An Introduction to Point-Set-Topology (Part II)

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Lecture 42

Sum Theorem for Higher Dimensions

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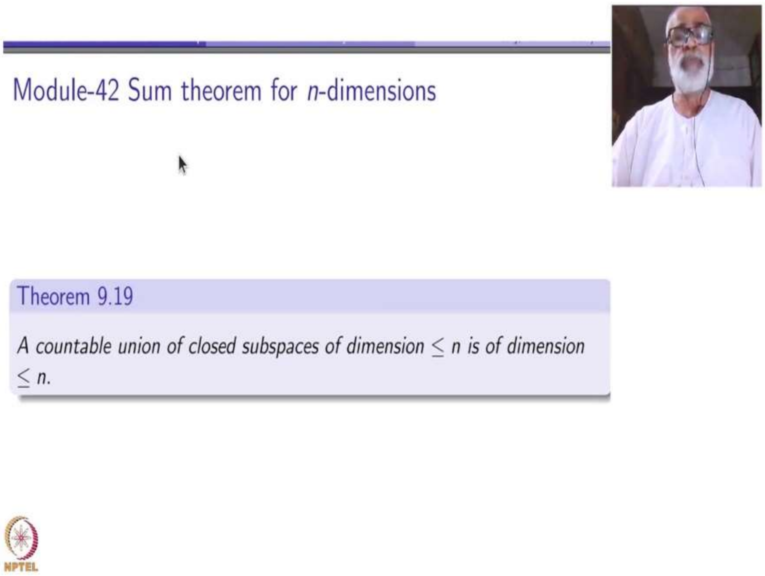

Welcome to NPTEL NOC on Point-Set-Topology Part II, module 42. So, we will now do some sum theorems for n-dimensions. The spirit is similar to what we did last time. Only, we are continuing getting stronger and stronger results here. A countable union of closed subspaces of dimension less than or equal to n is of dimension less than equal to n. In other words, taking countable union of closed subspaces does not increase the dimension. That is what one has to understand. So, I have a number of subspaces, where are they? They are all subspaces of a single separable metric space.

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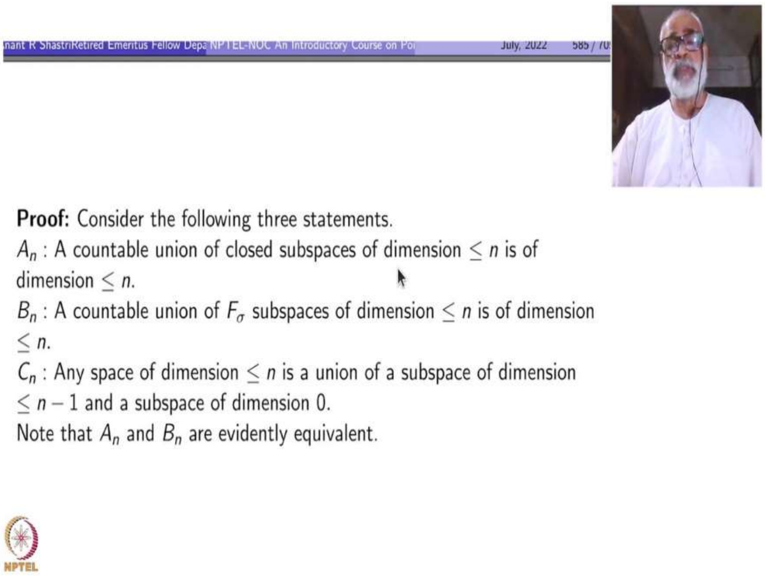

So, consider the following three statements A_n, B_n, C_n.

First A_n says that a countable union of closed subspaces of dimension less than or equal to n is of dimension less than or equal to n. This is just the statement of the above theorem which we want to prove. But we will make some more subsidiary statements here.

B_n says that a countable union of F_σ subspaces of dimension less than equal to n is of dimension less than equal to n. From closed subspaces, we have improved the statement slightly, to F_σ subspace. F_σ are countable union of closed sets, not necessarily closed themselves.

The third one C_n says any space of dimension less than or equal to n is a union of two subspaces, one subspace of dimension less than equal to $n-1$ and another subspace of dimension 0.

So, here nothing is said about the subspaces being closed, that is important here.

I said that B_n looks like an improvement on A_n. But it is not an improvement, they are the same. Because each F_σ is itself a countable union of closed sets, and then you are taking countable union of F_σ's so, that will be also a countable union of closed sets.

So, A_n and B_n are easily seen to be equivalent, but C_n is a somewhat strange thing here. Let us see what is the role of this C_n here.

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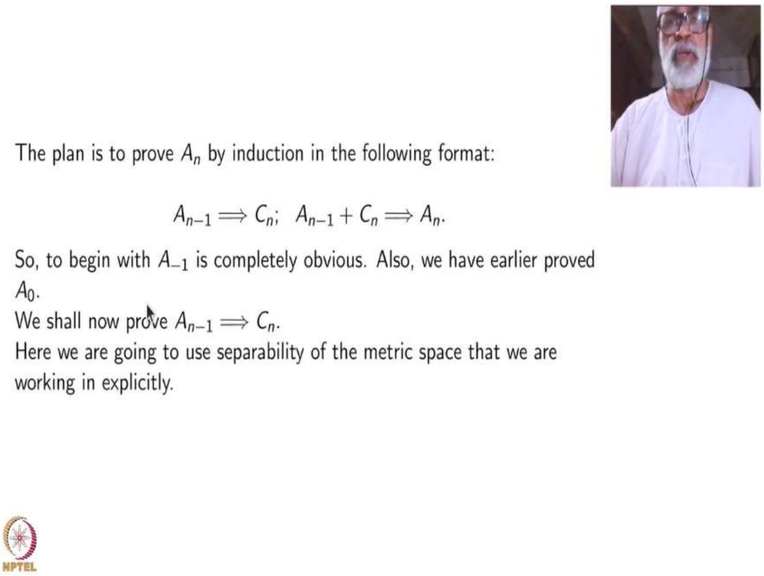

We have to prove A_n, by induction. How do we do this one? Here is a plan. We shall first prove that A_{n-1} implies C_n. Then we will use the induction hypothesis A_{n-1} and C_n to prove A_n. The statement B_n is only subsidiary here, it will play a subsidiary role. So, this is the plan.

So, to begin with, A_{-1} is completely obvious. If all spaces are of dimension minus less than equal to -1, so, they are all empty, countable union of empty sets is again empty and hence is of dimension -1. There is nothing to prove there. So, our induction starts at $n = -1$.

Also, we have earlier proved A_0 itself. Remember that. I told you, today, we are not going to prove any new phenomenon or anything. Same phenomenon we are using for higher dimension and so on.

So, countable union of closed subspaces of dimension 0 is of dimension 0. Actually less than equal to dimension 0 is what we have proved earlier. Equality follows easily. I do not have to actually prove this one because my induction starts with $n = -1$ and $n = 0$.

So, we should now prove that A_{n-1} implies C_n. Here, we are going to use separability of the metric space that we are working in explicitly.

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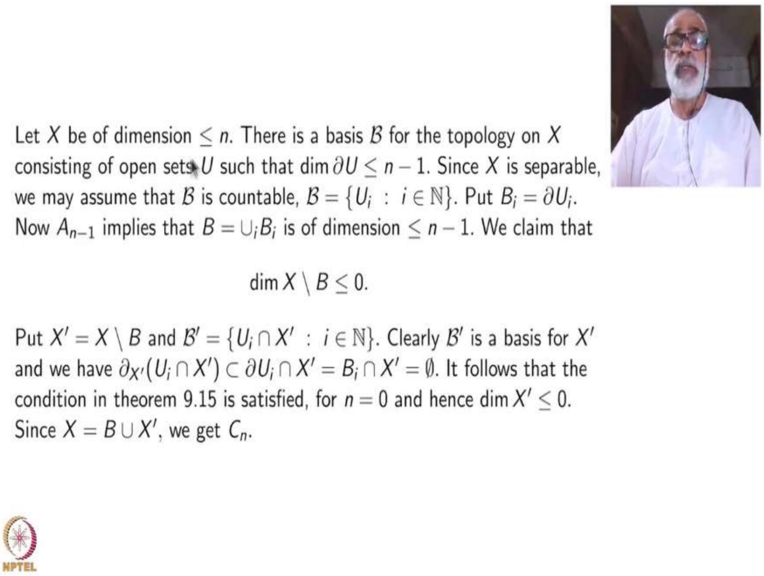

Let X be of dimension less than equal to n. That means there is a basis $\mathcal{B}$ for the topology on X consisting of open sets U such that dimension of the boundary of U is less than equal to $n-1$ for all U inside $\mathcal{B}$. Since X is separable, we may assume that $\mathcal{B}$ is countable. So, this is the role of separability here. So, I am just enumerating here $\mathcal{B} = \{U_i : i \in \mathbb{N}\}$.

Now, put B_i equal to boundary of U_i. All of them have dimension less than equal to $n-1$ by the choice of $\mathcal{B}$ here.

Now, the induction hypothesis A_{n-1} implies that you take B to be the union of all these B_i's, that is of dimension less than $n-1$.

We claim that the complement of B in X, namely dimension of $X \setminus B$ is less than or equal to 0.

So, how would I prove that? Take X' equals $X \setminus B$, this is just a temporary notation. And $\mathcal{B}'$ be the family of all $U_i \cap X'$. Clearly, because X' is a subspace of X, $\mathcal{B}'$ is a basis for X', in the subspace topology. And if you take $U_i \cap X'$ and its boundary in X', that is contained in the boundary of U_i in X intersected with X'. So it is just $B_i \cap X'$, and $B_i \cap X'$ are all empty. Why? Because X' is $X \setminus B$, all the B_i's have been thrown out.

So, it follows that the condition in earlier theorem is satisfied, namely, for $n = 0$. Therefore, dimension of X' is less than equal to 0. So, this is where we have used A_0. So, since X is B union X', we get C_n.

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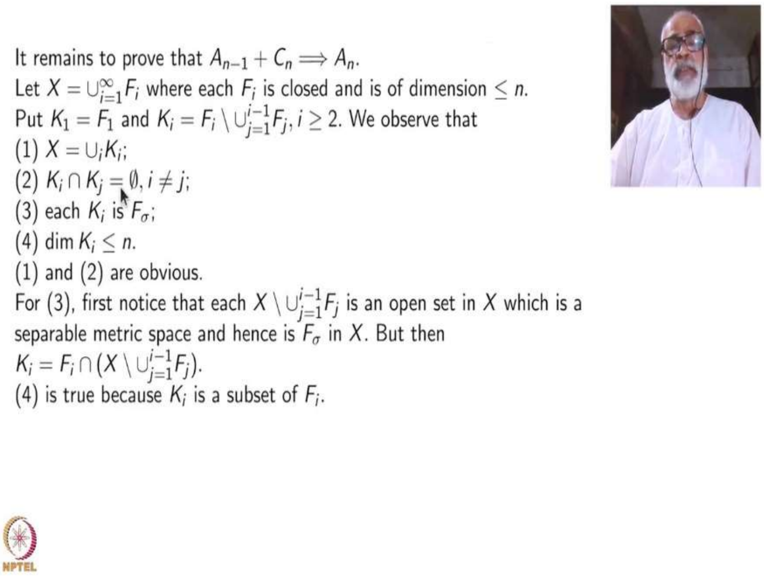

Now, it remains to prove A_{n-1} and C_n together implies A_n. That will complete the proof of the theorem.

So, start with X as a union of countably many closed subspaces F_i where each F_i is closed and dimension of each F_i is less than equal to n. So, inductively, let me define these K_i's: K_1 is F_1, K_2 is $F_2 \setminus F_1$, K_3 is $F_3 \setminus (F_1 \cup F_2)$, and so on. Throw away all the earlier sets F_j's from F_i to get K_i.

Then

(1) X is union of K_i's also. This kind of things you have seen several times.

(2) $K_i \cap K_j$ is empty for all $i \neq j$. This is the extra property we get, whereas that property is not there for the family $\{F_i\}$. By construction, our K_i and K_j will all be mutually disjoint.

(3) Each K_i is F_σ. Now, I cannot say that K_i is closed. Each K_i is F_i setminus some closed set and so, it is an open subset inside a closed subset. All spaces are metric spaces here. Any open subset of a metric space is F_σ. So, each K_i is F_σ.

This is where our B_n will play the role. That is all.

(4) So, dimension of K_i is less than or equal to n, because they are subspaces of F_i.

So, I have stating and explained these four properties of the family $\{K_i\}$, (1) to (4).

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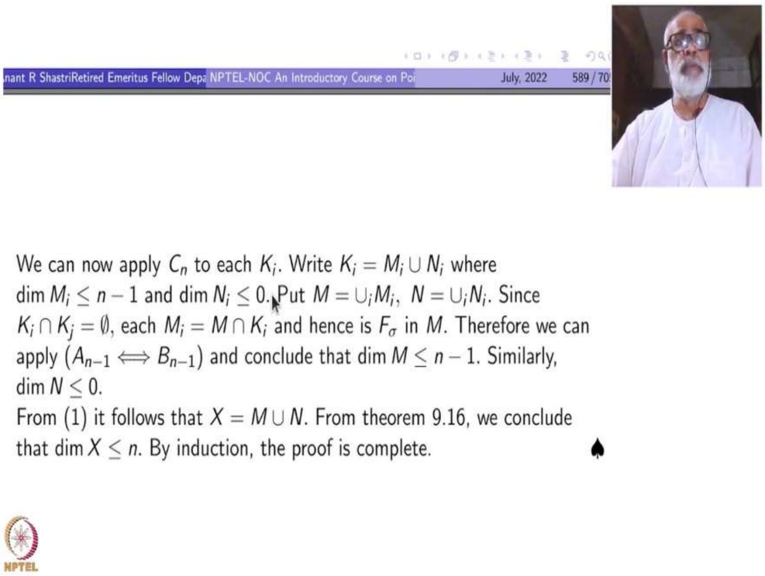

Now, we can apply C_n to each K_i because of (4). Write $K_i = M_i \cup N_i$, where dimension of M_i is less than $n-1$ and dimension of N_i is less than or equal to 0. Put M equal to the union of these M_i's, and N equal to union of N_i's. Since $\{K_i\}$ are all mutually disjoint, for each i, $M_i = M \cap K_i$, so, each of them is F_σ inside M. Therefore, we can apply A_{n-1} which is the same as B_{n-1} and conclude that dimension of M is less than or equal to $n-1$.

Similarly, each of these N_i dimension 0. Therefore, dimension of N, N being a countable union is less than equal to 0. From (1), it follows that X is $M \cup N$. What is M? Each K_i is written as the union of M_i and N_i, upon taking the union over i, we get X is $M \cup N$. From 9.16, whatever theorem, you conclude that dimension of X is less than equal to n now.

So, the inductive proof is completed. From a union of two subsets, we have proved the same thing for countable union.

Next, we relax the condition of closedness.

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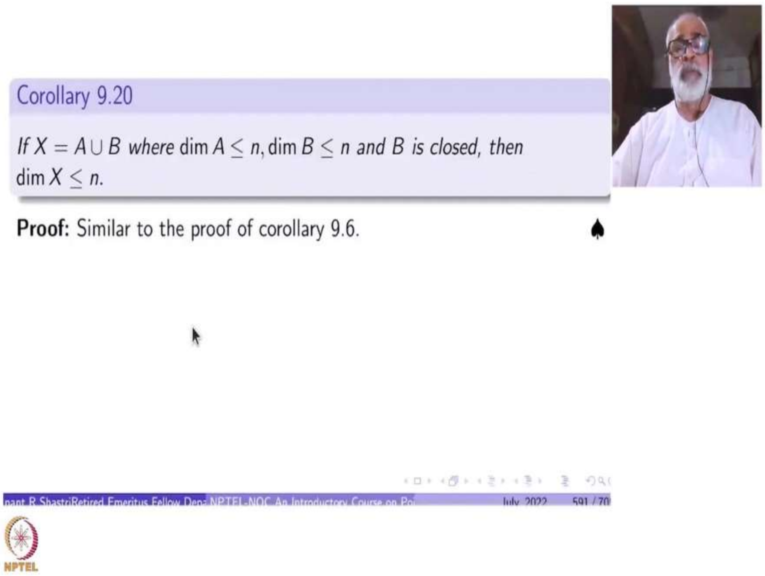

If X is a union of two subsets, each of them of dimension less than or equal to n, and one of them is closed, then dimension of X is less than equal to n.

Same kind of result. Instead of dimension 0, we are now working with dimension n. It is similar to what we proved in 9.6. Convert the non closed set into a countable union. We do not convert that itself, instead, take the compliment of B inside X and convert that. That is all. So, the proof is similar to that one.

Now, a special case here.

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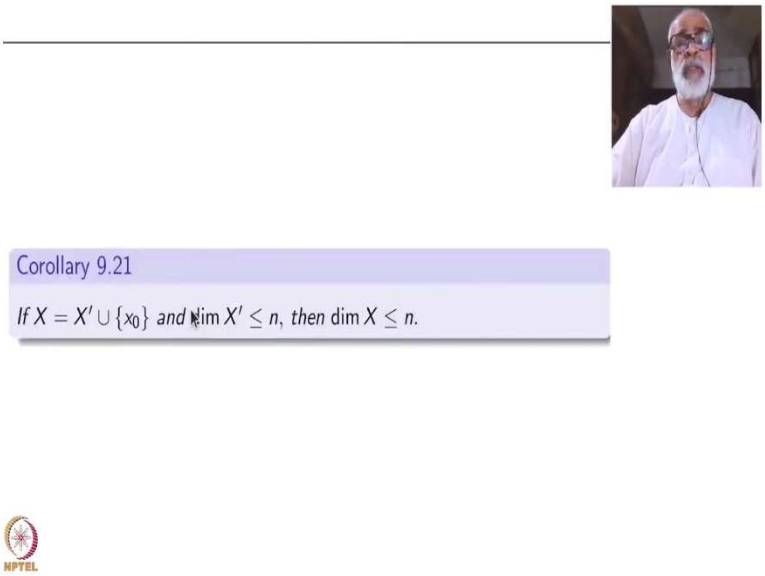

If X is X' union a singleton, and dimension of X' is less than to n, then dimension of X is less than equal to n.

Of course here, you have to assume that X' is non empty. This result is important only if X' is non empty after all.

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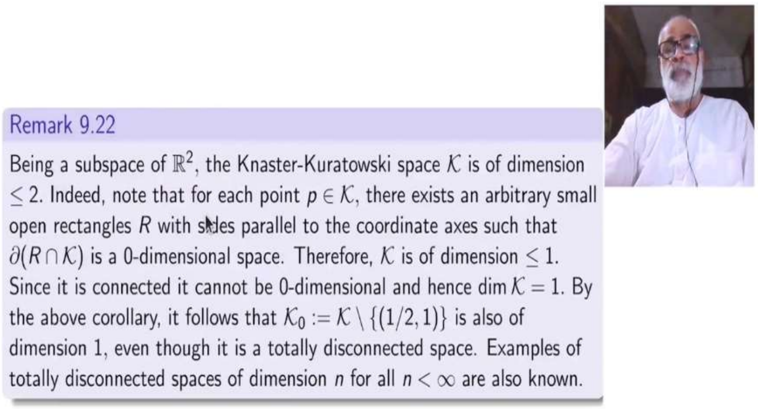

So, here is a remark now, on our famous and popular example Knaster-Kuratowski space $\mathcal{K}$, that is a subspace of $\mathbb{R}^2$. Remember that we have proved that it is a connected space. In any case, being subsets of $\mathbb{R}^2$, it is of dimension less than equal to 2.

Now, note that for each $p \in \mathcal{K}$, there exists an arbitrary small open rectangle R with sides parallel to the coordinate axes such that the boundary of $R \cap \mathcal{K}$ is a 0-dimensional space. Remember how the Knaster-Kuratowski space is constructed.

There is this apex point $p = (1/2, 1)$. From there you are taking lines joining p to the points inside the Cantor set. But those lines are all perforated. Either they consist only of rational numbers or they consist only irrational numbers depending upon what point you are taking inside the Cantor set. So, use that property to see that the boundary of any such rectangle intersection $\mathcal{K}$ is 0-dimensional.

Therefore, $\mathcal{K}$ is of dimension less than equal to 1. Now, what are the possibilities for the dimension? -1 is not possible because $\mathcal{K}$ is non empty. A 0-dimensional space (with more than one point) cannot be connected. So, it is not 0-dimensional either. Hence, the dimension of $\mathcal{K}$ is equal to 1.

Now, you apply this corollary with X' equal to $\mathcal{K}_0$, namely, the space obtained by subtracting the apex point $p = (1/2, 1)$ from $\mathcal{K}$.

We claim its dimension is 1. Why? First of all, dimension -1 is not possible because it is non empty. If it is of dimension 0, then after adding this point p, this will be also of dimension 0, which is not the case. Therefore, this must be already of dimension 1.

However, what is this $\mathcal{K}_0$? It is a totally disconnected space.

So, that is some surprise. Quite often, people mistakenly think that totally disconnected spaces are of dimension 0. At least in our definition, it does not behave like that. The totally disconnected spaces can be of higher dimension indeed. Examples of totally disconnected spaces of dimension n for any finite n, are also known, though, we are not going to discuss these examples anyway.

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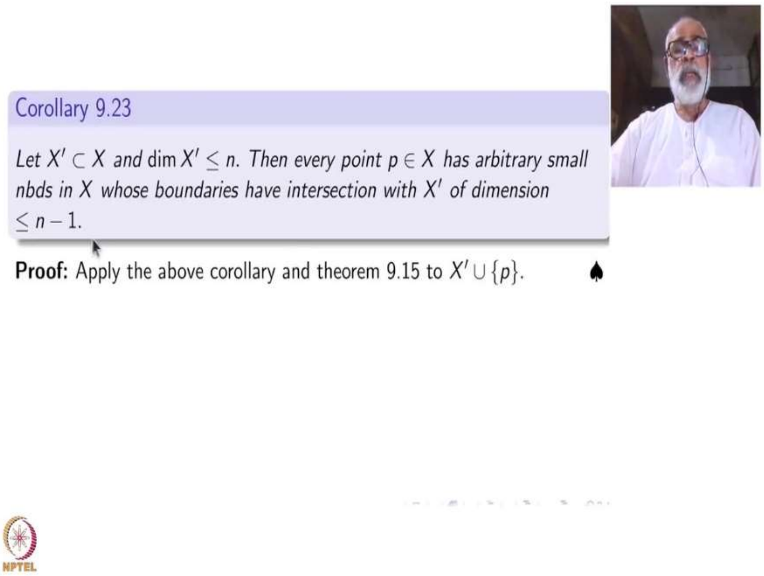

Another corollary: Take a subspace X' of dimension less than equal to n of a space X. Then for every point $p \in X$ (note that I am making a statement about all points in X itself) has arbitrary small neighborhoods in X, whose boundaries have intersection with X' of dimension less than equal to $n-1$.

The statement is obvious if p belongs to X'. But this is now for all points inside X itself. They are also have such a neighborhood. Only thing is, the entire boundary of these neighbourhoods may not be of dimension less than or equal to X, but only the part intersected with X'. For this, you have to apply the above corollary and theorem 9.15 to $X' \cup \{p\}$, that is all.

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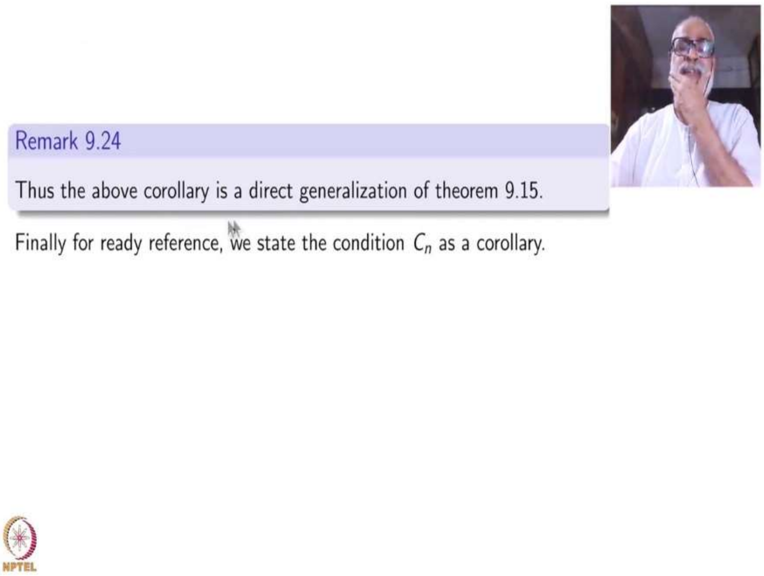

Thus, the above corollary is a direct generalization of 9.15. So, let me just recall this 9.15. Because I have written, several times, I am referring to this one. See, this is what we had. The global characterization of a subspace in terms of what is happening inside X itself. Dimension of X' is less than equal to n if and only if for every point in X, there are arbitrary small neighborhoods W such that dimension of boundary of W intersected with X' is less than equal to $n-1$.

So, this is the theorem that we have been using there. For ready reference, we state the condition C_n as a corollary, because we have actually proved it right? A_{n-1} implies C_n. and we have proved A_i for all i.

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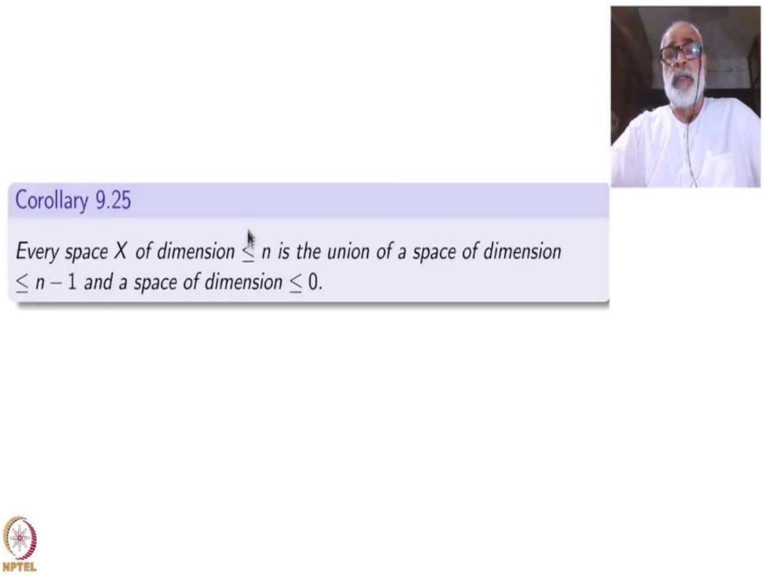

So, let us have that one because it is not just a subsidiary statement any more. So, let us state it separately. What is it?

Every (non empty) space X of dimension less than equal to n is the union of a space of dimension $n-1$ and a space or dimension less than equal to 0.

Once again, if you take X to be an empty set, this itself is of dimension -1, there is no space of dimension -2. So, you may resolve this logical difficulty by assuming X to be non empty or say that the statement is true vacuously.

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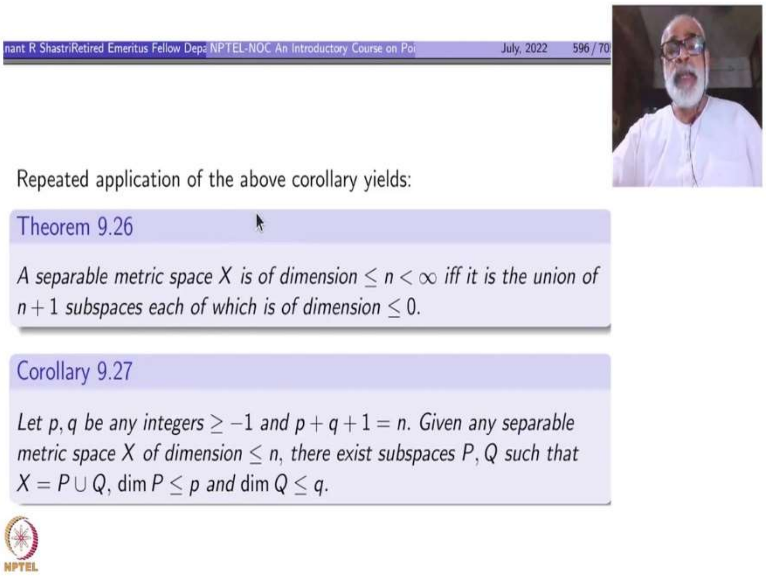

Repeated application of this is yields interesting result. Namely, if we have a separable metric space of dimension less than equal to n, (but this n must be finite, that is important, then and then only) it is the union of $n+1$ subspaces each of which is of dimension less than or equal to 0.

Next: Let p and q be any integers bigger than equal to -1. Just put $p+q+1=n$. (It is a definition of n, that is all.) Given any separable metric space X of dimension less than equal to n, (that is, $p+q+1$, that is all), there exists subspaces P, Q such that X is union of P and Q, dimension of this P is less than or equal to p, dimension of Q is less than or equal to q.

This is the generalization of writing X as a union of $n+1$ subspaces of dimension 0. You can break the n into any way you like, a partition of n into $p+q+1$, and then you will have this theorem. So, how do you do that? That is not very difficult. You have to use this one cleverly, inductively, viz., Theorem 9.26.

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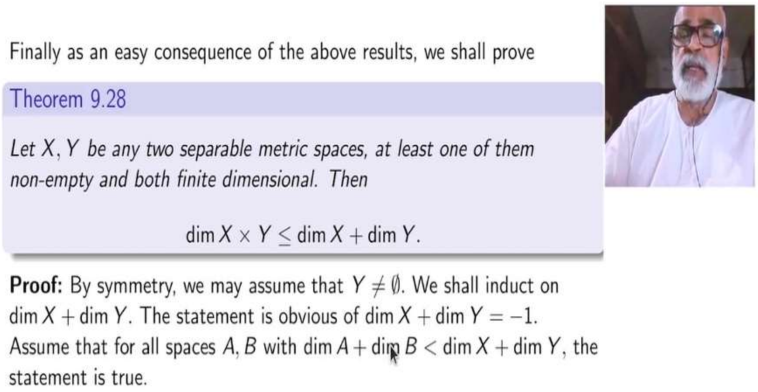

Finally, as an easy consequence of the above results, we shall prove the following. Now, we have come to the products. X and Y are separable metric spaces, at least one of them non-empty. (If you assume both of them are non-empty, well, there will be some problem, you will see. Suppose, one of them is non-empty, the other one is empty. What happens? The product will be empty space. So, what is the statement?). Assume that they are both finite dimensional. Then, the dimension of the product is less than equal to dimension of X plus the dimension of Y.

So, if both of them are empty, what happens? The RHS wll be -2. But these always minus. The statement is wrong because there is no space of dimenson less than or equal to -2. Only for that reason, we have to assume that at least one of them is non-empty, so that I get RHS is at least -1. Suppose X is empty and Y is non empty. The product is empty and the RHS is at least -1. So that case will be take care of.

So, we can assume that both of them are non empty also. We shall induct on dimension of X plus dimension of Y as usual. (The statement is obvious and is verified, if dimension of X plus dimension of Y is -1 already.)

So, assume that all spaces, A and B with this property viz., dimension of A plus dimension of B is less than dimension of X plus dimension of Y, satisfy the theorem.

So, this is the induction hypothesis. Having verified it for -1, now, we are assuming that dimension of X plus dimension of Y is positive, and the statement is true for all pairs of spaces with the sum of their dimension strictly less than this. So, that is the inductive hypothesis.

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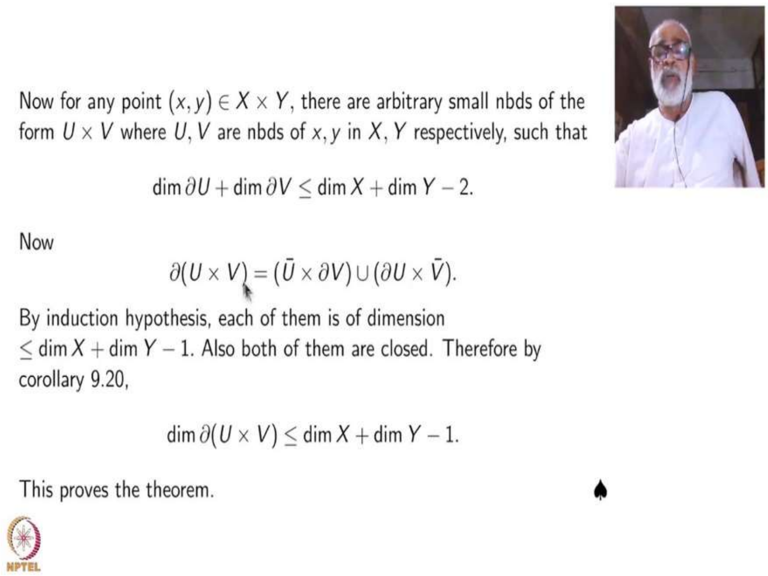

For any point $(x, y) \in X \times Y$, there are arbitrary small neighborhoods, $U \times V$, where U, V are neighborhoods of x and y in X and Y respectively such that dimension of the boundary of $U \times V$ is less than or equal to dimension of X plus dimension of Y minus 1.

Now, what is boundary of $U \times V$? You can take the closure here, boundary of $U \times V$ is same thing as boundary of $\overline{U} \times \overline{V}$. Then you will get the boundary to be $\overline{U}$ cross boundary of V union boundary of U cross $\overline{V}$. By induction hypothesis, each of them is of dimension less than equal to dimension of X plus dimension of Y minus 1.

Also, both of them are closed. Note that for each of them, only dimension of one of the factors goes down by 1. In boundary U cross $\overline{V}$, the dimension of the first factor is one less where as for $\overline{V}$ the dimension may be equal to that of Y. Also, both of them are closed. By, Corollary 9.20 boundary of $U \times V$ is of dimension less than equal to dimesnion of X plus dimension of Y minus 1. So, the theorem is proved.

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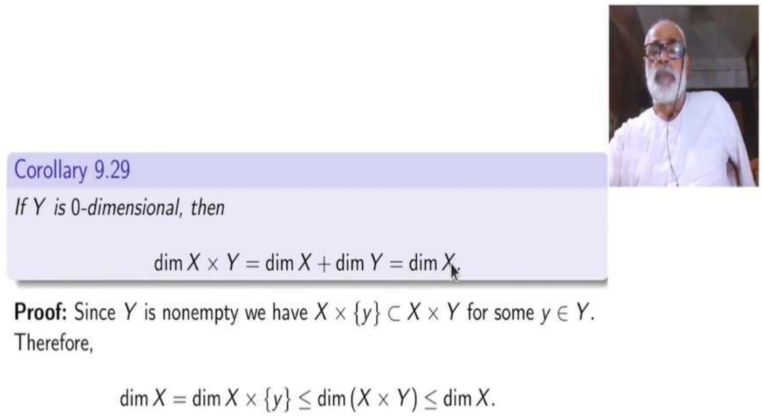

Special case. If Y is 0-dimensional, then dimension of $X \times Y$ is dimension of X plus dimension of Y, which is just dimension of X. So, taking product with a 0-dimensional space does not increase the dimension.

As a special case, we only get dimension of $X \times Y$ is less than or equal to dimension of X. But I have Y is non-empty because it is a 0-dimensional, and for any point $y \in Y$, we have X homeomorohic to $X \times \{y\}$ which is sitting inside $X \times Y$. Therefore, dimension of X is same thing as dimension of $X \times \{y\}$ which is subspace of $X \times Y$, and hence dimension of X is less than or equal to dimension of $X \times Y$, which is of course, dimension of X. So, equality occurs.

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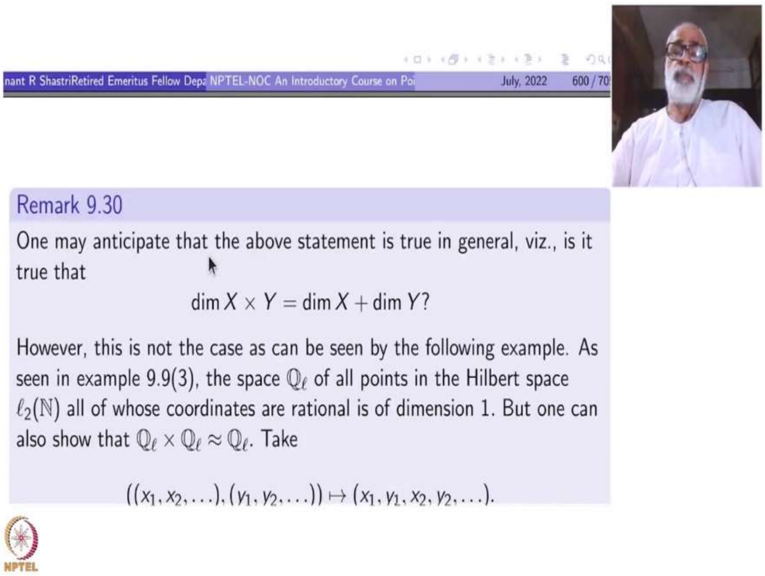

One may anticipate that the above statement is true in general just like dimension of $\mathbb{R}^n \times \mathbb{R}^m$ is equal to $n + m$. So, it will be a nice thing. So, one may anticipate the above statement is true. So, you can ask this question? Is the dimension of $X \times Y$ equal to dimension of X plus dimension of Y, in general? This is not the case, as soon as you take some huge spaces. That is necessary but the meaning of huge is just respect to dimension here, unfortunately. You do not have to go to infinite dimension. Infinite dimension, equality holds automatically. It happens just with some spaces of dimension 1 itself.

So, what is an example? Example is our favorite example. Namely, $\mathbb{Q}_\ell$, the space of all the points with rational coordinates inside the Hilbet spaces $\ell_2(\mathbb{N})$ space. So, this is of dimension 1. This is what we have proved earlier. But now, you can take $\mathbb{Q}_\ell \times \mathbb{Q}_\ell$ and show that it is again isomorphic to $\mathbb{Q}_\ell$ itself. By the obvious kind map: Begin with $(x_1, x_2, \ldots, x_n, \ldots)$, $(y_1, y_2, \ldots, y_n, \ldots)$ (and interlace them), map it to $(x_1, y_1, x_2, y_2, \ldots)$.

In fact, you can use this trick to show that product of of any number of copies of $\mathbb{Q}_\ell$, any number of finite number of copies, is again isomorphic to $\mathbb{Q}_\ell$. So, this space is of dimension 1, but after taking the product the dimension not add up.

So, let us stop here. Next time, we will launch a program to prove that dimension of the Euclidean space $\mathbb{R}^n$ is actually equal to n. Thank you.