An Introduction to Point - Set - Topology (Part II)
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Lecture No: 40
Definition of dimension and examples

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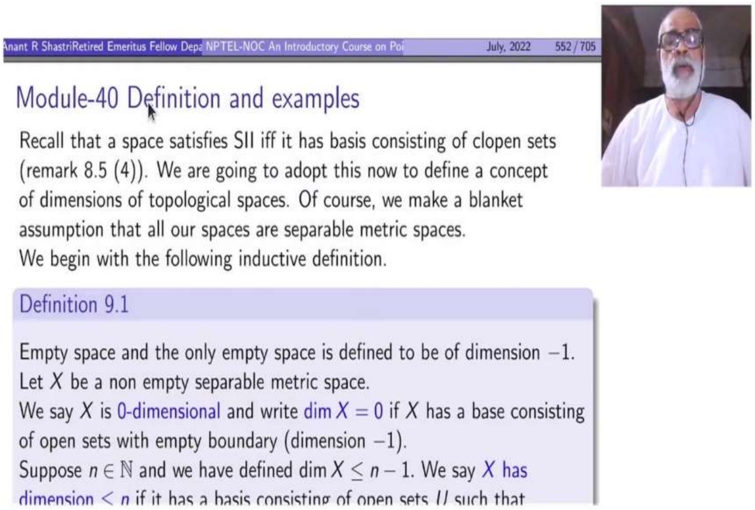

Hello welcome to NPTEL NOC an Introductory Course on Point Set Topology part 2, module 40. So, far we have prepared ourselves with the so called Global separation properties for the launch of dimension theory proper. That is what we are going to do today. Recall that a space satisfies SII if and only if it has a basis consisting of clopen sets. We are guided by the observations that we have in the classical study of euclidean sapcs. The real line with its usual topology, in which every point has a fundamental system of neighborhoods such that the boundaries are just two points, a discrete space, a 0-dimensional space.

In $\mathbb{R}^2$, every point has a fundamental system of neighbourhoods, say, open discs whose boundaries are circles which are 1-dimensional and so on. We now make an inductive definition of the dimension. The key property is this SII.

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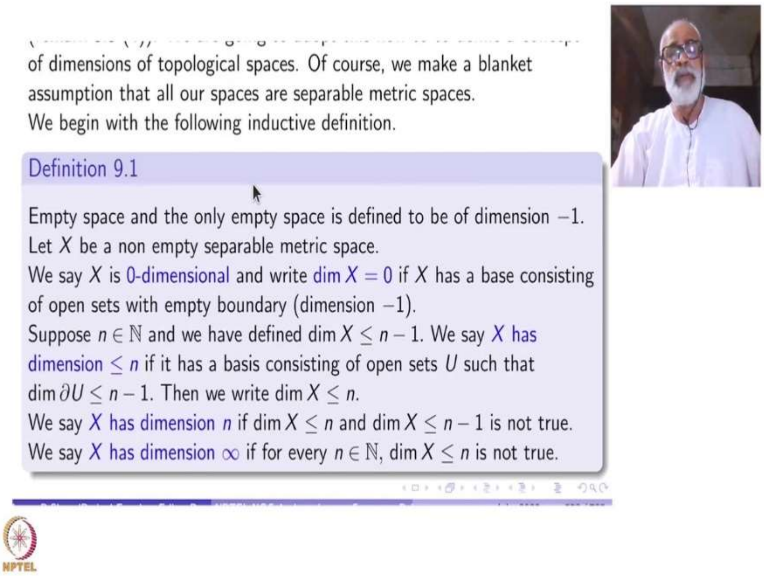

So, we begin with the definition of dimension -1 which is nothing but an empty space. Only empty space is defined to be of dimension -1.

Of course, in all other cases, we start with a non empty separable metric space just to remind you, without mentioning it again and again.

We say X is 0-dimensional (and then we write dimension of X equal to 0) if X has a base consisting of open sets with empty boundary. You see these basic elements of this base are open sets without boundary, is same thing as saying that the space X satisfies SII. `Empty boundary' can be termed as dimension -1, in our inductive definition.

So, this is the key, we will do it like this. The space X is 0-dimensional if it has a base consisting of boundaries of dimension -1.

Now, suppose n is a some positive integer and we have defined what is the meaning of dimension X is less than or equal to $n-1$. We say X has dimension less than or equal to n, (the next one), if it has a base consisting of open sets U such that the dimension of the boundary of U is less than or equal to $n-1$. In that case we write dimension of X is less than or equal to n.

Note that so far, I have not defined dimension of a space as a number. What I have defined is the entire phrase `dimension of X less than or equal to n'. What is the meaning of that, namely, X has a base consisting of open sets such that the boundaries of each of these open sets is of dimension less than equal to $n-1$. This makes sense by induction since we have defined dimension equal to -1 completely.

Now, we say X has dimension exactly equal to n, if first of all the dimenson of X is less than or equal to n as defined earlier and in addition, dimension of X is not less than or equal to $n-1$. This, I can rephrase as follows: take the least integer n for which dimension of X is less than or equal to n holds. That is called the dimension of X. Then if you take any number $k<n$, then it is not true that dimension of X is less than or equal to k.

So, this n is the least integer such that dimension of X is less than or equal to n. So, that least integer is called dimension of X. Finally, we say X has dimension infinity, if for every n, dimension of X is less than or equal to n is not true.

Let us take a little time to recapitulate what exactly this definition means, just logically.

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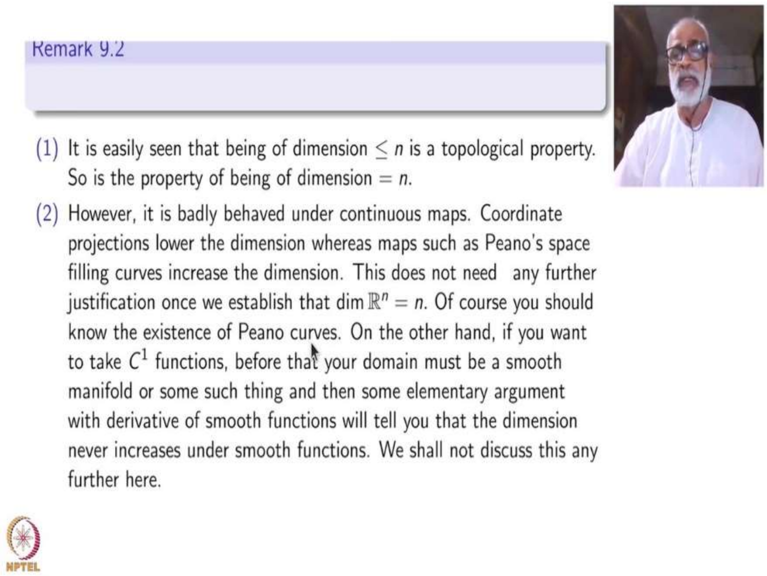

For example, you can easily see that dimension less than equal to n is a topological property, because it has been defined in terms of the existence of a base of a particular topological property. If a space homeomorphic to the given space, that space will also have the corresponding base with the corresponding topological property. Under the homeomorphism open sets will go to open sets and their boundaries will go to boundaries. So, inductively if f from X to Y is a homeomorphism and dimension of X is less than or equal to n, we can prove that dimension of $f(X)=Y$ is less than equal to n.

Inductively. So, where does the induction start? When you have an empty set and anything homeomorphic to an empty set is also an empty set, that is all so, there is nothing more than that.

However, under an arbitrary continuous function, the dimension is not behaved well. One would expect that dimension would go down under a continuous function, namely, image of a continuous function will be of dimension lower than the domain. That expectation is true only if you qualify this function a little more than just being a continuous function. Quite often, a C^1 function will do that job. Analytic functions, polynomial functions they all do that they will never increase the dimension. Of course, we know that the dimension can go down very easily. For example, you can take a constant function, a coordinate projections from $\mathbb{R}^n$ to $\mathbb{R}^{n-1}$, etc. Singleton space is of dimension 0, $\mathbb{R}$ is of dimension 1, and so on. That is easy. So, what is weird is that just if you take a continuous function, then dimension of the image may be smaller. You must be knowing the existence of what are called Peano's curves. The Peano's curves are such that they are continuous functions from a closed interval to the product of two closed intervals, to the product of three closed intervals, to the product of any number of closed intervals.

In fact, there are continuous surjective functions from a closure interval into the Hilbert cube itself. We shall not discuss those functions here, but just stating the fact for the sake of information.

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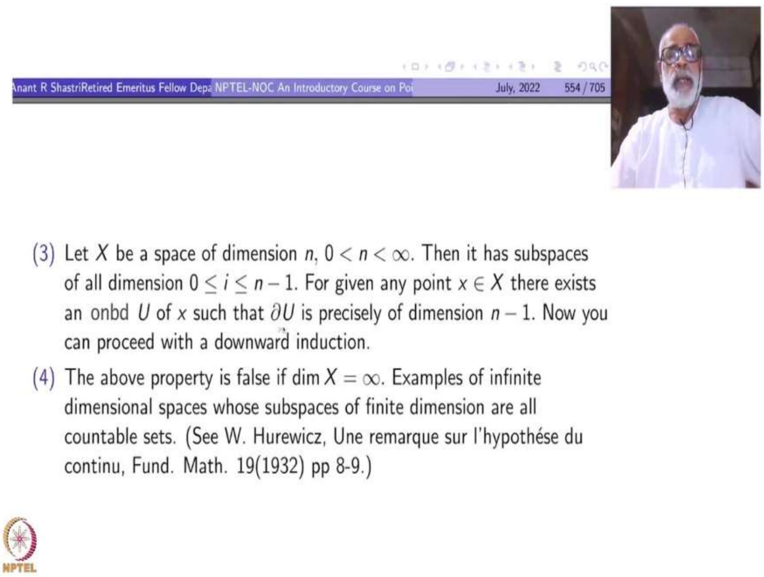

Let X be a space of dimension n for some finite number $n > 0$. Then it has subspaces of all dimensions less than or equal to n. Of course empty set is a subspace of dimension -1, so, I can include -1 also.

How to show this? There exists a point $x \in X$, with a fundamental system of neighbourhoods U such that boundary of U is precisely of dimension $n-1$. For each point there is a neighborhood we have the whole as a fundamental system of neighborhoods with this property.

So, boundary of U is dimension $n-1$. So, I have got a subspace of dimension $n-1$, but now we apply it to a point in the boundary. So this way, you keep going down. So, you get all the dimensions $n-1, n-2, \ldots, -1$.

The above property is false if X is of dimension infinity. One would expect that a space of dimension infinity will have subspaces of all finite dimensions. That is not the case. There are Examples of infinite dimensional spaces all of whose finite dimensional subspaces are countable sets. We will see easily that every nonempty countable space is of 0-dimensional. These examples are in paper of Hurewicz himself. So, if you are interested in you can look at it and I have given the reference here.

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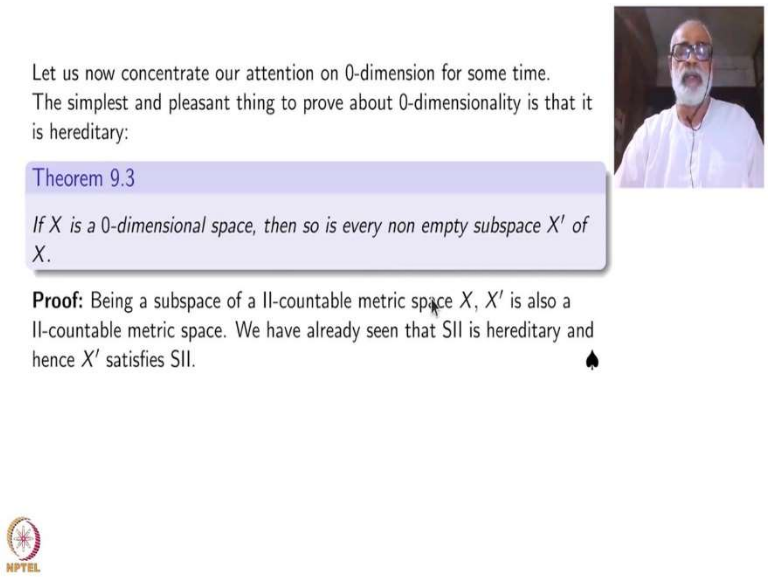

Let us now concentrate our attention on 0-dimension for some time. Slowly, you we will see that whatever you are doing for 0-dimension will be useful in the development of higher dimensions also. So, let us first concentrate on 0-dimension.

So, the first result is: If X is a 0-dimensional space then so is every non empty subset X' of X with the subspace topology.

The proof is very easy.

Being a subspace of a second countable metric space, first of all X' is also the second countable metric space, so, it is qualifies for the definition of dimension. but you have already seen that SII is hereditary.

So, X' already satisfy SII. That closes the argument. X' must be of dimension 0 because we have assumed that is non empty. If it were empty of course, then the dimension is -1.
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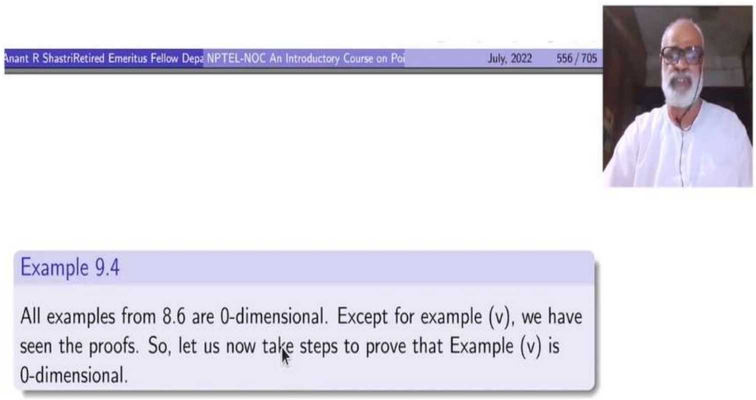

All examples in our previous chapter fit for this. Right in the beginning you can look into example, 8.6. They are all 0-dimensional. Except one, we have proved all of them, that they are 0-dimensional in the sense that they satisfy SII. We did not call them 0-dimensional at that time, in the new definition SII is the same thing as 0-dimension. So, only thing that we are left to do is to prove that example (v) is 0-dimensional, that it has a base consisting of empty boundary.
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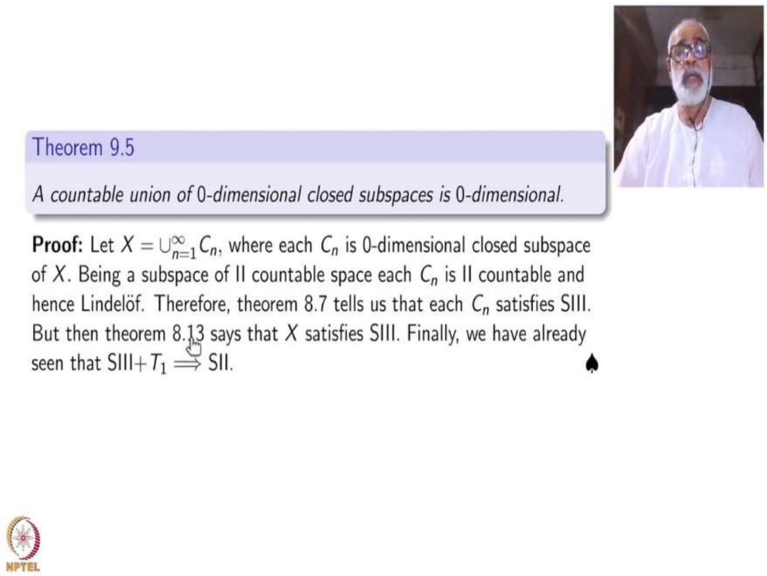

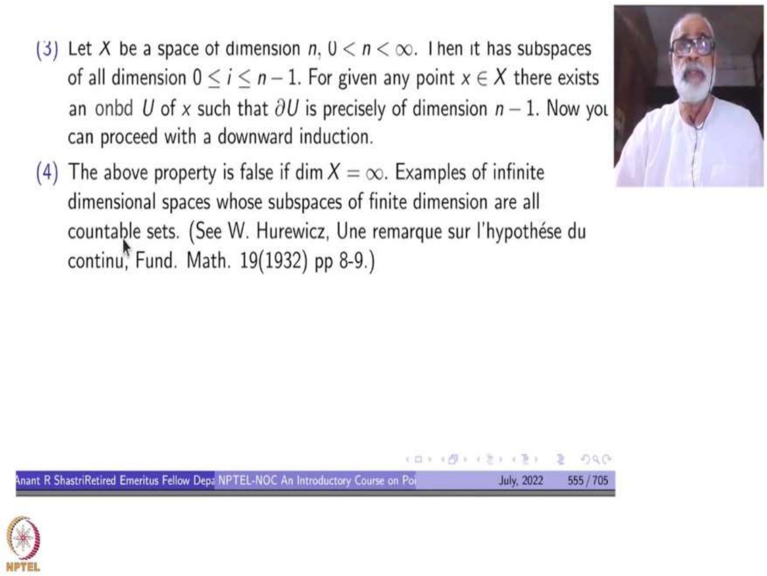

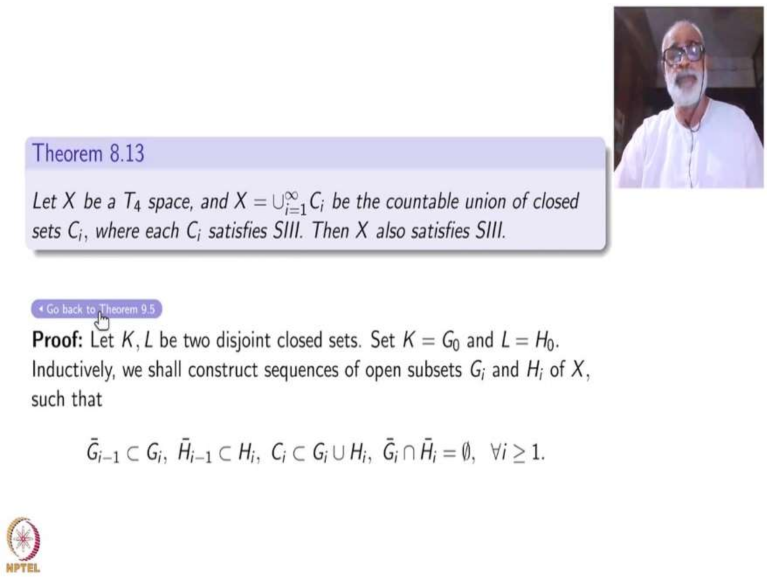

So, let us prepare to prove that one. And let us have some more theorems which will be more useful. I am just proving a theorem that will help in that example.

A countable union of 0-dimensional closed subspaces is 0-dimensional.

Just now, we had quoted a paper of Hurewicz in which you have a space, every finite dimensional subspace of which is countable. So, if you assume that these subspaces are countable, then it follows that they are actually of dimension 0 by the above theorem. Of course, the space is T_1 and hence every singleton is closed. And a countable set is a countable union of singletons. So, that is the consequence of the theorem now.

Proof is easy again. If X is the union n ranging from 1 to infinity of C_n, where each C_n is 0-dimensional closed closed subspace, being subspaces of a second countable space, each C_n is second countable and hence each of them is Lindelof. Therefore, our earlier theorem 8.7 tells us that each C_n satisfies SIII. Remember SII implies SIII under Lindelofness. That was a theorem. But then, another theorem says that X satisfies SIII, because it is countable union of these things. So, that was another theorem, I have just quoted here.

If X is a T_4 space, X is a countable union of closed sets each satisfying SIII then X also satisfies SIII.

So, that was a theorem. So, you see, all these background material we have prepared so that our life becomes easier here. So what we have got here is a countable union of 0-dimension closed subspaces is 0-dimensional. Of course, now it looks easy, but we have to use both these theorems.

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Corollary 9.6

Union of two 0-dimensional subspaces one of them is closed is 0-dimensional.

Proof: Let $X = C_1 \cup C_2$, C_1, C_2 being 0-dimensional and C_1 being closed. $X \setminus C_1$, being an open set in a metric space, is a F_σ set, say $X \setminus C_1 = \cup_{i=1}^{\infty} C_i'$, each C_i' being closed in X (slide no. 185). Being a subset of C_2, each of these C_i' is 0-dimensional. Now

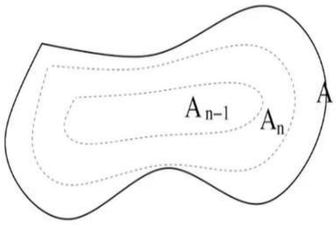
$$X = C_1 \bigcup (\cup_{i=1}^{\infty} C_i')$$

Note that for each A and each n, we have

(iii) $A_n \subset \bar{A}_n \subset A_{n+1} \subset A$ and each A_n is open. Moreover, if A is open then

$$A = \cup_n A_n = \cup_n \bar{A}_n$$

(This is what we meant by countable union of 'disc-like' subsets in the remark above.)



Proof: Let (X, d) be a pseudo-metric space. Introduce the following notation: For every subset A of X and $n \in \mathbb{N}$, let

(i) $A_n := \left\{ x \in X \ : \ d(x, A^c) > \frac{1}{2^n} \right\}$.

Use triangle inequality to check that

(ii) $d(A_n, A_{n+1}^c) \geq \frac{1}{2^n} - \frac{1}{2^{n+1}} = \frac{1}{2^{n+1}}$. (Exercise.)

Note that for each A and each n, we have

As a corollary: If we have union of just two 0-dimensional subspaces and one of them is closed, then the union is 0-dimensional. It is not a direct consequence. But how do we use the previous theorem? countable union of closed sets that is the key. So, union of two things, one of them is closed, all that you have to do is the other one, you must be able to write it as a countable union of closed subsets. That is all, being 0-dimensional, all those subspaces will be 0-dimensional also.

So, that is what we will do now, start with X equal to union of two subsets C_1 and C_2, one of them is closed both of them are 0-dimensional. So, let us assume C_1 is closed. Look at $X \setminus C_1$. That is an open subset of a metric space. Every open subset of a metric space is a countable union of closed sets, an F_σ set. So, write $X \setminus C_1$ equal to union i ranging from 1 to infinity of C_i', where each C_i' is closed in X.

You must remember this one. While proving that a metric space is paracompact, one of the things was to write an open set as a countable union of closed subsets. So, I will show you what it was just to recall. This was precisely this A_n's. Remember, these A_n are defined like this, the set of all points x in x such that $d(x, A^c)$ is bigger than $1/2^n$. $\overline{A_n}$ will be got by allowing equality as well. And union of all A_n's will be equal to the open set A. That is how we can write an open set as a countable union of closed sets.

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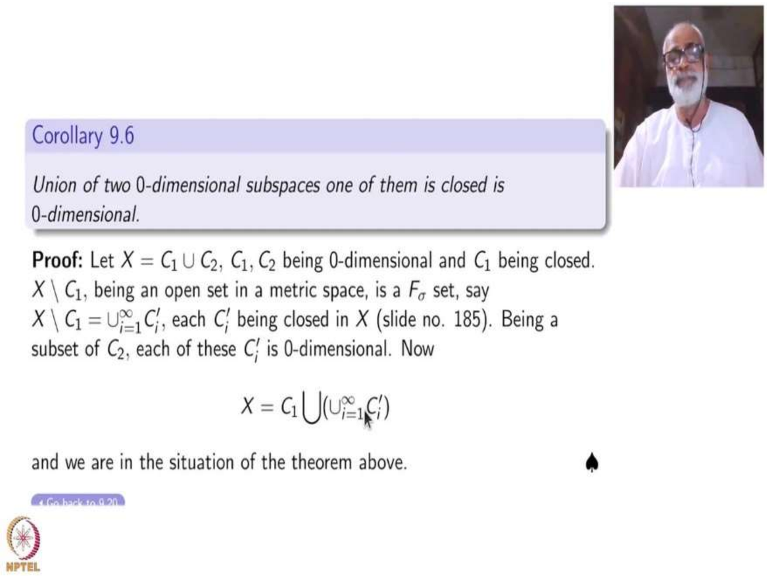

Now each C_i' being a subset of C_2 is 0-dimensonal. So, now you can apply the previous theorem to conclude the corollary.

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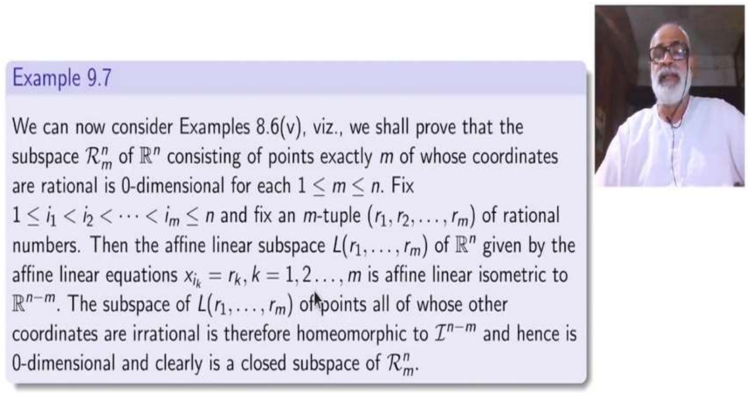

So, now, we come to the example, example 8.6 the fifth one there. Namely we shall prove that the subspace $\mathbb{R}^n_m$ of $\mathbb{R}^n$ consisting of points exactly m of whose coordinates are rational is 0-dimensional. In fact, the cases $m = 0$ and $m = n$, we have already proved. But we are not going to use that explicitly. We can directly prove this no problem. So, choose indices $i_1 < i_2 < \cdots < i_m \leq n, m$ of them, and fix them.

Not only that, next you fix rational number $r_1, r_2, \ldots, r_m$ also. Then look at the affine linear subspace which is denoted by $L(r_1, \ldots, r_m)$ of $\mathbb{R}^n$ given by these m equations, the (i_k)-th coordinate is equal to r_k for $k = 1, 2, \ldots, m$. That is clearly homeomorphic to $\mathbb{R}^{n-m}$, being a copy of $\mathbb{R}^{n-m}$ under a translation a shift coordinates that is all. So, inside this, the subspace consisting of points all of whose other coordinates are irrational is therefore homeomorphic to the subspace of $\mathbb{R}^{n-m}$ with the same property, namely $\mathbb{I}^{n-m}$.

We have proved that this is already 0-dimensional. Clearly this subspace is closed in $\mathcal{R}^n_m$, being given by a finite set of linear equations.

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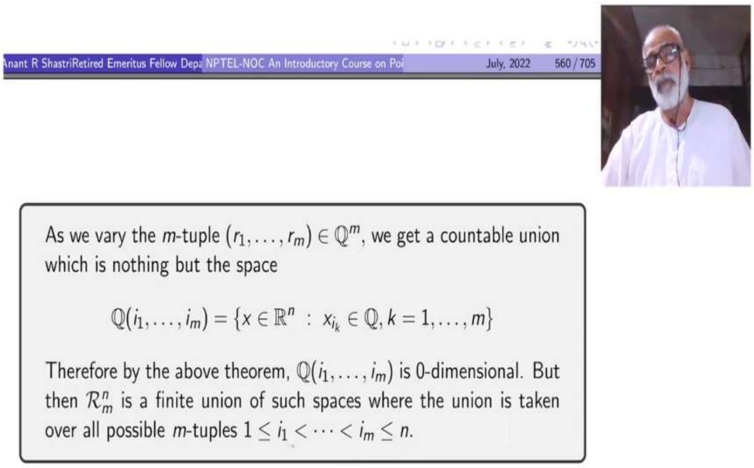

As you vary the m-tuple $(r_1, \ldots, r_m)$ over $\mathbb{Q}^{n-m}$, you will get all elements of $\mathcal{R}^n_m$ which have rational coordinates exactly at these coordinates $(i_1, i_2, \ldots, i_m)$. That is a countable union because $\mathbb{Q}^{n-m}$ is countable. Therefore the subspace $\mathbb{Q}(i_1, \ldots, i_m)$ is 0-dimensional. So, now you take another union, but this time a finite union of $\mathbb{Q}(i_1, \ldots, i_m)$ as these m-tuples vary over all possible strictly increasing functions s from $\{1, 2, \ldots, m\}$ into $(1, 2, \ldots, n-1)$, to obtain the space $\mathcal{R}^n_m$.

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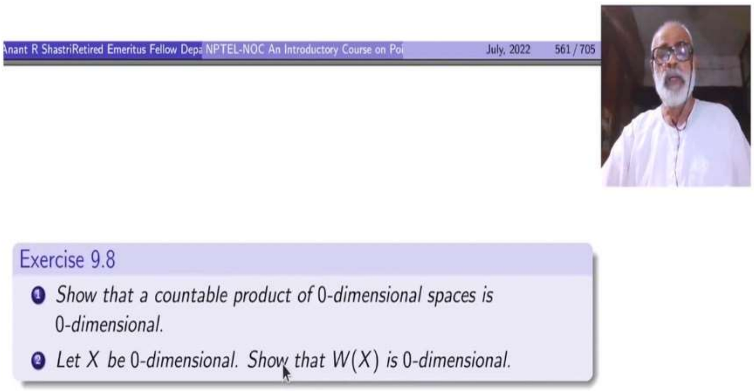

So, here are some elementary exercises you can try them on your own. Of course, there will be some TA's to help you if you do not get it. Show that a countable product of 0-dimensional space is 0-dimensional.

Next, suppose X is 0-dimensional, show that its Wallman compactification is 0-dimensional. This may be a little challenging but if you think a little then you will get it.

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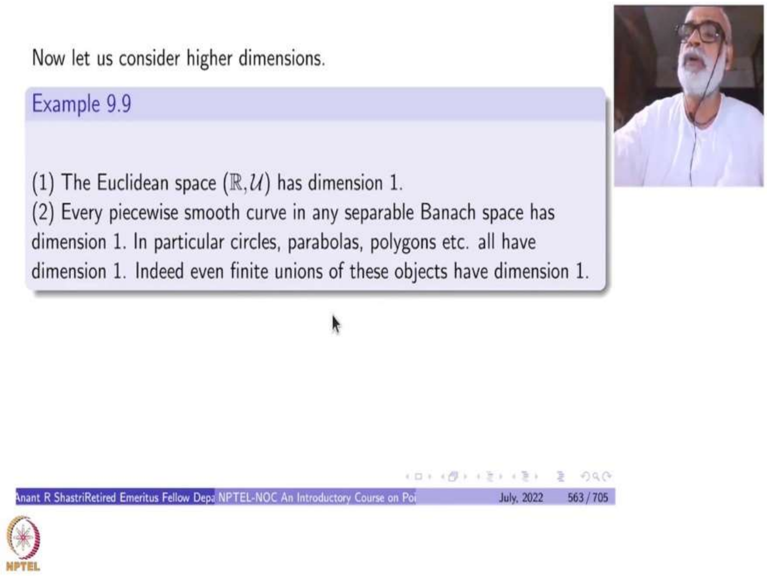

Let us now go to higher dimension. So, to begin with of course, we will have some examples. The first example as such should be our motivating example. Namely, the real line with the usual topology has dimension one as I have pointed out. The fundamental system of neighborhoods for $\mathbb{R}$ is the set of all bounded open intervals. The boundary of an open interval is just a two point set. which is 0-dimensional. So, that qualifies $\mathbb{R}$ to be dimension one.

Every piecewise smooth curve in any separable Banach space has dimension one. Why? because look at the smooth parts they are open parts, so they have dimension one being homeomorphic to open intervals. That is enough for us, why? On each of components of the smooth parts, by inverse function theorem, there is a diffeomorphism and hence is homeomorphism to an open interval in $\mathbb{R}$.

So, in particular circles, parabolas, any polygon etcetera, all of them have dimension 1. In fact countable union of these things are also 1-dimensional inside $\mathbb{R}^2$, $\mathbb{R}^3$ and so on because each of circles, parabolas etc they are all given by polynomial equations. So, they are closed subsets. That is why countable unon of these things will also be 1-dimensional.

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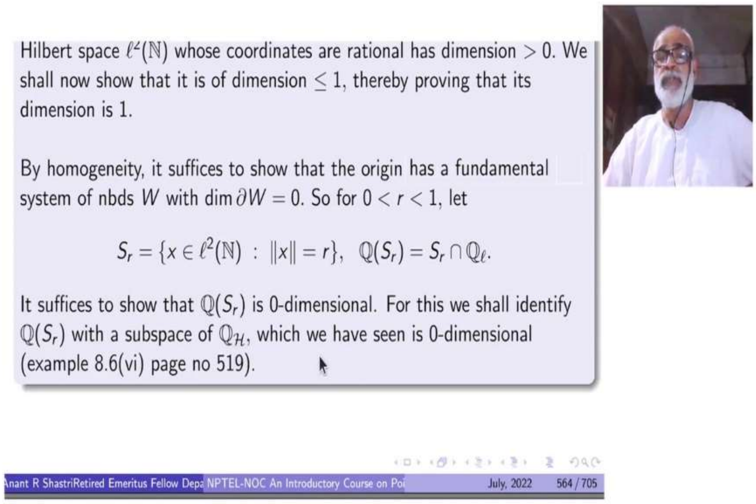

In example, 8.6(viii), we have seen that the subspace $\mathbb{Q}_\ell$ of all points in the Hilbert space $\ell^2(\mathbb{N})$, whose coordinates are all rational is of positive dimension. We shall now show that it is actually of dimension less than or equal to 1 and hence dimension is equal to 1. So, by homogeneity, (homogeneity is what, any point can be moved to any other point by a self homeomorphism inside of $\ell^2(\mathbb{N})$) it is enough to consider one single point say origin has a fundamental system of neighbourhoods W, with boundary of W having dimension 0. That is what we want to show. (If you have a fundamental system of neighborhoods which are clopen sets then the space itself would be of dimension 0.)

So, for $0 < r < 1$, let us have this notation: S_r is the set of all x belonging to ℓ^2 such that the $\|x\| = r$, i.e., nothing but summation x_i^2 is equal to r^2. Take $\mathbb{Q}(S_r)$ to be $S_r \cap \mathbb{Q}_\ell$, namely, all points x of S_r with each x_i being rational.

It suffices to show that $\mathbb{Q}(S_r)$ is 0-dimensional. Why? because S_r is the boundary of a system of neighborhoods which form a local base at 0 for the space ℓ^2 here. Intersected with $\mathbb{Q}_\ell$, they form a fundamental system of neighborhoods at 0 for $\mathbb{Q}_\ell$ with their boundaries S_r intersection with $\mathbb{Q}_\ell$. That is want we are interested in. So I will show that these are 0-dimensional. For this we shall identify $\mathbb{Q}(S_r)$ with a subspace of $\mathbb{Q}_\mathcal{H}$ where $\mathcal{H}$ is our Hilbert cube, and $\mathbb{Q}_\mathcal{H}$ is the set of all those points with all the coordinates rational. This we have seen is 0-dimensional just before this example, viz., in example 8.6 (vi).

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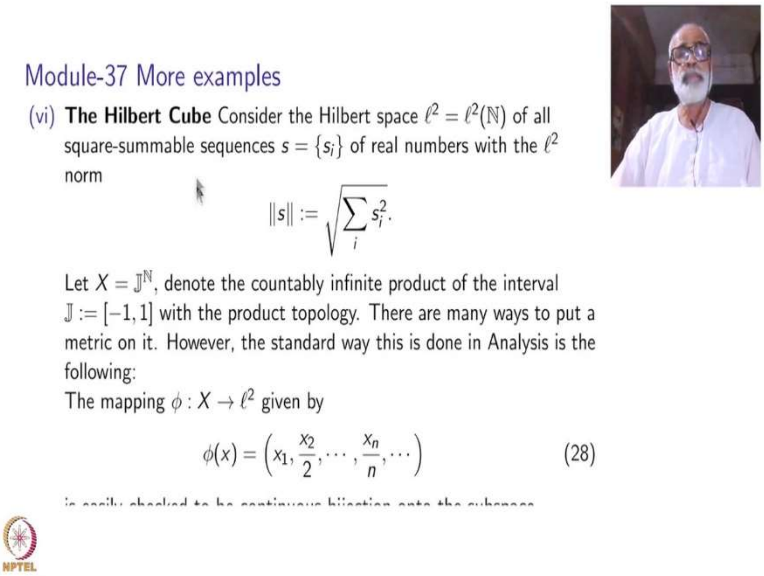

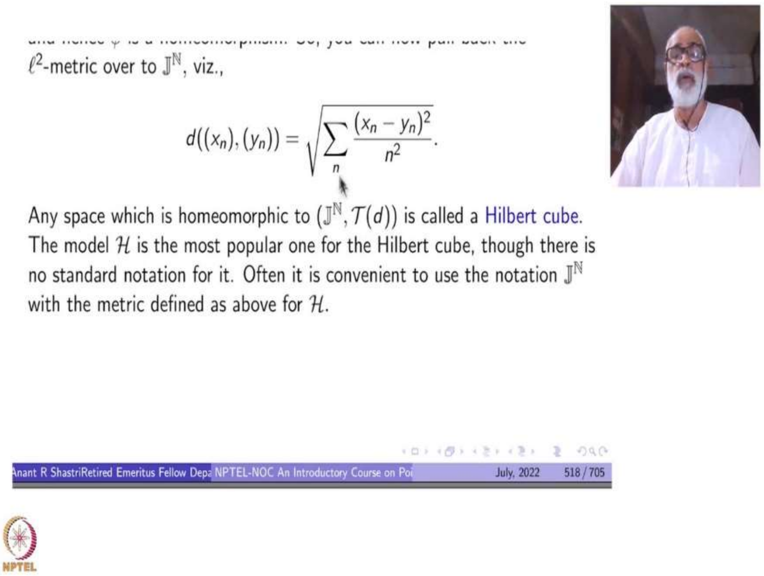

So, this was the example where we showed that $\mathbb{Q}_{\mathcal{H}}$ here is $\mathbb{J}^{\mathbb{N}}$. Remember that $\mathbb{Q}_{\mathcal{H}}$ was homeomorphic to this $\mathbb{J}^{\mathbb{N}}$, with the product topology and this $\mathcal{T}_d$ was the metric induced topology, etc.

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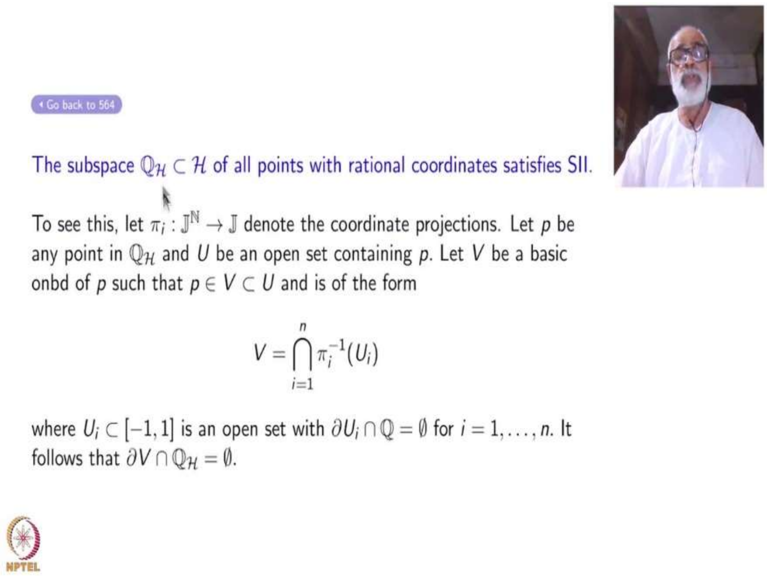

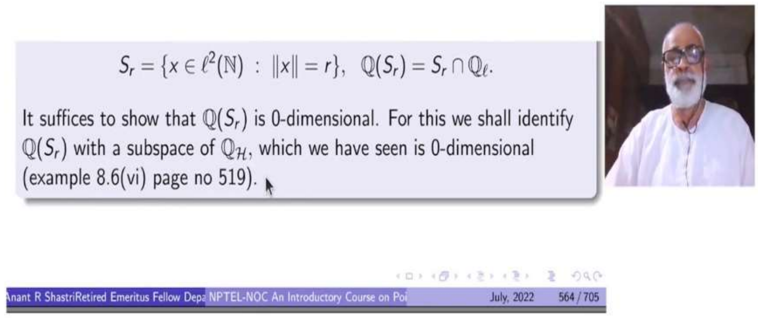

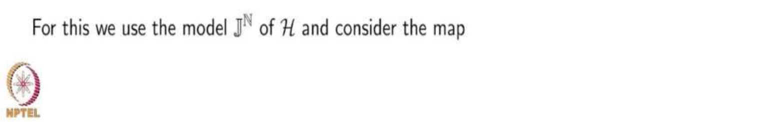

So, this $\mathbb{Q}_{\mathcal{H}}$ is 0-dimensional is what we have shown. So, we are going to show that this $\mathbb{Q}(S_r)$ is homeomorphic with a subspace of $\mathbb{Q}_{\mathcal{H}}$.

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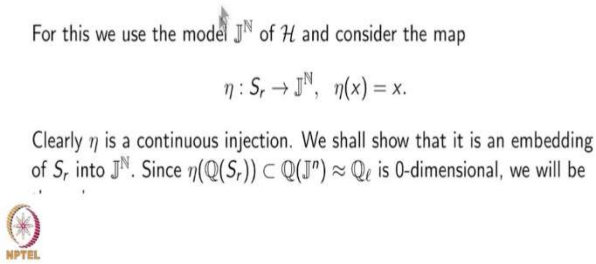

For this, we use instead of $\mathcal{H}$ we use this $\mathbb{J}^{\mathbb{N}}$, product of countably infinite copies of the closed interval $[-1, 1]$. So, consider the map η from S_r to $\mathbb{J}^{\mathbb{N}}$, which is just the identity map. Remember, this is the ball of radius r, r lies between 0 and 1, positive but less than 1. Therefore, each coordinate of a point x of S_r lies in $[-1, 1]$. (It is between $-r$ and r, actually.) So, this identity map makes sense, no problem.

Indeed, it is continuous, why? because any function into the product space is continuous if and only if the coordinate functions are continuous. In S_r as a subspace of ℓ^2, if you take just the i^{th}-coordinate, that is a continuous function all right. So, these are continuous functions. So, identity map we can call it as the inclusion. We should show that it is an embedding of S_r into $\mathbb{J}^{\mathbb{N}}$. Embedding means what, it is a homeomorphism onto the image. That is all I have to show. For then, $\eta(\mathbb{Q}(S_r))$ will go into the subspace with rational coordinates, which is nothing but $\mathbb{Q}_\ell$ and that is 0-dimensional; we will be done.

So, we have to show that this eta is an embedding. A continuous injection, when is it an embedding? You should either show that it is closed or it is open. Closed map is equivalent to open map because already it is an injective mapping. So that is all. So, let us try to show that it is a closed mapping.

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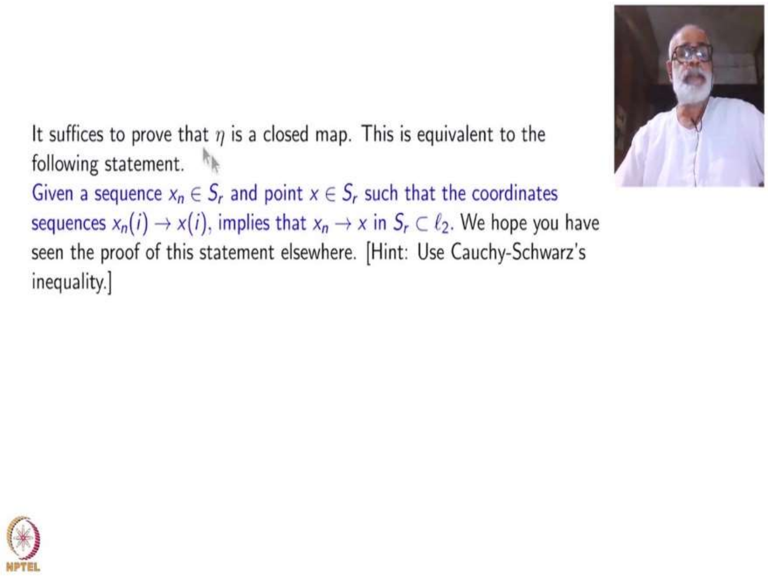

But this is equivalent to the following statement:

(Take a closed set inside S_r, its image is closed inside ℓ^2. That is what I want to show. Instead of been ℓ^2, the Hilbert cube, we have changed to the model $\mathbb{J}^{\mathbb{N}}$. So, inside $\mathbb{J}^{\mathbb{N}}$, it should be closed that is what I would show. It is the same thing as) taking a sequence inside the image, such that each coordinate function is convergent. That is the meaning of a sequence inside the product space is convergent. Then I have to show that the sequence is convergent in S_r itself.

So, correct statement I repeat, given the sequence $\{x_n\}$ in S_r and a point x in S_r such that the coordinates sequence $x_n(i)$ converges this to $x(i)$, it should imply that $\{x_n\}$ converges to x inside S_r, wherein I have to use the topology of ℓ^2 here. We hope you have seen the proof of this statement in the blue color, somewhere else. So, if not you can try it. I have given you a hint here: Use Cauchy Schwarz's.

Of course, if you do not get it, we will explain it to you. I think today it is enough. So, tomorrow we will continue with the study of higher dimensional spaces. Thank you.