is fixed, therefore $f(x)$ is fixed, therefore v is fixed. So, this is a quadratic in t with some coefficients $\|v\|^2$, $2v \cdot x$ and the constant term is $\|x\|^2 - 1$.

Now, where does x belong to? x is inside the closed ball $\mathbb{D}^n$. Therefore, $\|x\|^2$ is less than or equal to 1. So $\|x\|^2 - 1$ is less than or equal to 0. It may be negative. It follows that the discriminant of this quadratic is non-negative and identically zero, if and only if this $\|x\|^2$ is equal to 1. Not only that, identically 0 means the other one, this coefficient $v \cdot x$ must be also 0.

$v \cdot x$ is 0 mean what? x and v are perpendicular to each other. So, therefore, what we have is, norm of $f(x)^2$ will become norm of $x^2 + \|v\|^2$, which is bigger than 1, which is absurd. So, therefore, the discriminant is strictly positive, and hence the two roots are continuous. So, what I am saying, discriminant is non-negative and identically zero if and only if this happens.

If this happens, there will be a problem. So it must be strictly positive. That means the 2 roots are continuous functions of x. Whenever something is 0 when the roots are equal and so on, there will be a problem about continuity, which root is chosen and so on. So, here the discriminant is strictly positive. So there is no problem about that. So, continuity of the roots follows. What we have to take? We have to take the non-positive root of this. And that will give you the required solution, which is a continuous solution. So, g will be in terms of that, because put that value t, this right-hand side will be $g(x)$.

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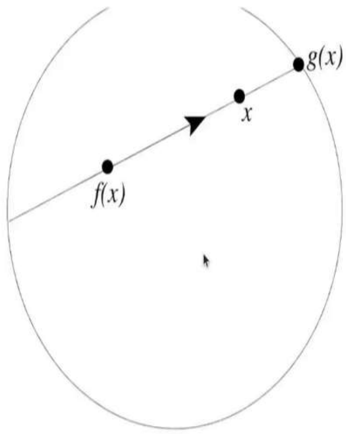

Figure 11: The sphere is not a retract of the disc

So, here is the picture. (Proof is completed for many people by the picture itself.) $f(x)$ is here, x is here, I am extending the line segment towards $f(x)$ to x and get $g(x)$. By chance, if x is already on the boundary, that $g(x)$ is equal to x. So, that is the beauty of this construction. This g is serves as the required function r in our statement. That is the whole idea. So, we have completed the proofs of (a) implies (b) and (b) implies (a).

So, here again, what we have proved? Not (b) implies not (a). If there is a map like this, not (a) implies not (b). Earlier, not (b) implies not (a), not (a) implies not (b), that is what we have proved.

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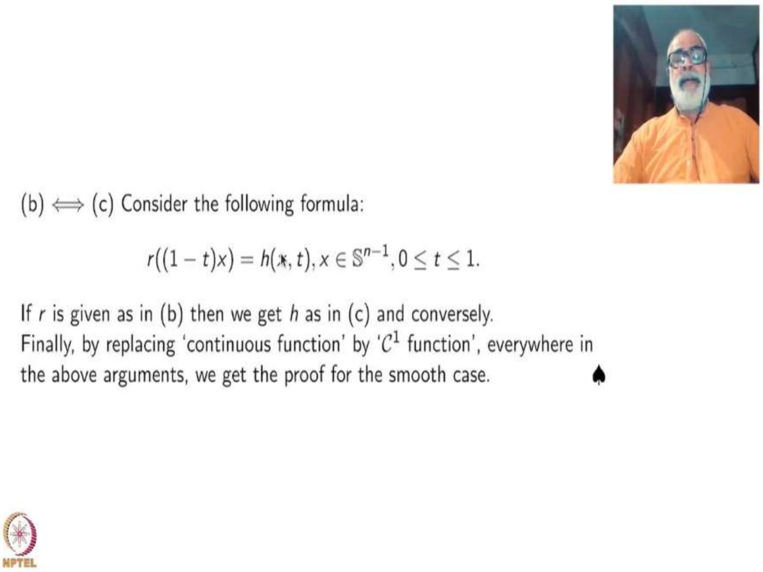

Now equivalence of (b) and (c) are very easy. Look at this formula $r((1-t)x) = h(x,t)$, where x is on $\mathbb{S}^{n-1}$ and t is between 0 and 1.

So, $1-t$ is also between 0 and 1. So, this is an element of the closed disk, x is an element of the boundary sphere. r times this one will make sense for all functions defined on $\mathbb{D}^n$. I am just putting it equal to $h(x,t)$. On the other hand, if I know $h(x,t)$, I could have defined r by this formula.

So, this is the formula which you will use to define either side, if you know the other side. If r is given as in (b) then h will be a continuous function as in (c). If r is continuous iff h is continuous. continuous. $h(x,1)$ is always the constant $r(0)$. $h(x,0)$ is x iff $r(x) = x$. h is of course taking values in $\mathbb{S}^{n-1}$. Therefore, r will always take values in $\mathbb{S}^{n-1}$ and vice versa.

In fact, they are analytic functions. Roots of a polynomial wherever, they are positively defined, strictly defined, they are analytic functions. So, that will take care of that. Wherever I have, so here also, alpha is also a smooth function, x is equal $-x$, is the inclusion map. So, if this is C^1, the composite C^1, so, what we get? f will be C^1 and so on.

So, you have r a retraction, iff h will satisfy that property viz., a homototy from the identity map of $\mathbb{S}^{n-1}$ to a constant. Therefore, this one formula proves both (b) implies (c) and (c) implies (b).

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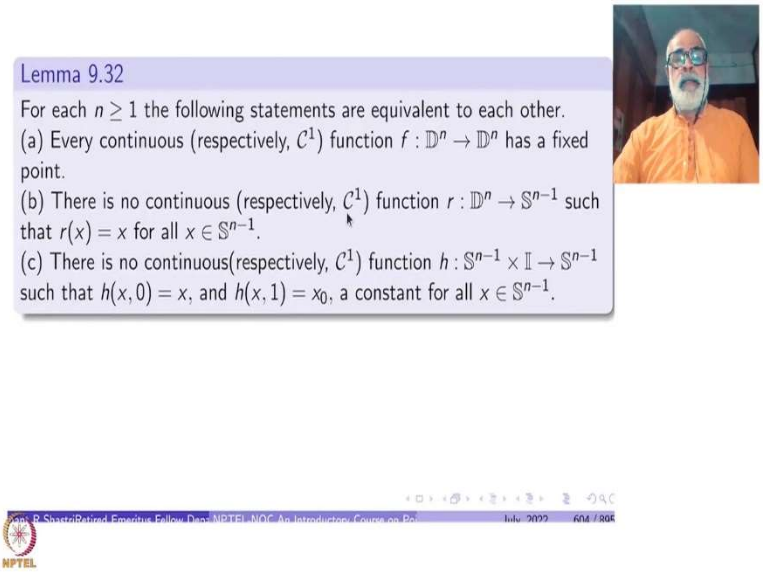

Now, comes the part wherein we take everywhere C^1 functions. If this r is C^1 then h is and vice versa. So, there is no problem in (b) implies (c). Here what happens? If the original function viz., f is a C^1 function, these roots are also C^1 functions. Infact roots always analytic functions. Roots of a polynomial wherever, they are defined, strictly defined, they are analytic functions. α is linear function and hence analytic. So, that will take care of that and hence the composite function g will be a C^1 function.

By the way, even if you just proof C^1 case here, you have to argue for continuous case separately, they have to be done separately. There is no other choice here. One does not imply the other. You have to do. both. Go through the whole thing separately, but that is easy anyway.

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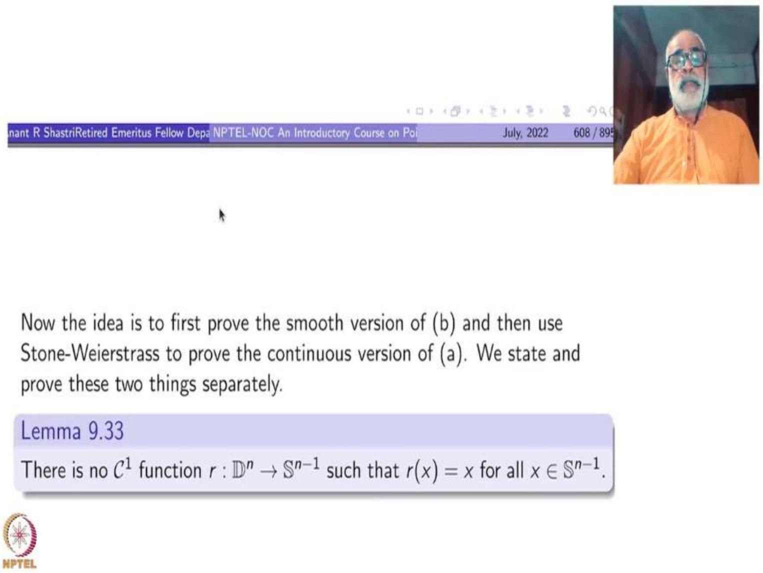

So, having prepared the ground for the general theorem now, the idea is to prove the smooth version of (b) and then use Stone-Weierstrass to prove the continuous version of (a). You see, you know this theorem, we refer to it like that. We state and prove these two things separately, no confusion. So, there is no C^1 function r from $\mathbb{D}^n$ to $\mathbb{S}^{n-1}$ such that $r(x)$ is equal to x for all x belonging to $\mathbb{S}^{n-1}$.

Now, this is an assertion. Earlier this was just a statement equivalent to two other statements. Now, we are proving this one. So, what is the proof?

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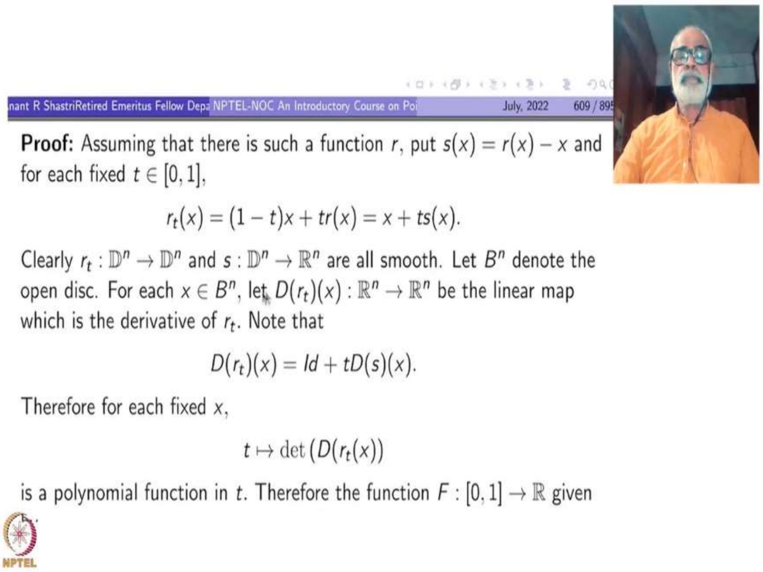

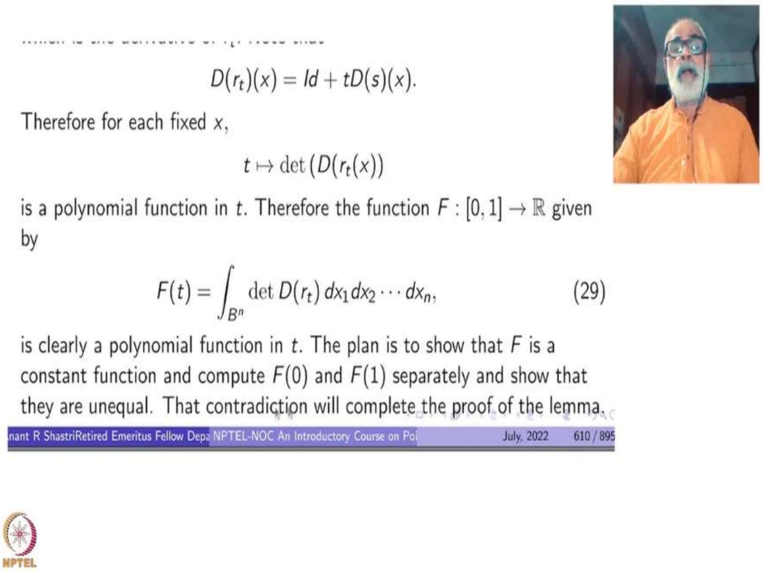

Assuming that there is such a function, let us tentability put $s(x) = r(x) - x$. And for each $t \in [0,1]$, let us put $r_t(x)$ equal to $(1-t)x + tr(x)$. So, joining the identity map and $r(x)$. This can be rewritten as $x - tx + tr(x)$ and rewrite the last two terms as $ts(x)$ because $s(x)$ is

$r(x) - x$. Just look at this formula. Each r_t is from $\mathbb{D}^n$ to $\mathbb{D}^n$ because r is from $\mathbb{D}^n$ to $\mathbb{S}^{n-1}$, x is identity. This is line joining the two points, you see. So, it will, convexity of $\mathbb{D}^n$ will tell you that is always inside $\mathbb{D}^n$. However, this difference $s(x)$ may go out also, you do not know where its values are.

So, this s is from $\mathbb{D}^n$ into $\mathbb{R}^n$ any way. Because I started with r as a smooth function, all these are smooth functions. Now, let us have a notation here B^n, you know, be the open disk, namely, set of all points x for which $\|x\| < 1$. For each fixed x inside B^n, take the total derivative of r_t at the point x, that is a linear map from $\mathbb{R}^n$ to $\mathbb{R}^n$. Note that the total derivative of r_t at any point is the identity map of $\mathbb{R}^{n+t}$ times the derivative of s at the point x. This is always a linear map from $\mathbb{R}^n$ to $\mathbb{R}^n$.

So, $D(r_t)$ is identity plus $tD(s)$. So, these derivatives are taken with respect to x. You may wonder what is happening to the variable t. Note that t is frozen here, in the notation r_t which is one single map for each t.

For each fixed x now, you look at the function t going to the determinant of this linear map, $D(r_t)(x)$, which is a linear map from $\mathbb{R}^n$ to $\mathbb{R}^n$. So, you can talk about the determinant of $D(r_t)(x)$, x is fixed now, one linear map is there, look at the determinant, but t is there. So the determinant of this identity plus some map $D(s)$, it will be a polynomial in t maybe of possible degree n. So t occurs in each entry here.

So, this is a polynomial function. Therefore, if I define the function F from the closed interval $[0, 1]$ to $\mathbb{R}$ by taking the integral of this function determinant of $D(r_t)(x)$, integrated on the entire of this open ball B^n. Remember that this function is a continuous function because we started a C^1 function and then we have taken one derivative, so, derivative is a continuous function. So this is just ordinary Riemann integration, an iterated integral, you may say.

So, there is a variable t involved here. Therefore, this will become a polynomial in t, because integrand is a polynomial in t. This polynomial, we are going to show that, is a constant function. What is that? Integral of the determinant function with respect to x, is a constant function of t. Then we will compute $F(0)$ and $F(1)$ separately and show that they are unequal. That is a contradiction. You started with some assumption and arrived at a contradiction, so the assumption must be wrong. That means there is no such function r. So that will complete the proof. So, now we have to compute $F(0)$ and $F(1)$.

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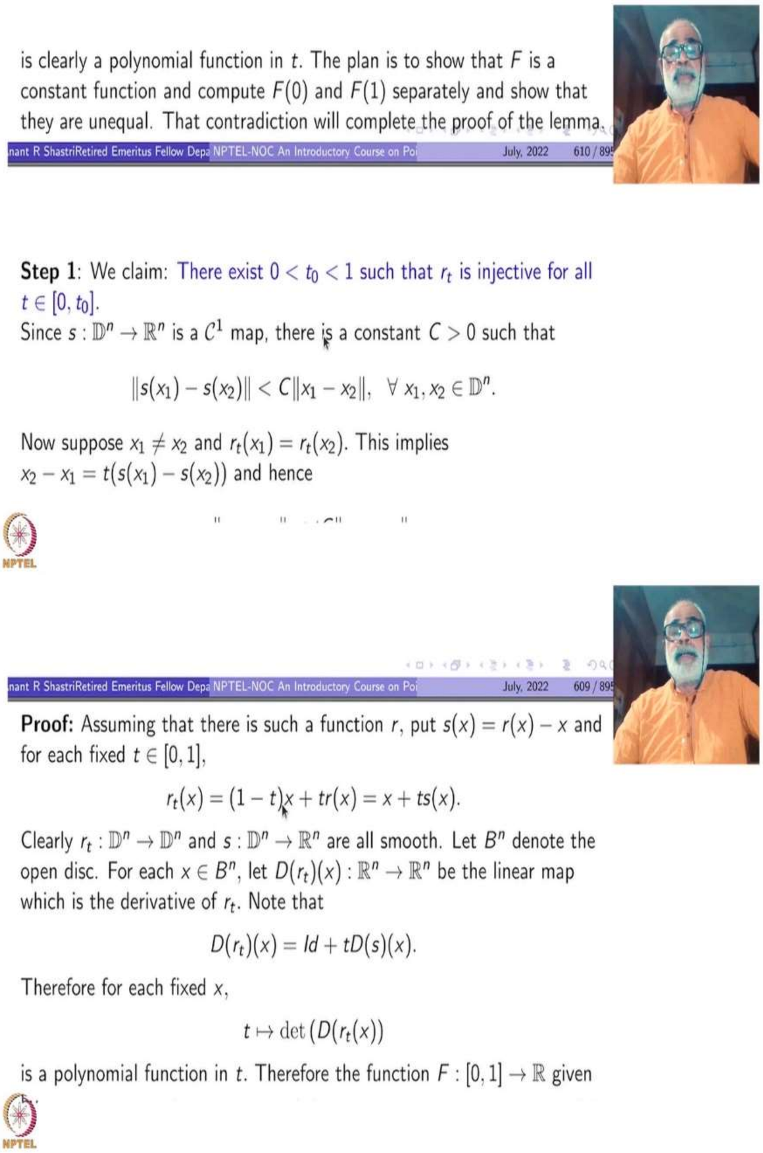

So, first claim is, there exists $0 < t_0 < 1$ such that this r_t is injective for all t between 0 and t_0 Look at $t = 0$, you have no control what is r_0? r_0 is just x. There, you are lucky. It is injective. So, we now say that there is some positive t_0, there is an interval of positive length $[0, t_0]$ here in which r_t is injective for all t. Look at the function s from $\mathbb{D}^n$ to $\mathbb{R}^n$, this is a C^1 map. Therefore, there is a constant c positive such that this condition holds, viz., $\|s(x_1) - s(x_2)\| < c\|x_1 - x_2\|$. This is true because $\mathbb{D}^n$ is compact, and s is C^1. So, we looked at the derivative, take the maximum of the derivative, its norm can be taken as C, that is the general statement here. This is some calculus. Now, suppose $x_1 \neq x_2$ and $r_t(x_1) = r_t(x_2)$. Remember, we have this formula right? $r_t(x)$ is $x + ts(x)$. Therefore, that will imply that $x_2 - x_1$ is equal to $ts(x_1) - s(x_2)$ because $r_t(x_1) = r_t(x_2)$. But then, $\|x_2 - x_1\|$ will be less than t times norm of this, where t some positive constant in $[0, 1]$. And Norm of this one is less than $C\|x_1 - x_2\|$.

But then, these two numbers are the same here. And there $x_1 \neq x_2$, This is a non-zero number. Non-zero number is less than or equal to some number C times the same number means this C must be bigger than 1. Once it is bigger than 1, I can choose $1/C$ as t_0. And then for $t < t_0$, we will, now have this. That is all. So, this inequality, this can happen only beyond this t_0, that is why we take something less than equal to t_0 this will never happen. That means that r_t is injective.

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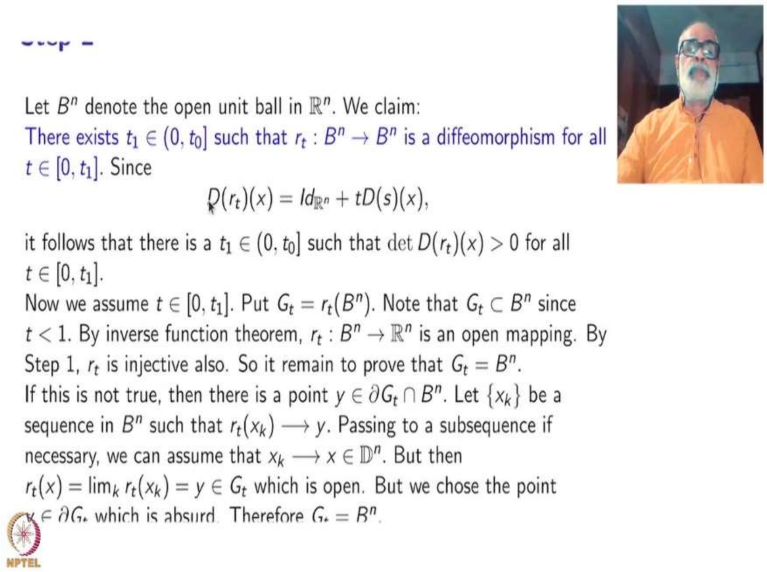

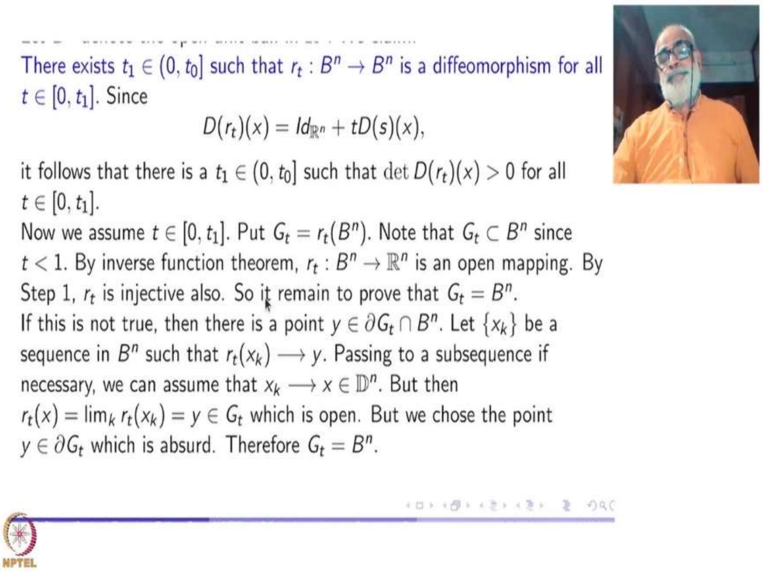

Now, let us go ahead. We claim that there exist a t_1 now, another constant, in the half open interval $(0, t_0]$, this time we do not want to go beyond of t_0, such that this function r_t from B^n to B^n on the open disk, is a diffeomorphism, for all t belonging to $[0, t_1]$. So, this is a claim now. It may not happen for all $t \in [0, t_0]$. Once you have found such a t_1, then you can actually take t_0 also equal to t_1. So, you have to find such a t_1, which is positive that is the whole idea.

Now, look at $D(r)(x)$ which we have seen is the identity of $\mathbb{R}^n$ plus $tD(s)(x)$. Therefore, it follows that there is a t_1 belonging to $(0, t_0]$, such that the determinant of $D(r_t)(x)$ is positive for all $t \in [0, t_1]$. Then I choose $t_1 < t_0$ such that the determinant of $D(r_t)(x)$ is positive for all $t \in [0, t_1]$. Because when you put $t = 0$, what is the determinant of $D(r_t)(x)$? It is determinant of the identity map, which is 1, by continuity of the derivative, as a function of t, determinant of $D(r_t)(x)$ must be positive in a small interval around 0.

Now, assume that t belongs to $[0, t_1]$. (Now, t_1 has been chosen. So, I say, this is good enough for getting the diffeomorphism.) Put G_t equal to r_t of B^n. Remember r_t is just a smooth map from $\mathbb{D}^n$ to $\mathbb{R}^n$, but we want to say that the image of the interior of $\mathbb{D}^n$, viz., $r_t(B^n)$ is contained in B^n. Why? Because $t < 1$. $(1 - t)x + ts(x)$ the convex combination of x with t belong less than 1.

By inverse function theorem, r_t from B^n to $\mathbb{R}^n$ is an open mapping. This is very important here. It is an open mapping into the whole of $\mathbb{R}^n$. By step I, r_t is injective also, because now we are taking t_1 to be less than t_0.

So, it remains to prove that G_t is equal to whole of B^n. That is only thing which we need to say that r_t is a diffeomorphism. So far what we have observed is that it is injective and open mapping, with the determinant of its derivative being positive. And so it is diffeomorphism onto G_t. So, if we show G_t is actually B^n, you are done.

Suppose G_t is not equal to the whole of B^n, we have already observed that G_t is inside B^n. So, if some open subset is not the whole space, then, there is a point y which is in the boundary of G_t but is inside B^n. (B^n is after all convex set. So, you can take a point inside G_t and a point in $B^n \setminus G_t$, the line segment joining them will intersect the boundary of G_t in some point. You take that point to be y.) Therefore, you can have a sequence $\{x_k\}$ inside B^n such that $r_t(x_k)$ tends to y.

Passing to a subsequence, if necessary, we can assume that $\{x_k\}$ itself converges to some point x inside $\mathbb{D}^n$ because $\mathbb{D}^n$ is compact. But then $r_t(x)$ is the limit of $r_t(x_k)$ which is equal to y. Because r_t is a smooth function actually, so it is continuous also.

If x is inside B^n then this means y is inside G_t which is open. But we have y in the boundary of G_t. So x is not in B^n. So, x must be in $\mathbb{S}^{n-1}$. But then $x = r_t(x) = y$ which is inside B^n. That is absurd. Therefore, G_t is equal to B^n.

So, in two steps, we have proved that we have located a positive number between $(0, 1]$ such that for all points t less than that r_t is diffeomorphism from B^n onto B^n.

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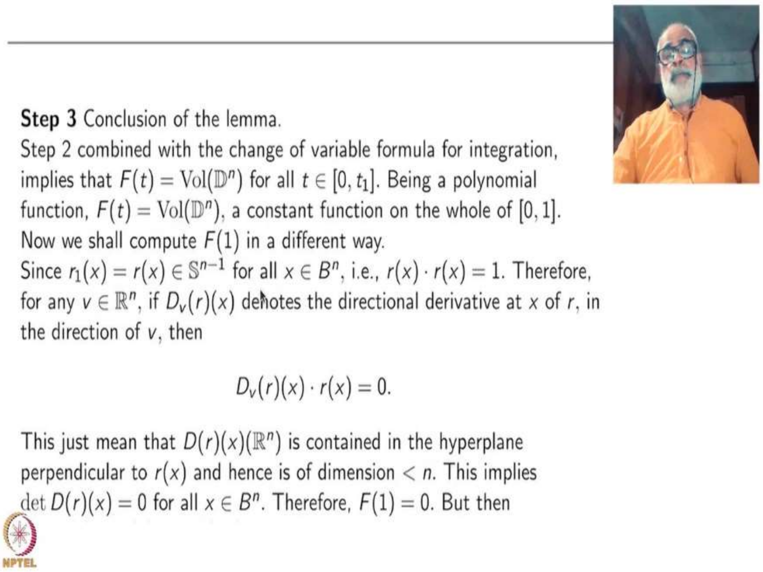

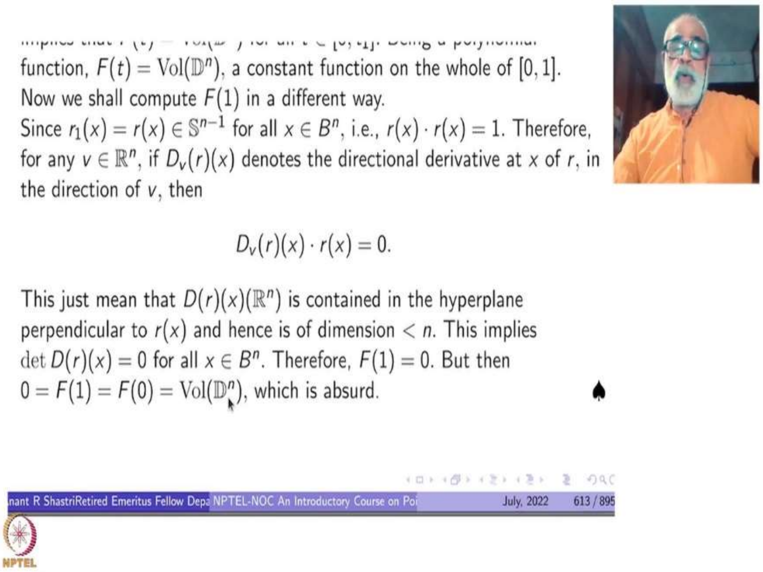

Now, we can conclude the lemma. Step two combined with a change of variable formula for integration implies wherever r_t is a diffeomorphism for all those t, $F(t)$ must be the volume of the whole of B^n (or of D^n, they are the same). Why? Look at this integrand. This is the determinant of $D(r_t)$. For $t \in [0, t-1]$, it is positive.

So, whenever you have an invertible function, a diffeomorphism, you have the change of variable formula. (So, we have used some good analysis here.) So, the volume of B^n is given by $F(t)$ for $t \in [0, t_1]$. But that is a constant. So, what is the meaning of this one? In this nonempty open interval, a polynomial function $F(t)$ is a constant. Therefore, this $F(t)$ must be equal to volume of this $\mathbb{D}^n$ on the whole of $[0, 1]$, in fact wherever it makes sense.

So, we have computed, in particular $F(0)$ which is the volume of B^n. Now, we shall compute F_1 in a different way and see that it is a different value, and that is a contradiction.

Look at $r_1(x)$. That is equal to $r(x)$ which is in $\mathbb{S}^{n-1}$ for all points inside B^n. Therefore, if you look at $r(x) \cdot r(x)$ that is equal to 1. that is the same thing as saying $r(x)$ is of unit length. Therefore, for any vector $v \in \mathbb{R}^n$, if you take the derivative of this equation in the direction of v, you get twice $D_v(r) \cdot r(x)$ equal to 0.

The directional derivative of the LHS in the direction of v is nothing but $D_v(r)(x) \cdot r(x) + r(x) \cdot D_v(r)(x)$. That is by the Leibniz rule. And that RHS is 0.

What does this mean? The entire image under this linear map, $D(r)(x)$ from $\mathbb{R}^n$ to $\mathbb{R}^n$, is perpendicular to $r(x)$. All the directional derivatives are all perpendicular to $r(x)$. Therefore, this entire linear space is perpendicular to $r(x)$, it is contained in the hyperplane perpendicular to $r(x)$. But that is of dimension less than n. So, this implies that the determinant of $D(r)(x)$ must be zero, because its rank is $n-1$ at most. It may be smaller also, I do not care. This happens for all $x \in B^n$. Therefore, the integral $F(1)$ will be 0, because on the right-hand side the integrand itself is 0. Therefore $F(1)$ is 0. Since F is a constant, we get an absurd result that the volume of B^n is zero!

Now, I shall leave the proof of the continuous version version of the lemma to you as an exercise, with the hint to use Stone-Weierstrass.

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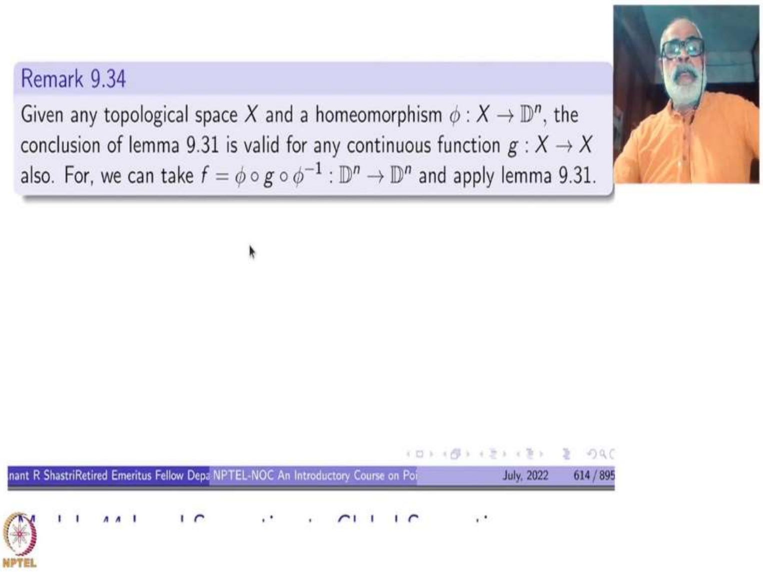

Given any topological space X and homeomorphism phi from X to $\mathbb{D}^n$, the conclusion of lemma 9.31 is valid for any continuous function g from X to X also, instead of C^1 functions from $\mathbb{D}^n$ to $\mathbb{D}^n$. For, we take f equal to $\phi \circ g \circ \phi^{-1}$, from $\mathbb{D}^n$ to $\mathbb{D}^n$ and apply theorem lemma 9.31. So, what I have done is I have just proved the C^1, C^1 version here. Now, you can apply Weierstrass theorem to get the continuous version.

As soon as any one of the three statements gets proved all the three statements get proved. In particular Brouwer's fixed point theorem, that is proved.

So, thank you.