An Introduction to Point-Set-Topology (Part II)
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Lecture No: 41
Dimensions of subspaces and Unions

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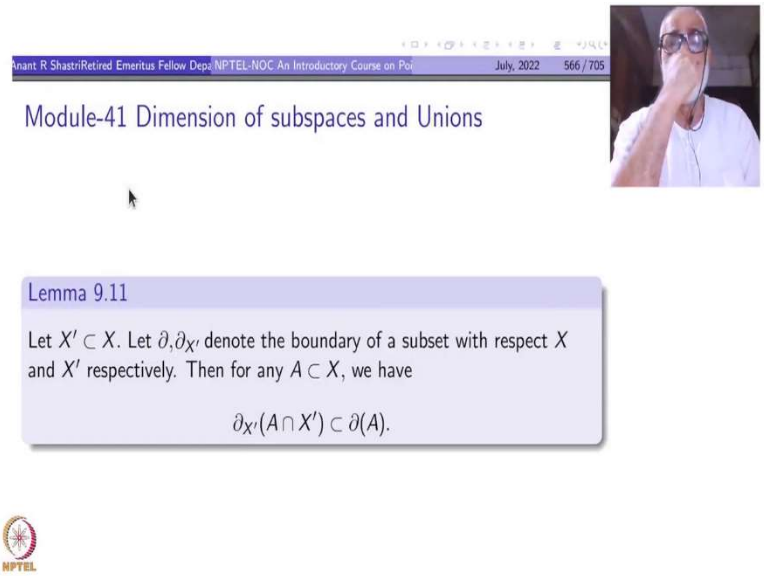

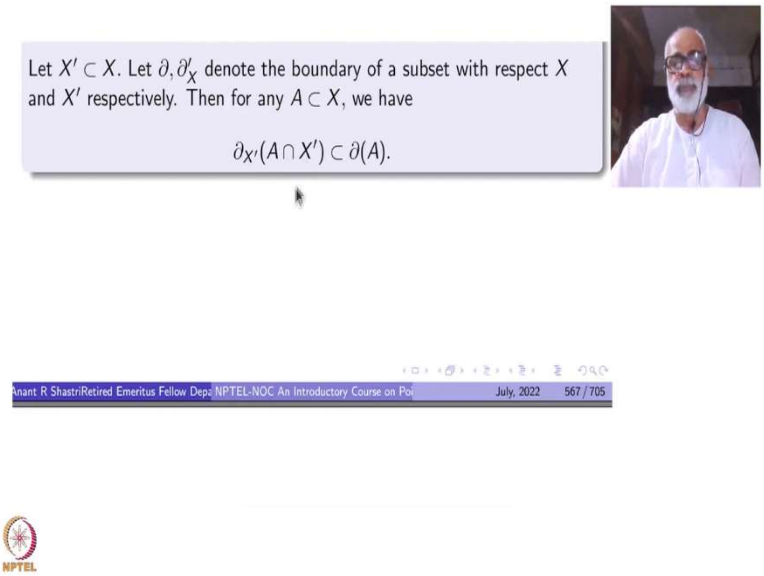

Hello, welcome to NPTEL NOC, an introductory course on Point-Set Topology, part-II, module 41, Dimension of Subspaces and Unions today.

Take a subspace of X, say X'. Let us have this notation, $\partial$and $\partial_{X'}$, I read them as boundary and boundary prime, respectively, denote the boundary a subset A of X', with respect to X and X' respectively.

Then for any A inside X, we have the boundary of $A \cap X'$, taken inside X', that is a subset of boundary of A without any decoration. This boundary just denotes the boundary inside the larger space X. So, this is an elementary result which you have seen in the first part itself, but now it becomes very crucial, so let us go through it a little carefully. Take a point, which is the boundary of $A \cap X'$, in the subspace X', you start with that.

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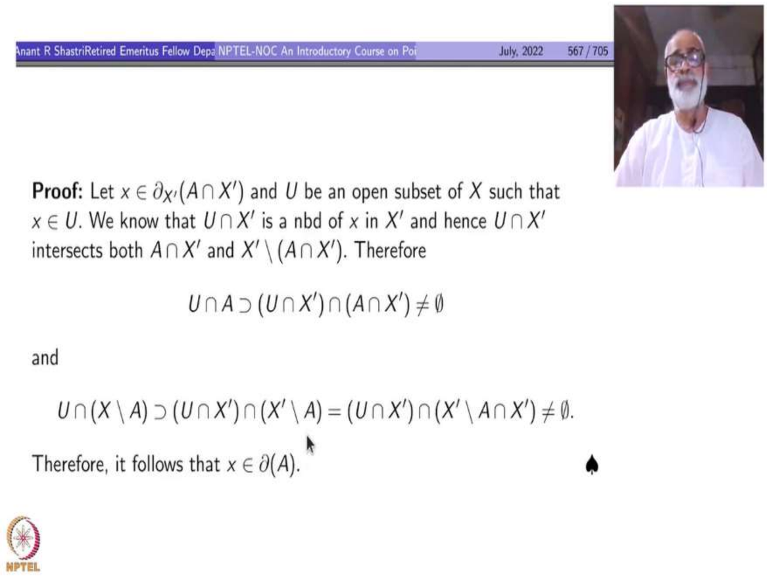

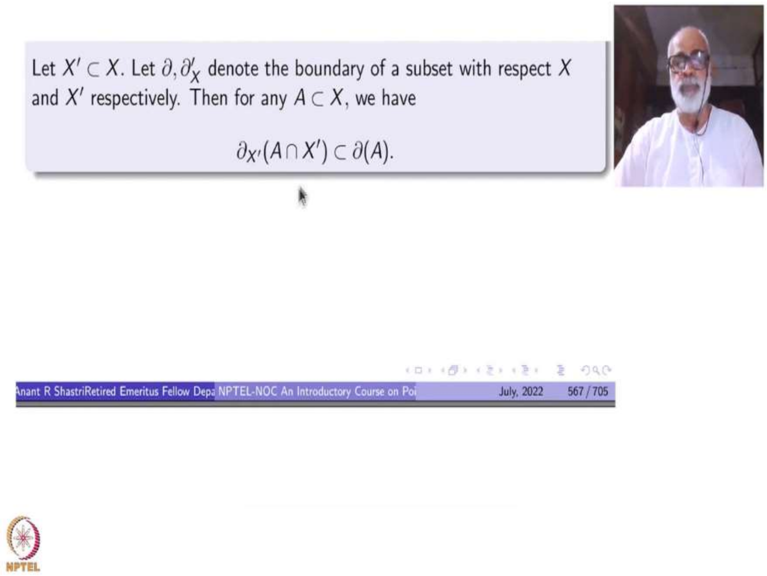

Take an open set U of X such that x is inside U. We know that $U \cap X'$ is a neighbourhood of $x \in X'$, and hence $U \cap X'$ intersects both $A \cap X'$ and its complement inside X', because it is the boundary point of $A \cap X'$.

Therefore, this $U \cap A$ which contains $(U \cap X') \cap (A \cap X')$ and hence has to be non-empty. So, we have proved that starting with any open neighbourhood of U of x in X, $U \cap A$ is non empty. What else we have to prove? We have to prove that A^c also intersects U. Then it would follows that x is a boundary point of A itself. So what is the intersection of U with $X \setminus A$? That contains $U \cap X'$ intersected with complement of A inside X'.

But the last set is $(X' \setminus A) \cap X'$. Therefore its intersection with $U \cap X'$ is non-empty as we have seen. therefore it follows that X must be inside boundary of A.

So, one way inclusion is true. In fact equality hardly occurs unless A itself is contained in X', or when X' is itself is closed and, so on. So, all that we need is one way inclusion here, so we will use this one heavily now.

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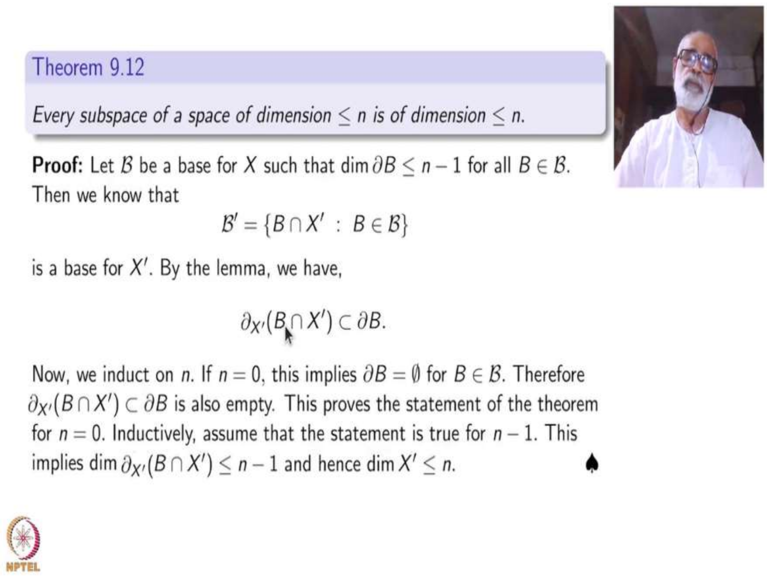

Every subspace A of a space of dimension less than or equal to n. When you take subspaces, dimension does not increase. So this is the theorem here now.

So, start with a base $\mathcal{B}$, for X such that dimension of the boundary of each member is less than or equal to $n-1$. That is the definition of dimension being less than equal to n. Then, we know that if you take $\mathcal{B}'$ to be the family of all $B \cap X'$, as B ranges over $\mathcal{B}$, that family is a base for X', because of subspace topology. By the lemma, we have boundary of $B \cap X$' in inside X' is contained inside boundary of B.

Now, we induct on n, if n is 0, this implies that boundary of B is empty for each B inside $\mathcal{B}$, therefore this $B \cap X$ is also empty. So this proves statement of theorem for n equal to 0.

Inductively, assume that we have proved the statement for $n-1$. But then that means the dimension of boundary of $B \cap X'$ is less than equal to $n-1$ for all $B \in \mathcal{B}$, because it is subspace of the corresponding B here.

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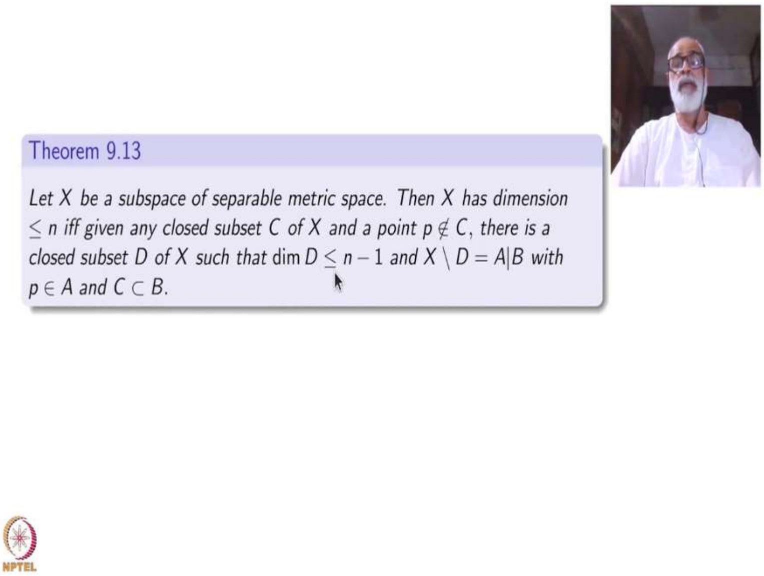

Next theorem is: Let X be a subspace of a separable metric space. Any space for that matter. X has dimension less than or equal to n, if and only if given any closed subspace C of X and a point p outside C, there is a closed subset D of X such that dimension of D is less than equal to $n-1$, and $X \setminus D = A|B$, a separation, with p inside A and C inside B.

Remember, what is SII? SII says that a closed set and a point can be globally separated by closed subsets. Now here we have something else, this is an extended SII. You may call it n dimensional version of SII. So, dimension of X is less than equal to n, you cannot expect closed sets and a point outside it to be separated by clopen sets, clopen sets means boundary is empty. Now, on that boundary we are going to put some condition, so this is elaborately stated in a different way here, you throw away a close subset of dimension $n-1$, then you have a separation.

When you have a clopen set, the boundary being empty played that role, that is why we did not have to bother it. Now here, we have to throw away some close subset, of course away from C and p, p and C you have to retain. So, $X \setminus D = A \cup B$, A and B are both closed and both open inside $X \setminus D$ and p is inside A, and C is inside B. So this is the generalized version of SII now you see. So, this is equivalent to having dimension less than or equal to n, just like SII was equivalent to having dimension 0. Let us prove this one.

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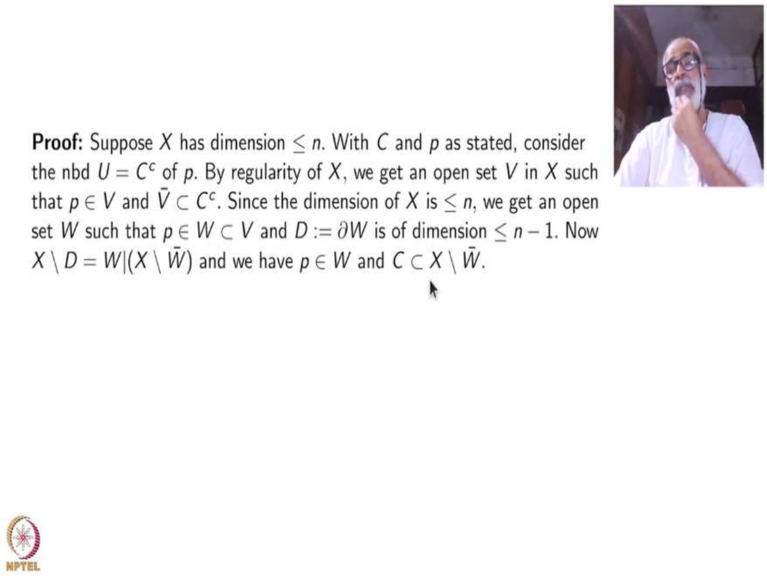

Suppose X has dimension less than or equal to n, with C and p as stated, means closed subset and a point outside. Take the neighbourhood U of p, which is complement of C. By regularity of X, because X is a metric space after all, we get an open set V in X such that p is inside V, and closure of V is inside complement of C, because p is inside complement of C, and complement of C is open. In between we have got this V and $\bar{V}$.

Since the dimension is less than or equal to n, we get an open set W inside this V, p belonging to W is containing side V, such that now you take D, what is D? I have to choose D to be the boundary of W. So, choose W such that D is of dimension less than or equal to $n-1$. So, this is the condition for X to be of dimension less than equal to n. Now, look at $X \setminus D$, D is a boundary of something, so it is a closed subset, so $X \setminus D$ is obviously union of W and $X \setminus \overline{W}$. Both of them are open, so both of them are closed in $X \setminus D$. Also p is inside W and C is inside $X \setminus \overline{W}$, because, because of what? the boundary $\overline{W}$ does not intersect C. $\overline{V}$ is contained inside complement of C, and W is contained inside V, so $\overline{W}$ is also contained inside $\overline{V}$, so C is contained inside $X \setminus \overline{W}$.

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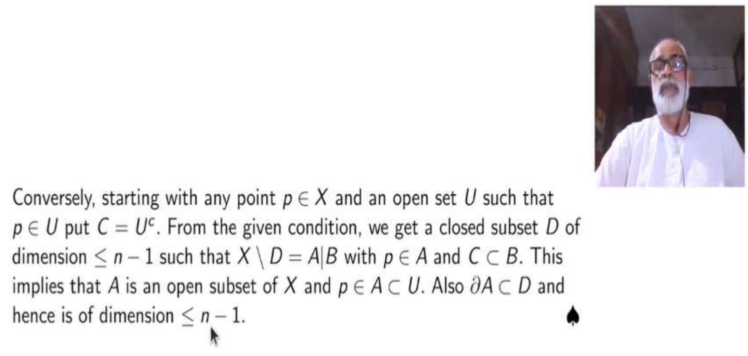
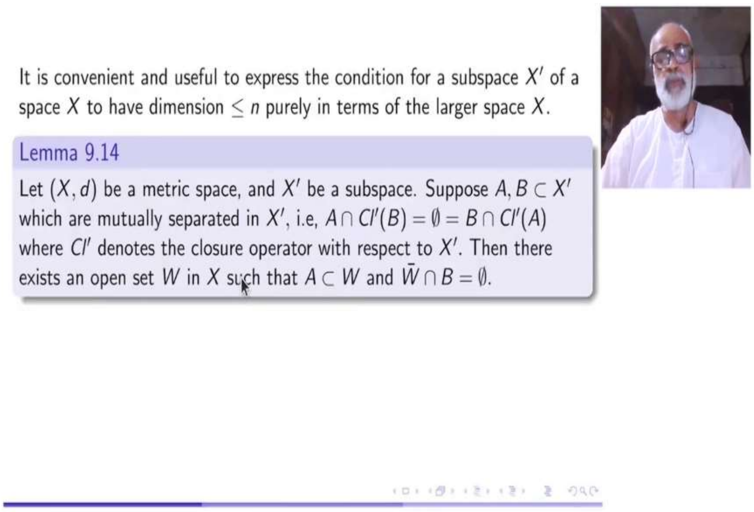


Conversely, starting with any point $p \in X$ and an open subset U such that p is inside U, put C equal to complement of U this time, so that C is closed subset and p is outside that. From the given condition, we get a closed subset D of dimension less than equal to $n-1$ such that $X \setminus D = A|B$, with p is inside A, and C is inside B. This implies A is an open subset of X, because D is a closed subset of X and A is open inside $X \setminus D$. And p is inside A, A is contained inside U, because it is disjoint from this C, that is all. Also, boundary of A will be contained inside D, and hence of dimension is less than equal to $n-1$. So, what we have done? Starting with any point and an open set we have produced a smaller neighbourhood of that point such that its boundary is of dimension less than $n-1$. So, this just means that at every point we have a local base for X consisting of elements members with their boundaries of dimension less than or equal to $n-1$. So that means that dimension of X is less than or equal to n. (Refer Slide Time: 12:54)



Now, it is convenient and useful to express the condition for a subspace X prime of space to have dimension less than equal to n, purely in terms of some condition on larger space X. We know what is the condition within the space, viz., there is a base satisfying with blah, blah, blah, So, we can say that it is the generalized SII condition. So, let us convert that condition purely in terms of X, so that would be useful for us.

So, this lemma says:

Start with a metric space X. (This is a general lemma.) Let X' be a subspace, suppose A and B are subsets of X' which are mutually separated in X'. (Remember, mutually separated means that $A \cap \overline{B}$ is empty, and $B \cap \overline{A}$ is empty. This time I am taking closures inside X', because there are two spaces involved here, you do not know where you are taking the closure if you just say closure of B or $\overline{B}$ and so on. I will use bar to denote the closures inside X, the standard one. For subsets of X', I will use this notation closure prime.)

Then there exists an open subset W in X such that A is contained inside W and $\overline{W} \cap B$ is empty.

So this is again a general result. While studying metric spaces, we have seen such proofs but let me recall it, because now it becomes crucial.

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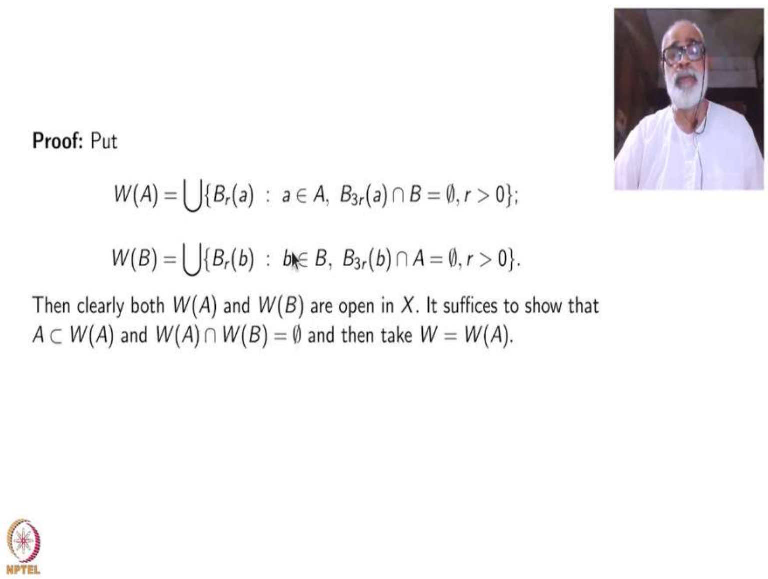

All that I have to do is use the metric properly. So, put $W(A)$ equal to union of all open balls $B_r(a)$, where a range is inside A and a condition on r, namely $B_{3r}(a)$ does not intersection B.

So, points of this $W(A)$, you know are far away from B. Of course, r should be positive this is just the definition of $W(A)$.

Similarly, let $W(B)$ be the union of $B_r(b)$, b ranges over B, same condition or r got by interchanging A and B, viz., $B_{3r}(b) \cap A$ is empty.

Then clearly both $W(A)$ and $W(B)$ are open subsets of X. Because they are unions of open balls. It suffices to show that, A is inside $W(A)$, $W(A) \cap W(B)$ is empty. Because, by symmetry, B is inside $W(B)$ and that is open, automatically it will imply $\overline{W(A)} \cap B$ is empty. After that you can take $W = W(A)$. We want W to be an open set containing A, such that its closure in X does not intersect B.

A set does not intersect an open neighbourhood around a set then its closure will not intersect that set, that is all.

So let us prove that A is inside $W(A)$. We have not yet used the hypothesis. What is the hypothesis? A and B are mutually separated, we have not used that one yet.

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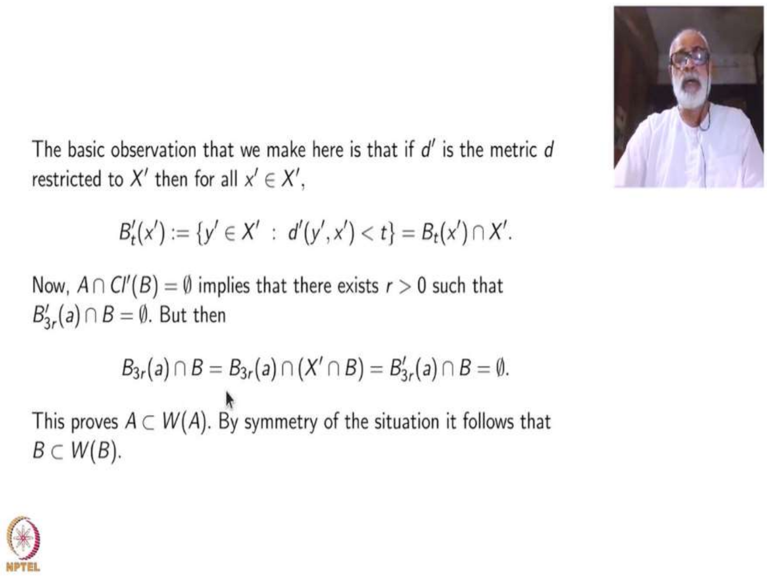

Proof: Put

$$W(A) = \bigcup \{B_r(a) \ : \ a \in A, \ B_{3r}(a) \cap B = \emptyset, r > 0\};$$

$$W(B) = \bigcup \{B_r(b) \ : \ b \in B, \ B_{3r}(b) \cap A = \emptyset, r > 0\}.$$

Then clearly both $W(A)$ and $W(B)$ are open in X. It suffices to show that $A \subset W(A)$ and $W(A) \cap W(B) = \emptyset$ and then take $W = W(A)$.

The basic observation we make here is the following. Let d' be the metric d restricted to X', (the same metric d on X, but points are taken inside X'), then for all $x' \in X'$, and for all positive t, we have $B'_t(x')$ the ball of radius t arond x' in X', that is the set of all y' in X', such that $d(x', y')$ is less than t, this ball is nothing but the standard ball in X around X' intersected with X'. I am just repeating the meaning of the restricted metric here.

Now, A intersection closure of B inside X' is empty implies that for every $a \in A$, there exists some r positive, such that the ball of radius $3r$ around a with respect to d' (everything working inside X' now) intersection B is empty. But then, $B_{3r}(a) \cap B$ is $B_{3r}(a) \cap (X' \cap B)$, (because both B and A are inside X') and that is the same as the $B'_{3r}(A) \cap B$ which is empty.

So, for each a, you have got a positive r as above. This proves that A is contained in $W(A)$. Interchanging A and B, this also shown that B is contained in $W(B)$.

Now, you have to prove that $W(A) \cap W(B)$ is empty.and that is why we are taking only balls of radius r, but condition is on the ball of radius $3r$. So that the triangle inequality will help you, that is all. So standard triangle inequality you have to employ.

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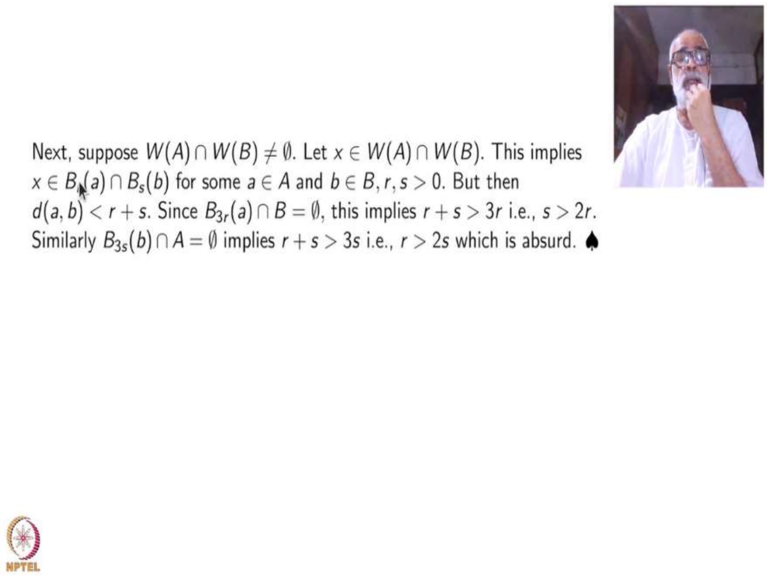

Suppose, you have a point in x both $W(A)$ and $W(B)$. That implies that this x is inside some $B_r(a)$ as well as in $B_s(b)$ for some r and s positive numbers and a and b inside A and B respectively. But then, distance between a and b is less than or equal to sum of the distances from a to x and x to b which is will be less than $r + s$. Triangle inequality here. But $B_{3r}(a) \cap B$ is empty, implies that $r + s$ must be bigger than $3r$. So s is bigger than $2r$.

Similarly, $B_{3s}(b) \cap A$ is empty implies $r > 2s$. That is absurd. There are different ways of getting contradiction once you know that the triangle inequality has to be used here.

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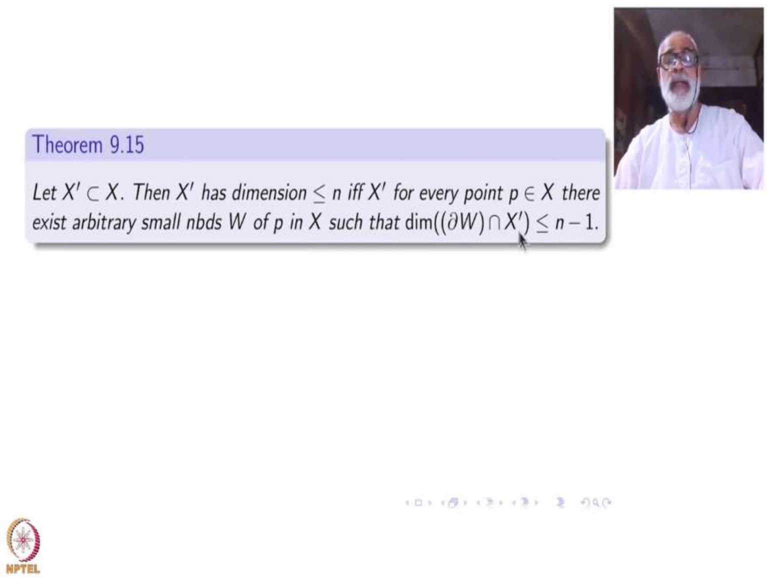

Now, let us go to dimension theory. Let X' contain X. (Recall that now I am working with separable metric spaces.) Then X' has dimension less than equal to n, if and only if for every point p belonging to X, there exist arbitrary small neighbourhoods W of p in X, (I want everything inside X now) such that dimension of the boundary of W taken inside X intersected with X' is less than or equal to $n-1$.

(That is as far as you can go, everything trying to do only in terms of X is not possible, somewhere X' must be involved! Able to put it only at the last moment. Everything inside X' that is true if X' is of dimension less than or equal n. From that I have to get a neighbourhood W of this point inside X with this property, so that is the gist of the theorem.

First look at the converse which is easier.

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Proof: Assume that the given condition is satisfied. Let $p \in X'$ and let U' be an onbd of p in X'. Let U be an onbd of p in X and $U' = U \cap X'$. By the given condition, there exists a nbd W of p such that $W \subset U$ and $\partial W \cap X'$ is of dimension $\leq n-1$. Put $V' = W \cap X'$. Then $U' \subset V'$ and we know that $\partial_{X'}(V') \subset (\partial W) \cap X'$ which is given to be of dimension $\leq n-1$. Therefore, from theorem 9.12, the conclusion follows.

Anyway, assume that the given condition is satisfied, here. Let p belong to X' and U be a neighbourhood of p in X'. Now, every neighbourhood in X' can be expanded to a neighbourhood of the point inside the larger space. By the definition of subspace topology, we get neighbourhood U of p in X such that U' is $U \cap X$.

Then by the given condition, we get a smaller neighbourhood W of p, such that W is contained inside U and dimension of $W \cap X'$ is of dimension less than or equal to $n-1$. This is the stated condition in the theorem. Now, you take $V' = W \cap X'$.Then V' is contained inside U', and we know that the boundary of V' inside X' is contained inside boundary of $W \cap X'$, by the above. Therefore, boundary of V' in X' is of dimension less than or equal to $n-1$ by theorem 9.2. Therefore we can conclude that the dimension of X' is less than or equal to $n-1$.

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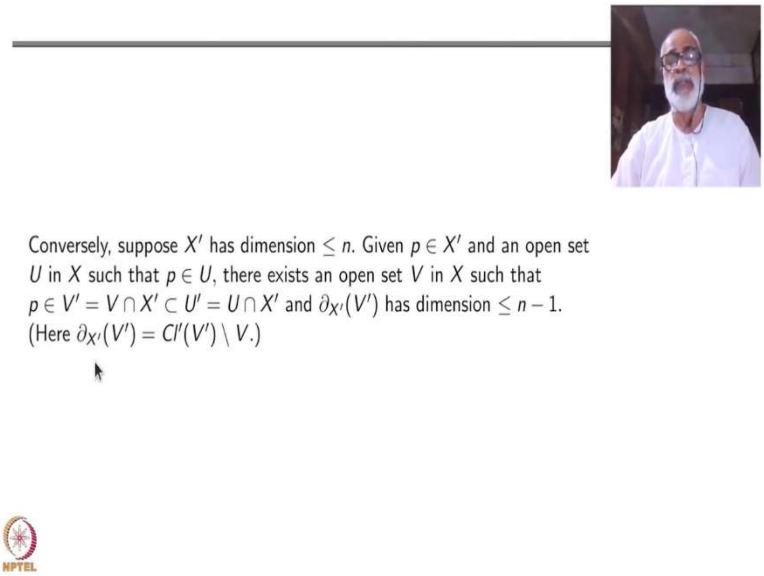

Conversely, suppose dimension of X' is less than or equal to n, given a point p inside X' and a neighbourhood, an open subset in the larger space X such that p is inside U, there are existing open subset V inside X, such that p inside V' which is $V \cap X'$ contained inside U', which is $U \cap X'$, and dimension of the boundary of this V' in X' is less than or equal to $n-1$. This is the meaning of dimension of X' is less than or equal to n.

Starting with an arbitrary neighbourhood, inside that I can find a neihbourhood V' with this property because dimension of X' is less than n. The boundary of this X' is taken inside X' now. You recall that this is nothing but the closure of V' inside $X' \setminus V'$, (which is same thing as subtracting V, does not matter) that is the meaning of this one.

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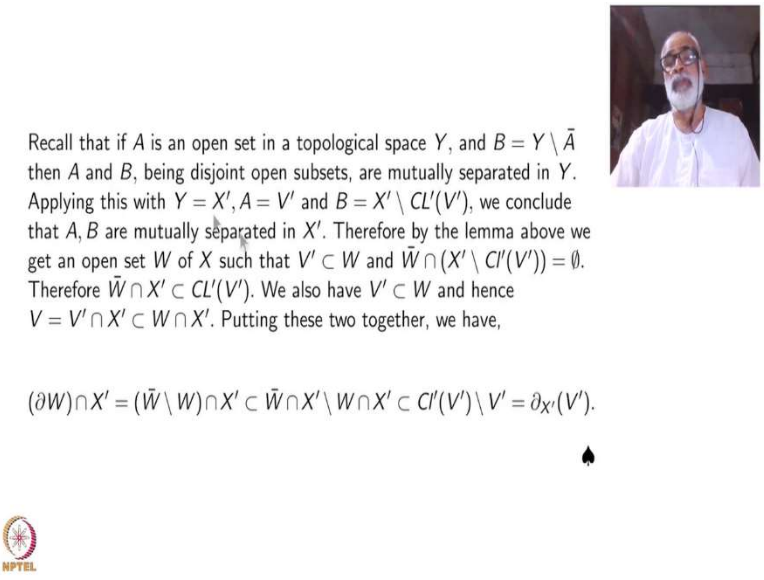

Now, recall that if A is an open set in a topological space Y, and B is $Y \setminus \bar{A}$, then A and B will disjoint open sets and hence they are mutually separated inside Y. Start with an open subset A, and take B to $Y \setminus \bar{A}$, so automatically, you know that boundaries of A and B are themselves disjoint.

Apply this, with Y equal to X' and A equal to V', and B equal to X' setminus the closure of V' inside X'. Then A and B are mutually separated and therefore, by the above lemma, we get an open subset W of X such that $A = V'$ is inside W and the $\overline{W} \cap X'$ setminus closure of V' in X' (that is B) is empty.

Therefore, $\overline{W} \cap X'$, if you take the only interaction that must be contained inside this subset which you have thrown away. Once you throw away this the intersection is empty, so this must be contained inside closure of V' inside X'. We also have V' inside W now as well as X' because $V'$is $V \cap X'$.

So, putting these two together, what we have is boundary of $W \cap X'$ will be equal to (boundary of W by definition is) $(\overline{W} \setminus W) \cap X'$, which is $\overline{W} \cap X'$ setminus (points of W have to be thrown away, that is the same as) $W \cap X'$. But this first one contained in the closure of V' setminus the second one contains the smaller set V' this is a subset of this one. But this is nothing but the boundary of V' taken inside X'. and that is precisely the statement in the converse part.

Note that it took some set topology here, you see, you have to do it carefully.

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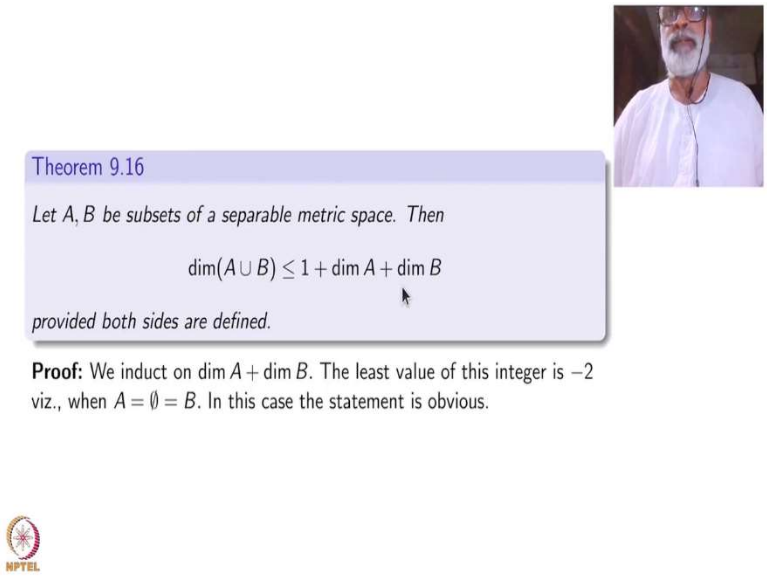

Now, we have a beautiful theorem here. Maybe not very beautiful because you may expect that dimension of the union is equal to the sum of the dimensions of the individual sets. However such sweeping things will not work. So, slight modification is necessary, so yet it is quite beautiful that is what I want to say.

A and B are subsets of a separable metric space (no other condition). Then dimension of the union $A \cup B$ is less than or equal to dimension of A plus dimension of B plus 1.

Only thing I am considering is the case when dimensions are finite. The statement is true other also with correct interpretation of the two sides of the inequality. For instance, suppose dimension of A is infinite or dimension B is infinite, then the inequality is obvious. I want to avoid all that discussion. I want to take the case wherein dimension of A plus B is itself is finite, dimension of A is finite, dimension of B is finite. You can examine what happens when they are infinite, no problem.

Let us do induction, on dimension of A plus dimension of B. The least value of dimension of A plus dimension of B (this occurs when both A and B are empty) is -2 and then (their union is also empty and hence is of dimension -1) $-2 + 1 = -1$, so this is okay.

You see even at this level, viz., when $A \cup B$ is empty, the inequality `dimension $A \cup B$ is less than or equal to dimenson A plus dimension B will not be true. So, the modification by adding 1 on the RHS was necessary.

You see even at the, even if that level empty set empty set, dimension of $A \cup B$ is empty is less than or equal to -1, -1 does not would have will not have made sense. See, -1 is not less than equal to -1 plus -1 we have to add one more, even at the level this 1 is necessary even at the very beginning. So, this case is over the least case.

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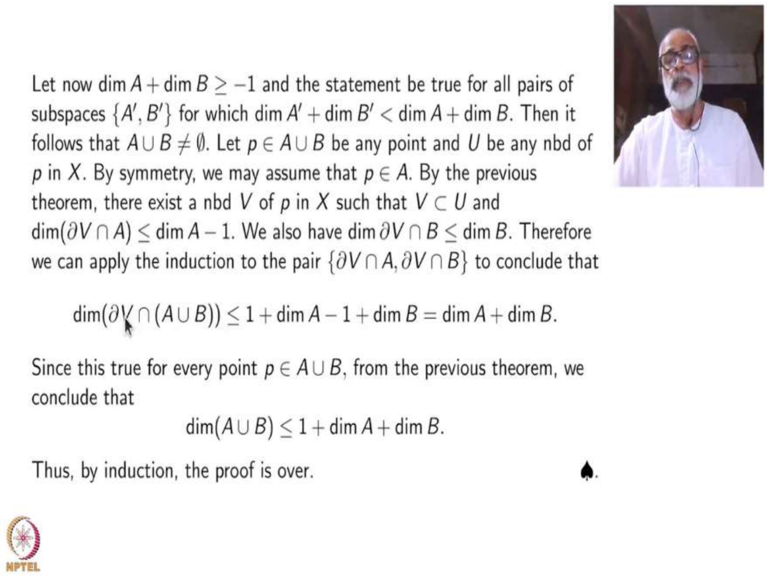

Now, assume that dimension of A plus dimension of B is greater than or equal to -1 (which just means that at least one of A or B must be non-empty that is all). Suppose the statement is true for all pairs of spaces, A', B', whenever this happens, namely for which dimension of A' plus dimension of B' is less than dimension of A plus dimension of B, this is the inductive hypothesis here. Not just for A and B, whenever you have the sum of the dimensions is less than dimension A plus dimension B, for such pairs of spaces, the inequality in the theorem must hold. That is the inductive hypothesis. Then, we want to prove it for one higher number.

First of all it follows that $A \cup B$ is non-empty, that is what I have told you already. So take p to be in the union and U be a neighbourhood of p. By symmetry, we may assume p is either in A or in B, so you assume p is in A just for writing down the proof.

By the previous theorem, there exists a neighbourhood V of p, such that V is contained inside U and dimension of $\partial V \cap A$ is less than equal to dimension of A minus 1. (The previous theorem enters here, you see everything is happening inside X except the last condition viz., dimension of $\partial V \cap A$ is of dimension less than equal to this one.) So, we also have dimension of $\partial V \cap B$ is less than dimension of B, because subspaces respect dimension.

Therefore, we can apply the induction to the pair ($\partial V \cap A$, $\partial V \cap B$), this is (A', B'), the sum total dimension is smaller than dimension of A plus dimension of $B + 1$. First one is strictly smaller than A minus 1, the other one is just less than, so the sum total will be less than dimension of A plus dimension of B.

Therefore, induction hypothesis should be applicable to these two subspaces $\partial V \cap A$, $\partial V \cap B$, to conclude that the dimension of the union of these two subspaces is less than or equal to 1 plus dimension of the first one is A minus 1, dimension of A minus 1 here, you see the other one is less than dimension of B. So, 1 and -1 cancel out and the sum total is fine.

Since, this is true for every point p inside the union, from the previous theorem, we conclude that dimension of $A \cup B$ must be 1 plus dimension of A plus dimension of B.

So, induction is quite easy here, once you have this theorem which gives you a criterion in terms of the ambient space $A \cup B$. I do not have to work with A or B separately. This boundary of V is taken inside $A \cup B$.

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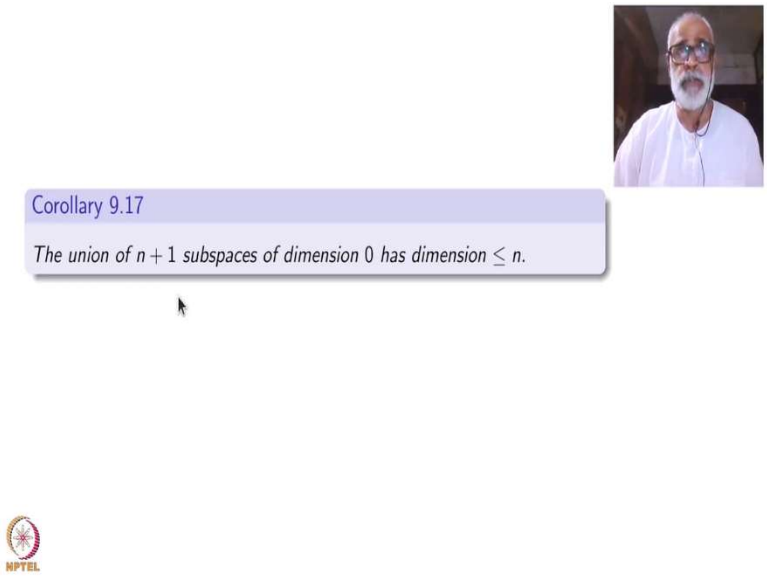

Now, we have another easy corollary, which is something funny here you can say.

Union of $n + 1$ subspaces of dimension 0 has dimension less than or equal to n.

You cannot say its dimension is 0. Dimension when you take union, may go up by one at a time. You know we have seen that, that is a statement of previous theorem.

So, this is an easy corollary to the previous theorem, namely, if you take two 0-dimensional spaces, the union is of dimension less than equal to dimension 1, take one more then the dimension is less than equal to $1 + 1$ and, so on it goes on, so you will get dimension less than or equal to n.

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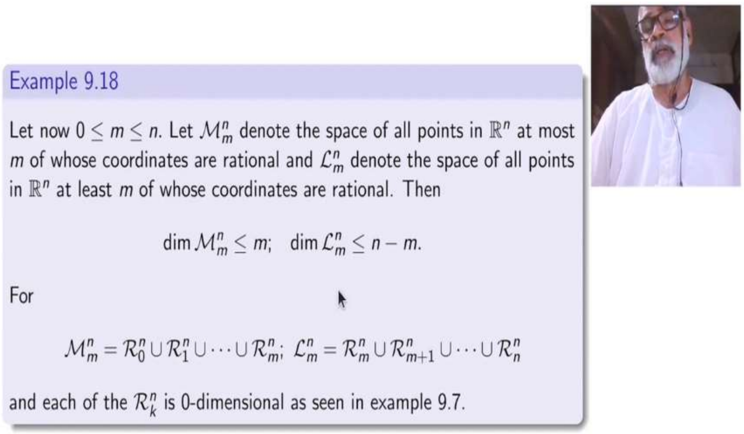

Now, here is an example now, start with any integer n, possibly positive otherwise there will not be anything left here, and take m to be smaller than that, you can take equality also. Let $\mathcal{M}$ upper n lower m denote the space of all points in $\mathbb{R}^n$, at most m of whose coordinates are rational. Recall earlier we have studied $\mathcal{R}^n_m$, which is the space of points in $\mathbb{R}^n$ with exactly m coordinates rational.

Now, you have to be very careful here, that we are taking at most m of whose coordinates are rational. And let $\mathcal{L}^m_n$ be the space of those points with at least m of whose coordinates are rational. Their union is the whole space $\mathbb{R}^n$.

The claim is that the dimension of $\mathcal{M}$ is less than equal to m, and dimension of $\mathcal{L}$ less than or equal to $n - m$. How do you do that?

Let us look at the space $\mathcal{M}^n_0$. So you can start with $m = 0$, i.e., no coordinate is rational. It is fullly, all the all coordinates are irrational. We know that this is $\mathcal{R}^n_0 = \mathbb{I}^n$ and is 0-dimensional. Then, take $\mathcal{R}^n_1$, the space with exactly one coordinate is rational, and so on, upto $\mathcal{R}^n_m$. $\mathcal{M}^n_m$ is the union of these $m + 1$ spaces which we have studies earlier, they are 0-dimensional. So, dimension of $\mathcal{M}^n_m$ is less than n, Exactly similarly we have the other way around for $\mathcal{L}^m_n$. Here at least m of the coordinates are rational, so take $\mathcal{R}^n_m \cup \mathcal{R}^n_{m+1} \cup \ldots \mathcal{R}^n_n$. How many $n - m + 1$ of them all of them 0-dimensional. Therefore, the dimension of the union is less than or equal to $n - m$. We will stop here, we will continue again next time, some theorems for n-dimension spaces, thank you.