An Introduction to Point-Set-Topology (Part II)
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Lecture 4
Implicit and Inverse Function Theorems

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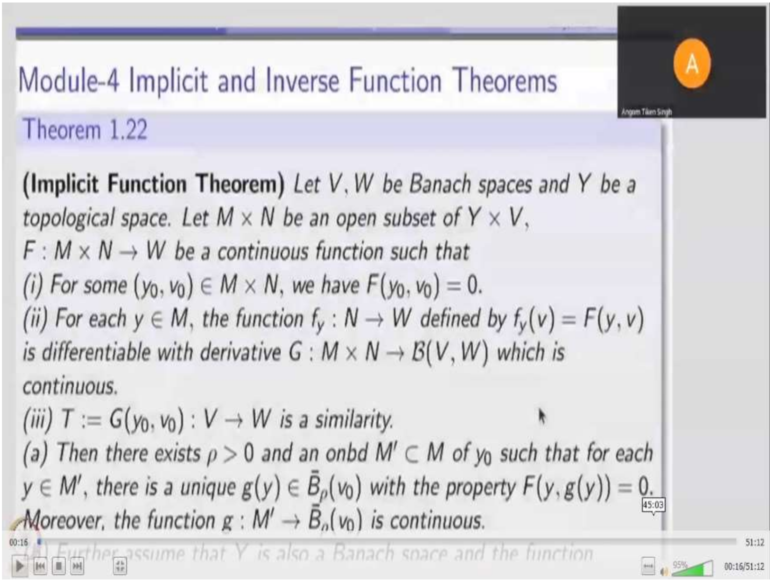

Hello, welcome to NPTEL NOC, an introductory course on Point Set Topology, Part II, Module 4. So, in past three modules we have been preparing for a proof of implicit and inverse function theorems. Last time we also saw the statement of the implicit function theorem.

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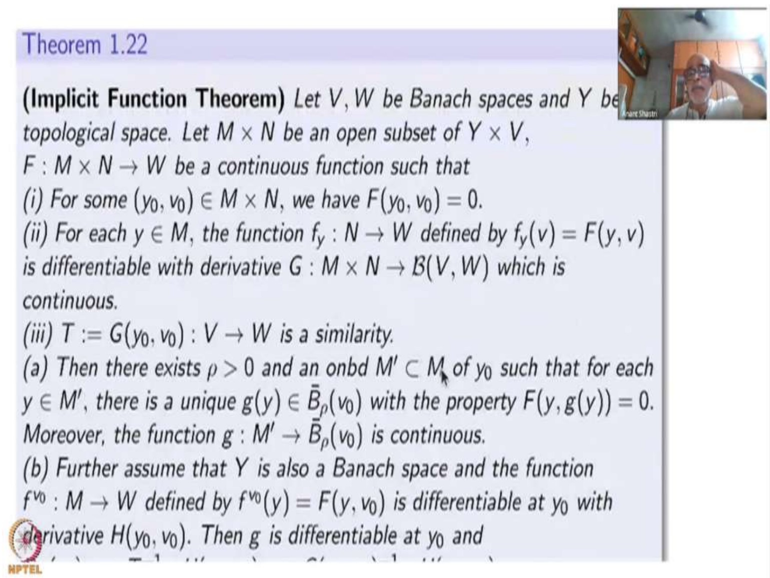

I have explained the statement last time. Let me just recall this one.

Now, only V and W are Banach spaces and Y is any topological space. $M \times N$ be an open subset of $Y \times V$ and F from $M \times N$ to W be a continuous function. Then it is assumed to satisfy three more conditions. What are they?

(i) For some point (y_0, v_0) inside $M \times N$, $F(y_0, v_0)$ is 0.

(ii) For each $y \in M$, the function f_y which sends v to $F(y, v)$ is differentiable as a function from N to W and the associated derivative function G from $M \times N$ to $\mathcal{B}(V, W)$ is continuous.

(iii) And the derivative $G(y_0, v_0) = D(f_{y_0})(v_0)$ is an isometry of V to W.

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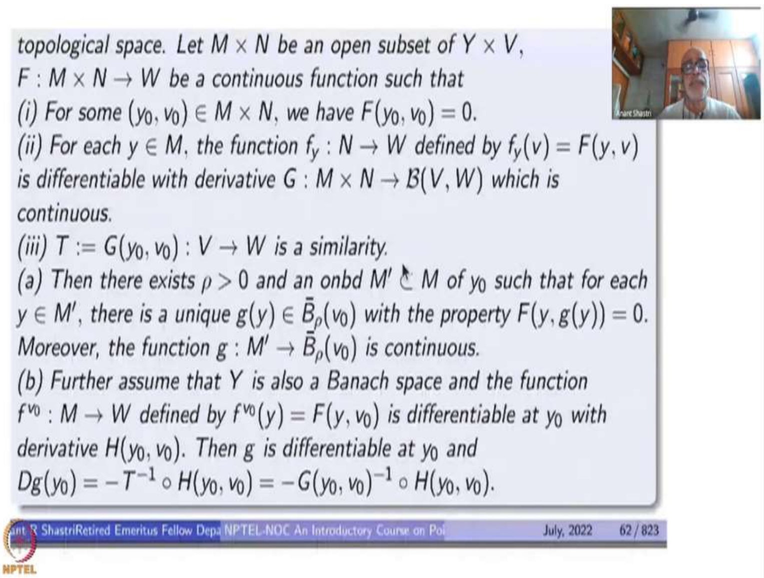

(a) Then there exists ρ positive and an open neighborhood to M' of y_0 inside M, and the function g from M' into the closed ball of radius ρ around v_0 such that such that $F(y, g(y)) = 0$ for all $y \in M'$. Moreover, this function g from M' to $\bar{B}_\rho(v_0)$ is continuous. That is the first conclusion.

The second conclusion, conclusion (b) requires one more hypothesis; namely, assume that Y is also a Banach space and the function F restricted to $v = v_0$ from this M to W, namely, $f^{v_0}(y) = F(y, v_0)$ is differentiable at y_0, and its derivative denoted by H. Then g will be differentiable at y_0 and the derivative of g is given by $-T^{-1}H$. So, let us start proving this one now. There are a number of steps to be taken so that we understand what is going on. So, the proof is broken up into smaller steps.

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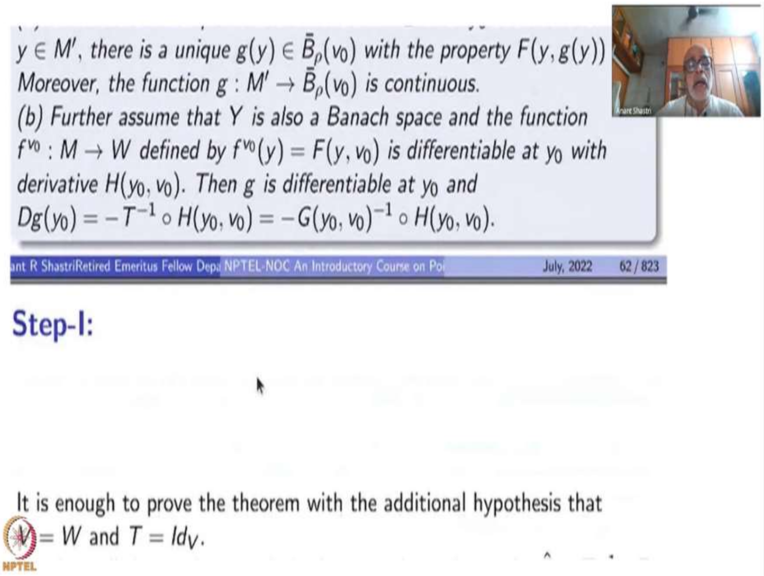

The first step is I want to make a simplification in the statement as well as in the proof. Namely, I would like to reduce to the special case when this T is the identity map. How can I do that? Namely, by composing with T^{-1} from W to V, and as if we are working now all the time inside V. From W we come back to V via T^{-1}. Keep coming into V. This is what I want to do. So, how do I do that? So, as follows.

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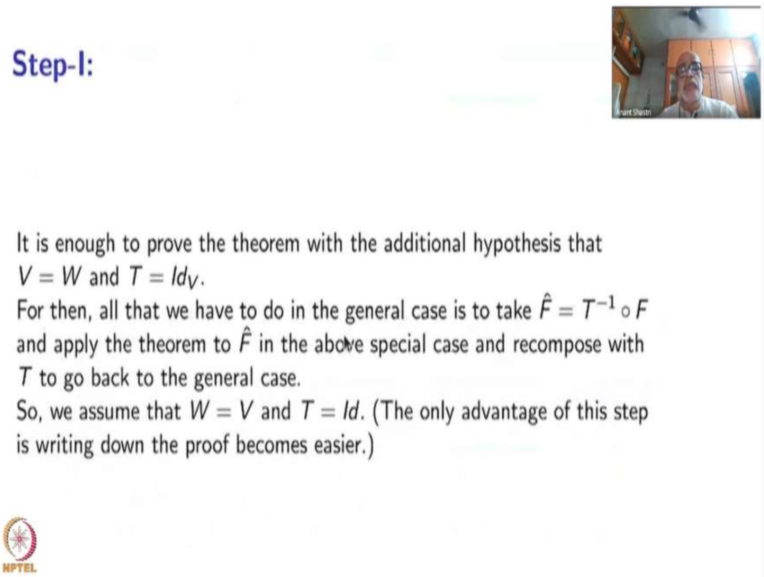

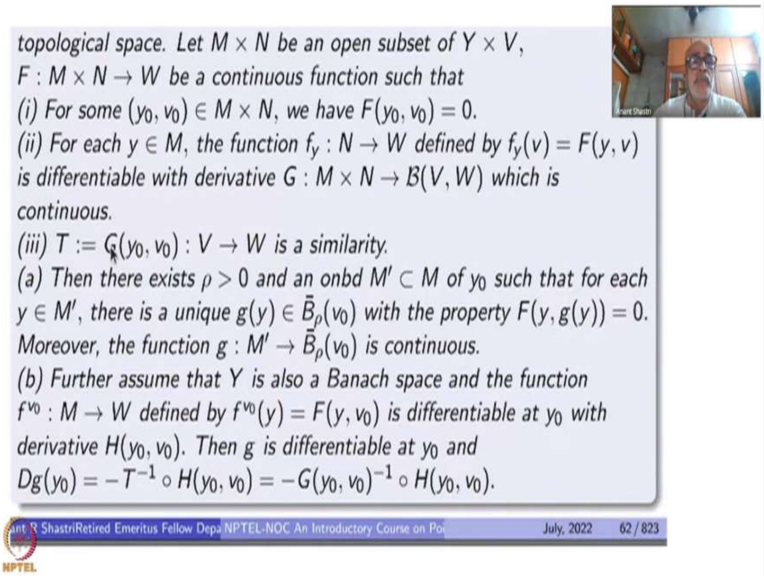

Namely, replace F by $\hat{F}$ which is $T^{-1}F$. Remember F was from $M \times N$ to W. Now, $\hat{F}$ will be from $M \times N$ into V itself. So, that is all. Now, suppose we have proved this theorem for $\hat{F}$, in this special case. Then we can go back to the original statement again composing by T.

So, if we replace F by $\hat{F}$, then the its derivative $\hat{G}(y_0, v_0)$ will be equal to $T^{-1} \circ T$ which Identity map of V, because the derivative of a linear automorphism is itself. The derivative of a composition is by chain rule, is the composite of the derivatives. So, $\hat{G}(y_0, v_0)$ will be equal to $T^{-1} \circ T$ which Identity map of V.

So, this is what we want to assume, $V = W$ and $G(y_0, v_0)$ is Id_V, so that writing down the proof will become easier. I am not have to keep on writing T here that is all. So, that is the first step. As we have proved it for the Special case we know that it is true for the general case also.

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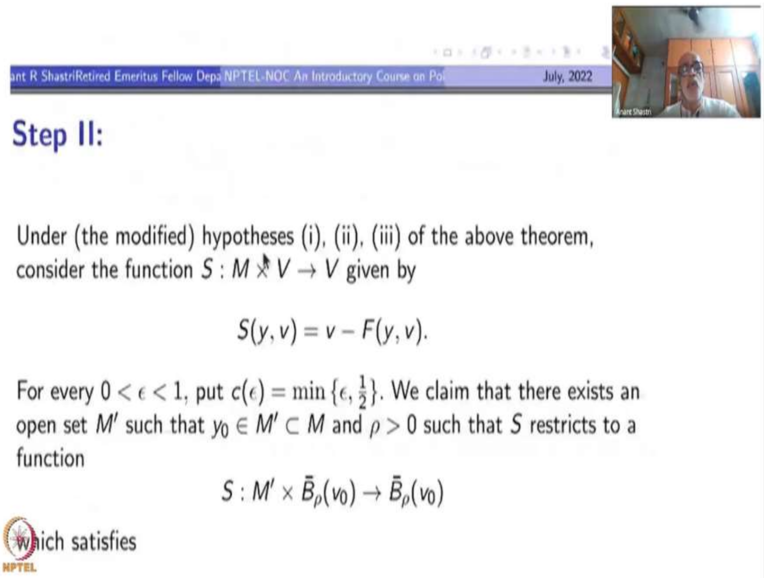

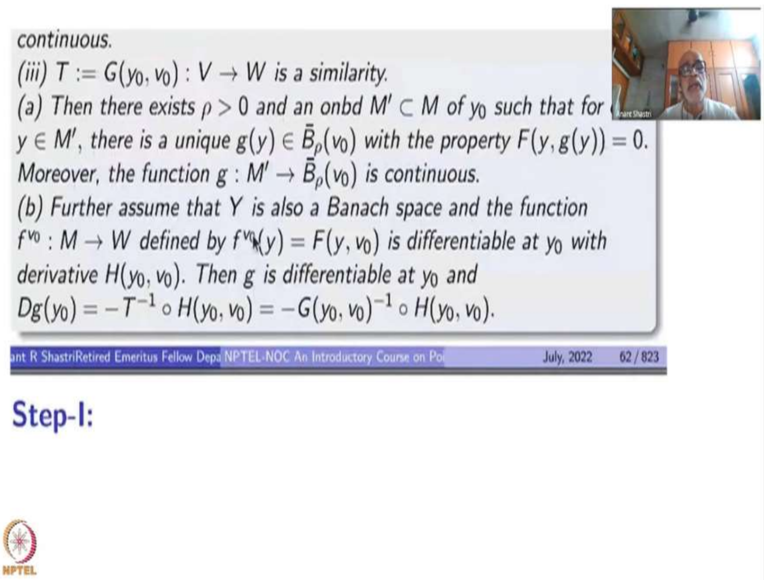

In the second step, now, we have the modified hypothesis. All 1, 2, 3 are all modified, namely, $W = V$ etc. and F is from $M \times N$ to V, etc. Now, you take a new function S defined by $S(y, v) = v - F(y, v)$. F is a function from $M \times N$ to V. So, S is also a function from $M \times N$ to V.

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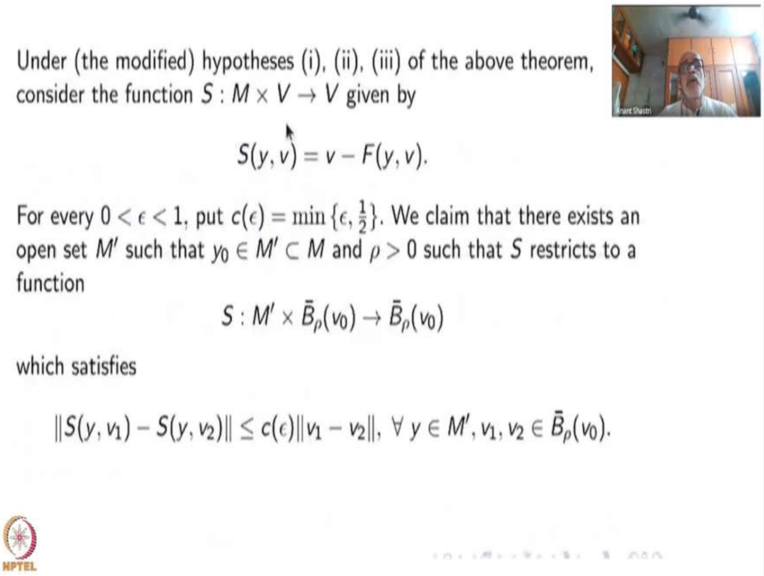

For every ϵ between 0 and 1, let us have this short notation $c(\epsilon)$= minimum of ϵ and $1/2$. First we claim that there is an open subset M' such that y_0 is inside M' and M' is contained inside M and there is a positive ρ such that S restricted to M' cross the closed ball goes inside the closed ball. And satisfies this inequality. This is our second step. Part of this, remember, is the existence of this M' and ρ.

This was a part of (a) right? But the conclusion is not exactly same as in (a). It is apparently a weaker conclusion. We are not yet saying that there is a unique g and etc. g has not yet appeared. The first thing is the new function S has this property namely, the $\|S(y, v_1) - S(y, v_2)\| \leq c(\epsilon)\|v_1 - v_2\|$, for every $y \in M'$, v_1 and v_2 inside the closed ball.

Remember, this was nothing but a uniform contraction. So, we are going to apply contraction mapping which was done in the first module. Remember that? So, first we have to claim this one. So, first let us get the proof of this part. How to prove this one?

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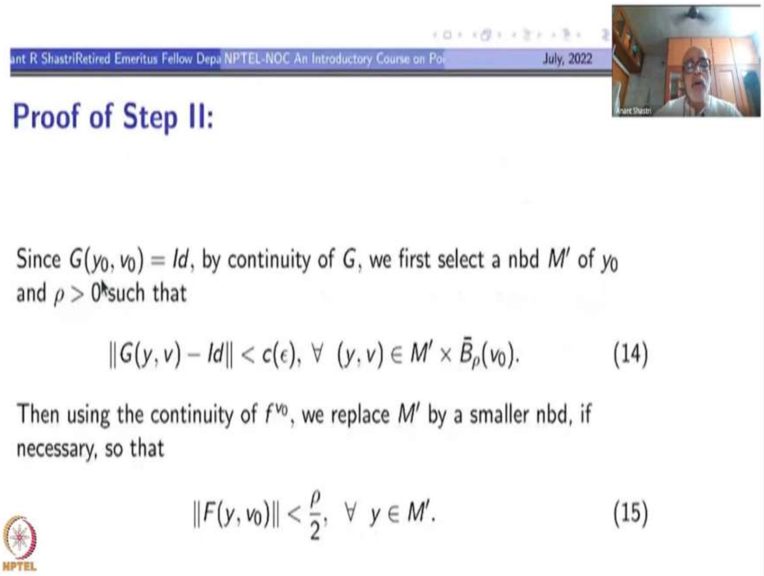

$G(y_0, v_0)$ is identity of V now. (Earlier it was just a similarity T). In the new hypothesis, it is identity. Using continuity of G, we first select a neighborhood M' of y and a ρ positive such that $G(y, v)$ minus identity is less than $c(\epsilon)$. So, this is where the continuity of the partial derivative of F in the second variable that is used. This G was the differentiation of capital F, with respect to the second variable v.

So, by continuity of this, some neighborhood of y_0 and some neighborhood of v_0 will go inside the $c(\epsilon)$ ball. In the neighborhood of v_0, I can choose an open ball, of course, so that its closure is contained in that neighbourhood. No problem. But for neighborhood of y_0, since Y is some arbitrary space, I do not have any balls there yet. For every (y, v), inside M' cross this closed ball, this inequality holds.

Now, use the continuity of $f^{v_0)}$, to replace M' by a smaller neighborhood if necessary, so, that norm of $F(y, v_0)$ is less than $\rho/2$ for every $y \in M'$. So, there is a modification of M' at the second stage. This new M' will depend upon the ρ. The number role is chosen in the first instance. Then I am choosing M' sufficiently small so that another inequality is satisfies. So, the choice of M' will depend upon ρ. (If I was working with original T, then I had to take care of factor of $\|T^{-1}\|$ here. No, it is easier.)

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Fix $y \in M'$ and put $S_y(v) = S(y, v)$. Consider the derivative of S_y with respect to v:

$$D_v(S_y)(v') = Id - G(y, v').$$

Therefore

$$\begin{aligned} \|D_v(S_y)(v')\| &= \|Id - G(y, v')\| \\ &\leq c(\epsilon) \leq \tfrac{1}{2}, \ \forall \ (y, v') \in M' \times \bar{B}_\rho(v_0). \end{aligned}$$

Therefore

$$\|S_y(v_2) - S_y(v_1)\| \leq \frac{1}{2}\|v_2 - v_1\|, \ \forall \ (y, v') \in M' \times \bar{B}_\rho(v_0)$$

Step II:

Under (the modified) hypotheses (i), (ii), (iii) of the above theorem, consider the function $S : M \times V \to V$ given by

$$S(y, v) = v - F(y, v).$$

For every $0 < \epsilon < 1$, put $c(\epsilon) = \min\{\epsilon, \frac{1}{2}\}$. We claim that there exists an open set M' such that $y_0 \in M' \subset M$ and $\rho > 0$ such that S restricts to a function

$$S : M' \times \bar{B}_\rho(v_0) \to \bar{B}_\rho(v_0)$$

which satisfies

Since $G(y_0, v_0) = Id$, by continuity of G, we first select a nbd M' of y_0 and $\rho > 0$ such that

$$\|G(y, v) - Id\| < c(\epsilon), \ \forall \ (y, v) \in M' \times \bar{B}_\rho(v_0). \quad (14)$$

Then using the continuity of f^{v_0}, we replace M' by a smaller nbd, if necessary, so that

$$\|F(y, v_0)\| < \frac{\rho}{2}, \ \forall \ y \in M'. \quad (15)$$

Now, fix a $y \in M'$ and put $S_y(v) = S(y, v)$. Remember, what was $S(y, v)$, by definition, it is $v - F(y, v)$. So, S_y is a function of v, y is fixed. The derivative of S_y at v' with respect to v is nothing but, derivative of v is identity map on v' minus this part is $G(y, v')$. I am just using this formula, this definition here. This derivative of this one is identity minus the derivative of F, all right. So, $D(S_y)(v') = v' - G(y, v')$.

Therefore, the norm of this, which is less than equal to the norm of identity minus G of this one that is less than $c(\epsilon)$. See, G minus identity less than $c(\epsilon)$, that is (14). So, $c(\epsilon)$ is less than or equal to half because $c(\epsilon)$ is the minimum of ϵ and $1/2$. This inequality holds for every (y, v') belonging to $M' \times \bar{B}_\rho$. Here, once the derivative has less than or equal to $1/2$ we know that $S_y(v_2) - S_y(v_1)$ is less than or equal to this constant $1/2(v_2 - v_1)$. So, this is the mean value theorem that we have proved. So, what is this one? This constant is less than one. Therefore, this S_y is a contraction mapping.

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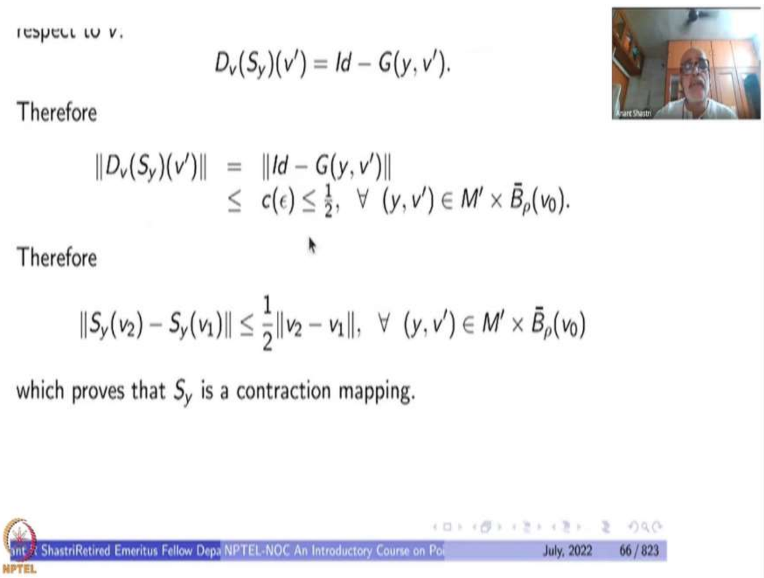

In order to apply the contraction mapping principle, we have yet to show that this S_y take the closed ball inside the closed ball. A closed ball in a Banach space is a complete metric space on its own. Then we can apply contraction mapping.

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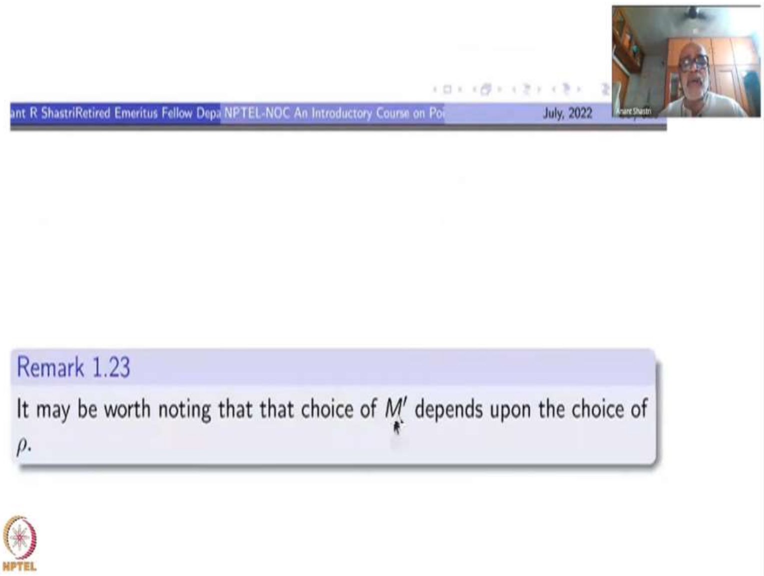

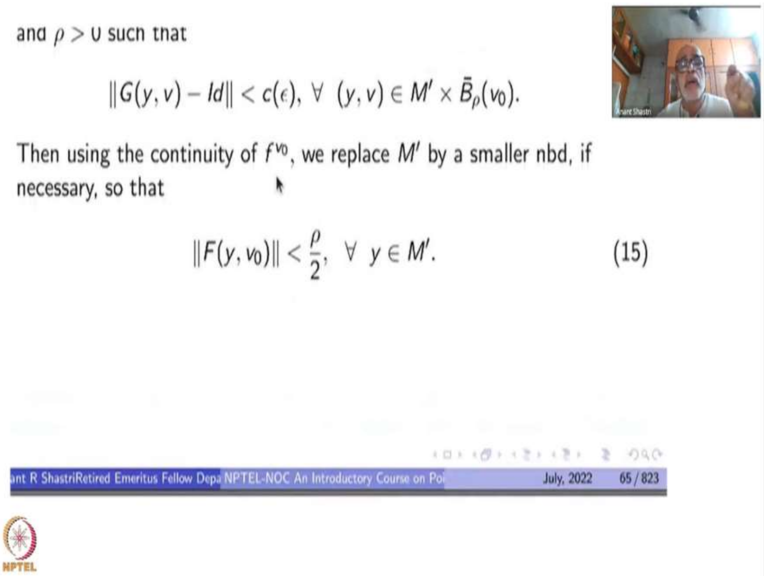

So, next step, we have to do this one. So, here I have made a remark, which I have already explained. So, we have to prove that the closed ball goes inside a closed ball and under S_y. So, that we can think of S_y as a contraction mapping inside this metric space which is a complete metric there. So, why this is true? Take any v such that $\|v - v_0\| \leq \rho$, that means a point of the closed ball of radius ρ around v_0. Then I should show that the norm of $S_y(v) - v_0$ is less that to equal to ρ. So, that will prove this statement.

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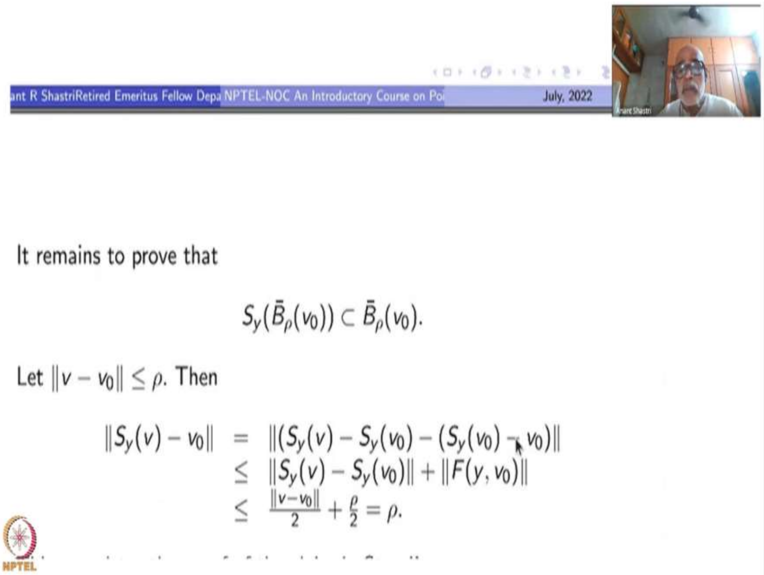

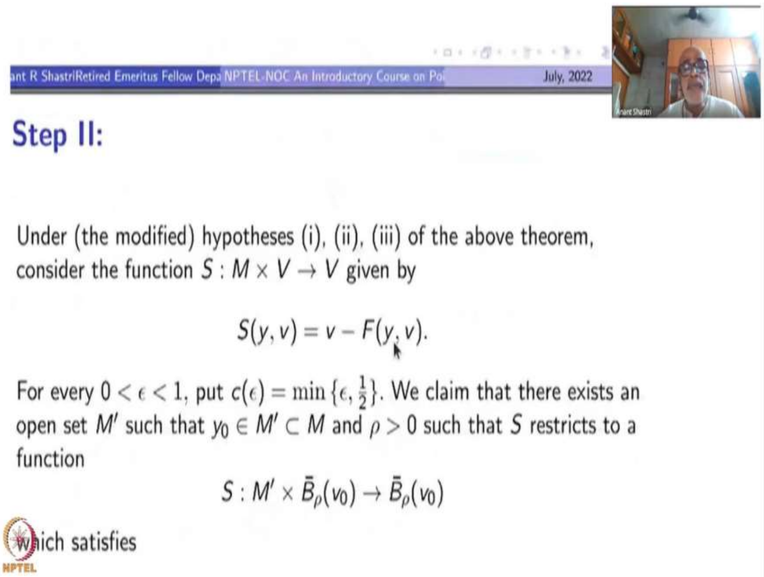

So, I am looking at the norm of this one. Now, here you add and subtract $S_y(v_0)$. The first term is less than or equal to $\|v - v_0\|/2$. The second term is norm of $F(y, v_0)$ which is less than or equal to $\rho/2$. Therefore LHS is les than or equal to ρ.

This is minus of minus that will plus. Norm when you take they are the same. $F(y, v_0)$ is this, let me bring it this one, this one on this side, so, v minus this one. What I am telling here $S(y, v_0)$ here $F(y, v_0) - v$ will be equal to $v - F(y, v_0)$ with a negative sign, the norm will be the same.

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$$S_y(\bar{B}_\rho(v_0)) \subset \bar{B}_\rho(v_0).$$

Let $\|v - v_0\| \leq \rho$. Then

$$\begin{aligned}
\|S_y(v) - v_0\| &= \|(S_y(v) - S_y(v_0) - (S_y(v_0) - v_0)\| \\
&\leq \|S_y(v) - S_y(v_0)\| + \|F(y, v_0)\| \\
&\leq \frac{\|v - v_0\|}{2} + \frac{\rho}{2} = \rho.
\end{aligned}$$

This completes the proof of the claim in Step II.

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respect to v:

$$D_v(S_y)(v') = Id - G(y, v').$$

Therefore

$$\begin{aligned}
\|D_v(S_y)(v')\| &= \|Id - G(y, v')\| \\
&\leq c(\epsilon) \leq \tfrac{1}{2}, \ \forall \ (y, v') \in M' \times \bar{B}_\rho(v_0).
\end{aligned}$$

Therefore

$$\|S_y(v_2) - S_y(v_1)\| \leq \frac{1}{2}\|v_2 - v_1\|, \ \forall \ (y, v') \in M' \times \bar{B}_\rho(v_0)$$

which proves that S_y is a contraction mapping.

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$$\|F(y, v_0)\| < \frac{\rho}{2}, \ \forall \ y \in M'.$$

Fix $y \in M'$ and put $S_y(v) = S(y, v)$. Consider the derivative of S_y with respect to v:

$$D_v(S_y)(v') = Id - G(y, v').$$

Therefore

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So, now, we can come to the proof of statement (a) that is the step three. V is a Banach space, every closed ball in it is a complete metric space. Therefore by step 2, we can apply the contraction mapping theorem to conclude that S_y has a unique fixed point. We define g from M' into $\bar{B}_\rho$ by the formula $S_y(g(y)) = g(y)$. For each y there is only one unique map, that is important, there is one unique point inside this ball with this property. By definition, this is equivalent to saying that $F(y, g(y)) = 0$, because $S(y)(v) = v - F(y, v)$.

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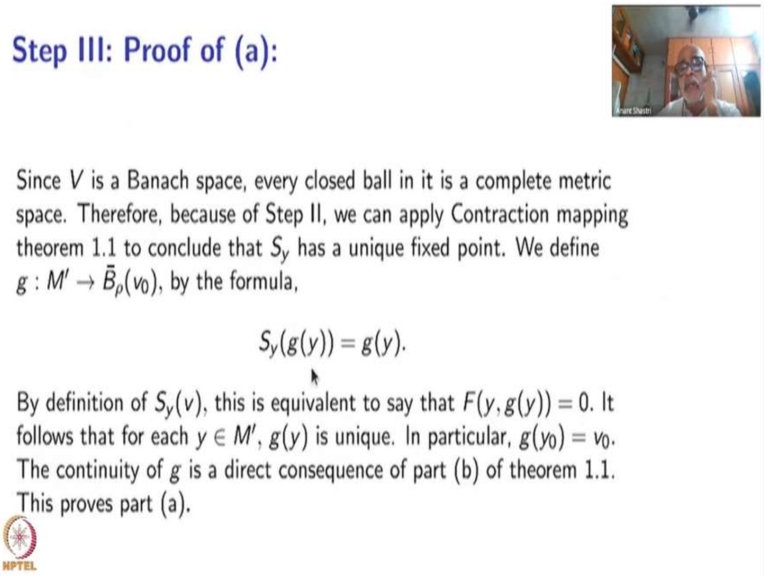

In particular $g(y_0) = v_0$, because we have assumed $F(y_0, v_0) = 0$. That was a starting hypothesis. The continuity of g is a direct consequence of Part (b) of the contraction-mapping theorem, which we have proved in the first module. Since S_y is continuous in y, continuity of g follows again what we have proved there. So, this proves part (a).

Now, the proof of Part (b). Assume that Y is a Banach space. We may assume that M' is a convex neighborhood of y_0, in both is (14) and (15), by replacing M' by a smaller set if necessary. This did not make sense earlier when Y was not a vector space.

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Step IV : Proof of part (b):

Since we now assume that Y is a Banach space, we may also assume that the nbd M' is convex in (14) and (15). Then by theorem 1.21, along with (14), it follows that

$$\|F(y,v) - F(y,v_0) - (v - v_0)\| \leq c(\epsilon)\|v - v_0\|, \ \forall (y,v) \in M' \times \bar{B}_\rho(v_0).$$

Take $y \in M'$ and put $v = g(y)$. Then

$$\|F(y,g(y)) - F(y,v_0) - (g(y) - v_0))\| \leq c(\epsilon)\|g(y) - v_0\|.$$

Fix $y \in M'$ and put $S_y(v) = S(y,v)$. Consider the derivative of S_y wit respect to v:

$$D_v(S_y)(v') = Id - G(y,v').$$

Therefore

$$\begin{aligned} \|D_v(S_y)(v')\| &= \|Id - G(y,v')\| \\ &\leq c(\epsilon) \leq \tfrac{1}{2}, \ \forall \ (y,v') \in M' \times \bar{B}_\rho(v_0). \end{aligned}$$

Therefore

$$\|S_y(v_2) - S_y(v_1)\| \leq \frac{1}{2}\|v_2 - v_1\|, \ \forall \ (y,v') \in M' \times \bar{B}_\rho(v_0)$$

which proves that S_y is a contraction mapping.

and $\rho > 0$ such that

$$\|G(y,v) - Id\| < c(\epsilon), \ \forall \ (y,v) \in M' \times \bar{B}_\rho(v_0).$$

Then using the continuity of f^{v_0}, we replace M' by a smaller nbd, if necessary, so that

$$\|F(y,v_0)\| < \frac{\rho}{2}, \ \forall \ y \in M'. \tag{15}$$

But once we know that Y is a Banach space, we can choose M' to be a convex neighborhood of the point y_0. So, having made that demand on M', let us continue now for the proof of this part, here.

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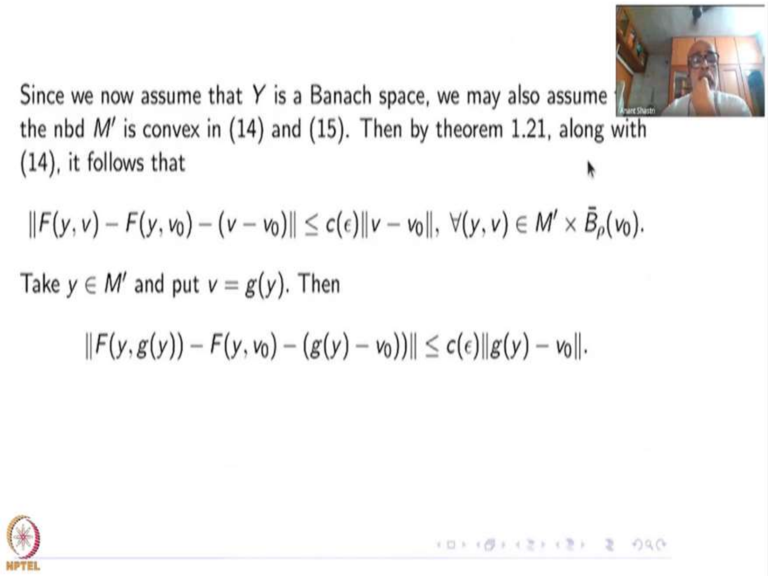

By theorem 1.21 now, because of convexity along with this hypothesis (14) what happens?

The norm of $F(y,v) - F(y,v_0) - (v - v_0)$ (recall T is identity here) is less than equal to $c(\epsilon)\|v - v_0\|$, for every $(y,v) \in M' \times \bar{B}_\rho$. So, this was the mean value inequality. We have proved this theorem 1.21. Put $v = g(y)$ in this formula.

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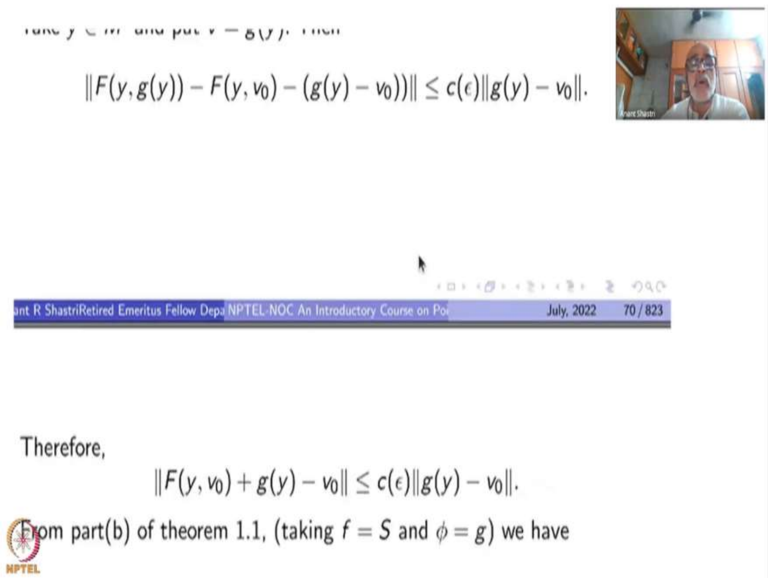

Then we have norm of $F(y, g(y)) - F(y, v_0) - (g(y) - v_0)$ is less than or equal to $c(\epsilon)\|g(y) - v_0\|$. Since $F(y, g(y)) = 0$, this becomes $\|F(y, v_0) + g(y) - v_0\|$ is less than equal to $c(\epsilon)\|g(y) - v_0\|$. So, because I am taking a norm I can convert all these negative signs into positive signs.

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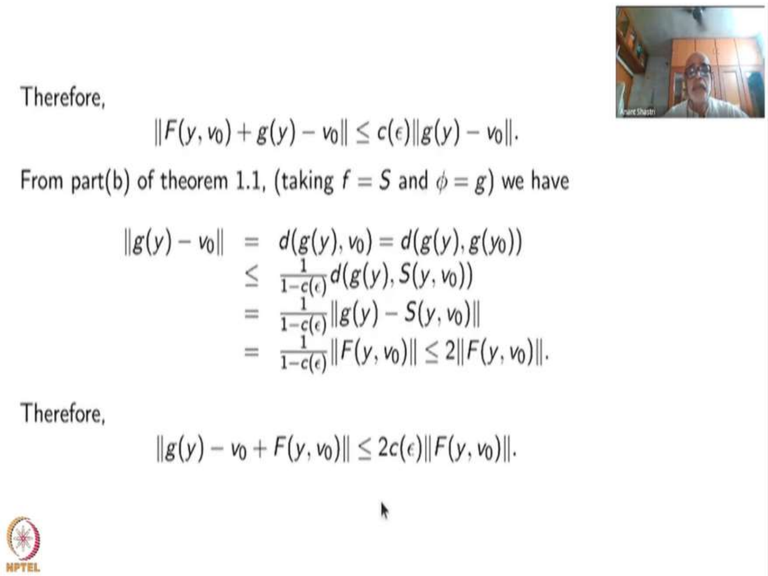

So, we now appeal to Part b of theorem 1.1 namely the continuity part. Taking f equal to S and ϕ equal to g, we get norm of $g(y) - v_0$ is equal to the distance between $g(y)$ and v_0. What is v_0? It is $g(y_0)$. Therefore this distance is less than or equal to 1 divided by $1 - \epsilon$ times distance between $g(y)$ and $S_y(v_0)$. This distance you can express as the norm of the difference. Therefore it is equal to norm of $F(y, v_0)$ and hence the LHS is less than or equal to 2-norm of $F(y, v_0)$.

See I have used full statement of Part b of theorem 1.1, namely the inequality that we have established there. $\|g(y) - v_0\|$ can be expressed in terms of the metric which was the notation used in the theorem 1.1. So I go back and forth with the norm and the distance. That is all. Finally, since $c(\epsilon)$ is less than or equal to $1/2$, I can replace the factor 1 divided by $1 - c(\epsilon)$ with 2.

Therefore, going back now, we get norm of $g(y) - v_0 + F(y, v_0)$ is less than or equal to $2c(\epsilon)\|F(y, v_0)\|$. All that we want is some constant here. It depends upon c. It may be three times four times that does not matter.

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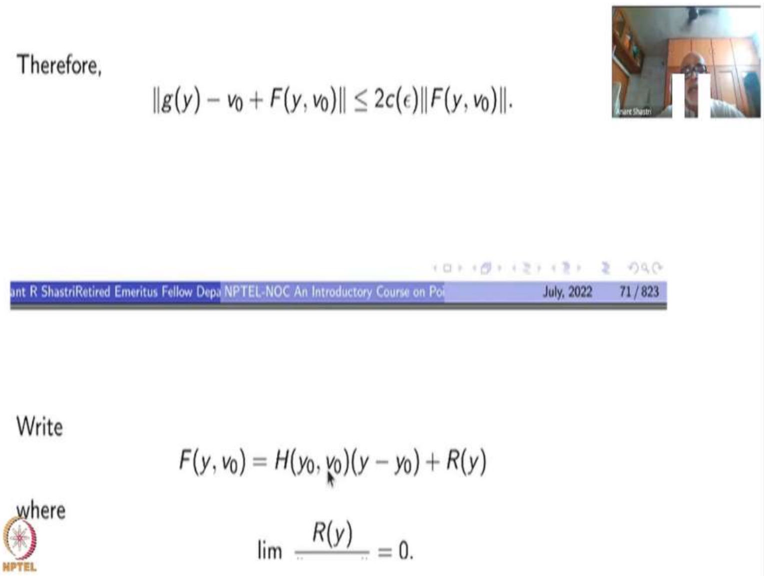

So, let us write $F(y, v_0) = H(y_0, v_0)(y - v_0)$ plus the remainder $R(y)$. Why we can write this way? Because this is H is the derivative of F with respect to the y-coordinate, v_0 is fixed here. $y - y_0$, I am taking because this is the derivative with respect to the y-coordinate here. So, $R(y)$ the remainder after the first term has a property that $R(y)$ divided by norm of $(y - y_0)$ tends to 0 as y tends to y_0. So, I am using the increment theorem here.

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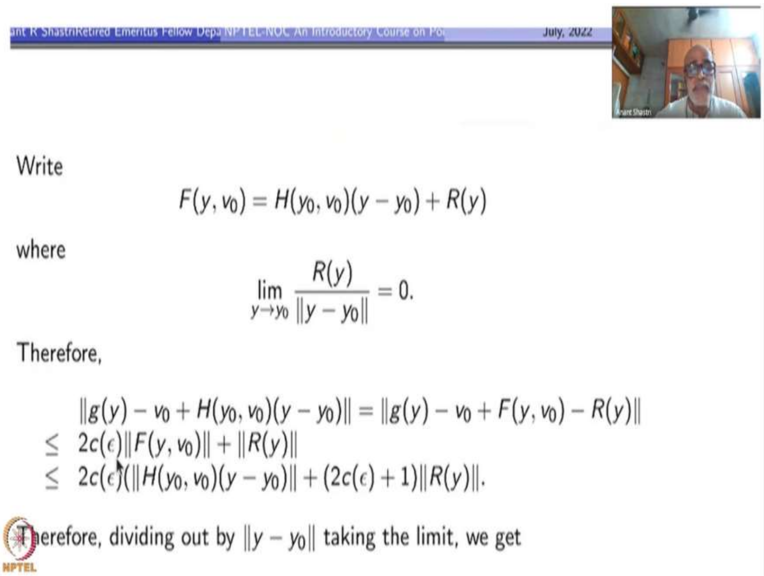

Therefore, if you use this inequality that we have established, what we get is the norm of $g(y) - v_0 + H(y_0, v_0)(y - y_0)$ is equal to norm of $g(y) - v_0 + F(y, v_0) - R(y)$ which is less than or equal to $2c(\epsilon)\|F(y, v_0) + \|R(y)\|$.

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$$\lim_{y \to y_0} \frac{\|g(y) - v_0 + H(y_0, v_0)(y - y_0)\|}{\|y - y_0\|} \leq 2c(\epsilon)\|H(y_0, v_0)\|.$$

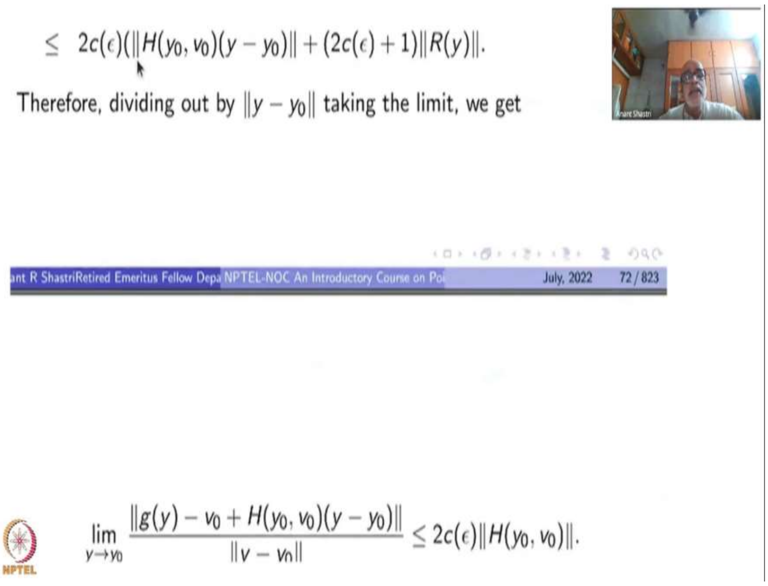
Since this holds for all $\epsilon > 0$, we conclude that g is differentiable at y_0 and $Dg(y_0) = -H(y_0, v_0)$.

$$\lim_{y \to y_0} \frac{}{\|y - y_0\|} = 0.$$

Therefore,

$$\begin{aligned} &\|g(y) - v_0 + H(y_0, v_0)(y - y_0)\| = \|g(y) - v_0 + F(y, v_0) - R(y)\| \\ \leq\ & 2c(\epsilon)\|F(y, v_0)\| + \|R(y)\| \\ \leq\ & 2c(\epsilon)(\|H(y_0, v_0)(y - y_0)\| + (2c(\epsilon) + 1)\|R(y)\|. \end{aligned}$$

Therefore, dividing out by $\|y - y_0\|$ taking the limit, we get



You divide out by $\|y - y_0\|$ and take this limit as y tends to y_0, that term will be less than or equal to $2c(\epsilon)\|H(y_0, v_0)\|$. Since this is true for all $\epsilon > 0$, it follows that this limit is zero. That proves that g is differentiable at y_0 with its derivative equal to $-H(y_0, v_0)$.

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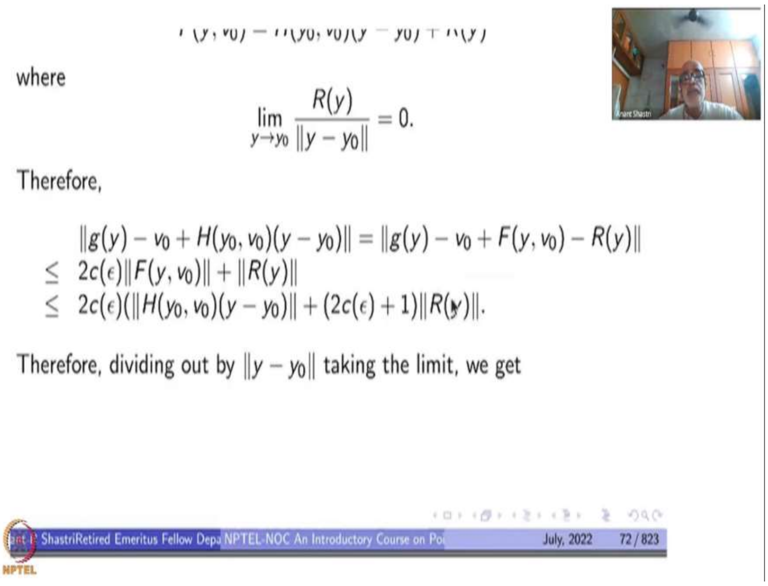

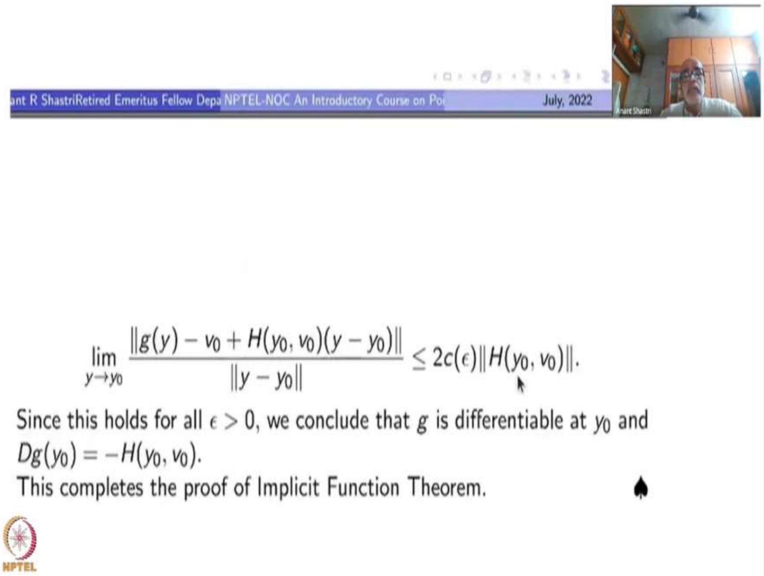

So, this completes the proof of Part b and thereby compute the proof of the implicit function theorem. I recall that in the original statement there was a T^{-1} here, but now, in the in the modify statement we have made T to be identity map that is why the T does not appear here. So, that is a proof of implicit function theorem.

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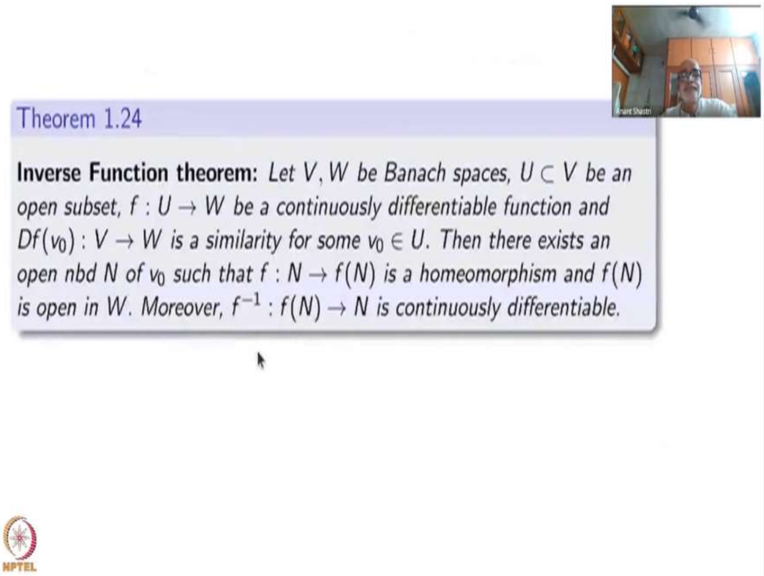

Now, let us go to inverse function theorem. That is one-step ahead but this is the crux of the business. This is the main thing that we want to prove finally.

Let V and W be Banach spaces, U is an open subset of the first Banach space V, f is a function from this open set U into W. The condition on F is that it is continuously differentiable function and its derivative at a particular point v_0 in U is a similarity. The conclusion is that there exist a neighborhood N of v_0 such that f from N to $f(N)$, its image is a homeomorphism onto an open set $f(N)$ in W. Moreover f^{-1} from $f(N)$ to N also continuously differentiable.

So, starting with just a continuously differentiable function which is such that at one point the derivative is invertible, we conclude that, in a small neighborhood the function itself is a homeomorphism actually a diffeomorphism, because inverse is also continuously differentiable. Moreover, the image is also open. Both N and $f(N)$ are open. N is open in V and $f(N)$ is open in W. The hypothesis that $D(f(v_0))$ is a similarity automatically implies that V and W are similar spaces. So, how does one prove this entire statement?

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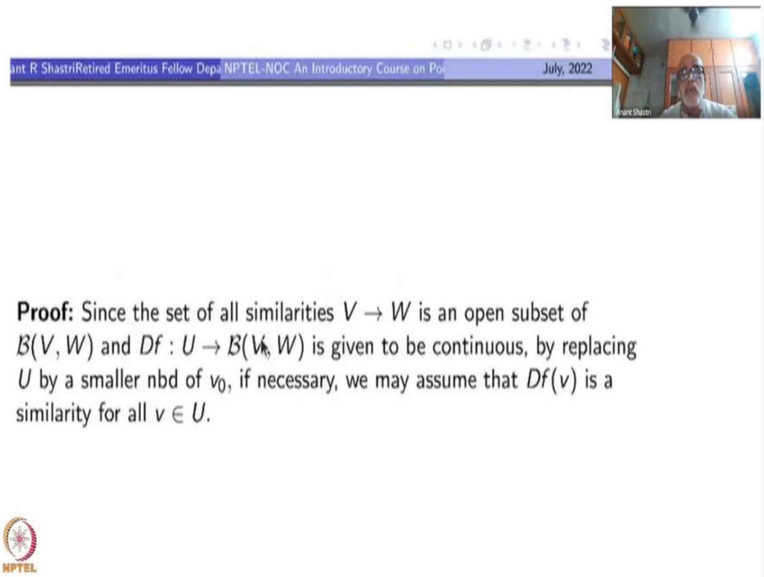

Recall the set of all similarities from one Banach space to another Banach space is an open subset of the continuous linear maps from V to W all continuous maps. This is what we have seen. The function $D(f)$ from U to $\mathcal{B}(V, W)$ is given to be continuous. Therefore, by replacing U by a smaller neighborhood of v_0, if necessary, we can assume that $D(f)(v)$ is a similarity for all $v \in U$. All that I did was to appeal to the fact that $D(f)$, is continuous and $D(f)(v_0)$ is in the open set of all similarities.

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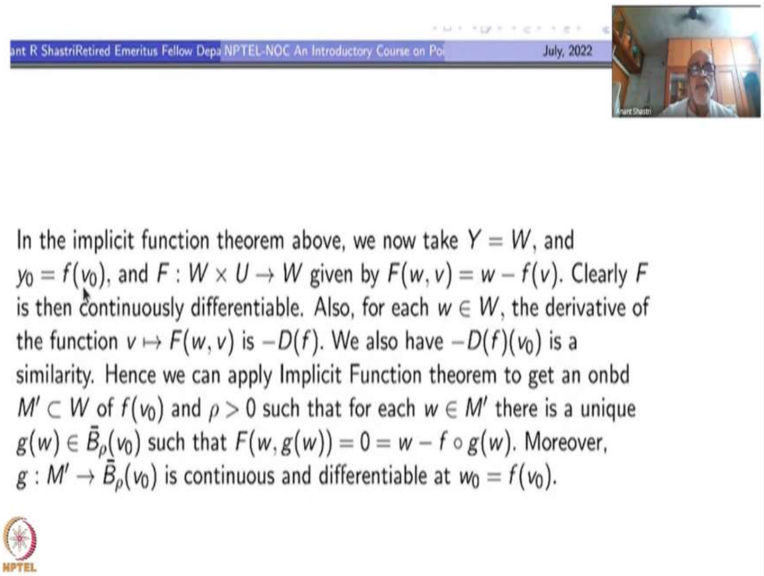

Next step: In the implicit function theorem above we take $Y = W$. So, we are in the part (b) already. Remember part (b) of implicit function theorem wanted that Y to be a Banach space. So, we are inside a very special case, viz, $Y = W$ itself. Take $y_0 = f(v_0)$ and F from $W \times U$ to W to be $F(w, v) = w - f(v)$. In fact, I am supposed to take some neighbourhood of y_0 but for that we are taking the whole of $Y = W$. And F is a very simple function. Clearly, F is continuously differentiable as a function of v that is what we wanted first of all.

In fact, this is continuously differentiable even in terms of w also. So, all the hypothesis that we needed are satisfied. First of all $F(y_0, v_0) = 0$. For each $w \in W$, the derivative of the function, namely, v going to $F(w, v) = w - f(v)$ with respect to v is of $-D(f)$. And $-D(f)(v_0)$ is a similarity. So, all the hypothesis of implicit function theorem are satisfied.

The first part says there is a neighborhood M' of $y_0 = f(v_0)$ in W, and a ρ positive such that each $w \in M'$, there is a unique $g(w)$(note that since $Y = W$, I am writing w from Y as well) belonging to $\bar{B}_\rho$ such that $F(w, g(w)) = 0$. But what is F? $F(w, g(w)) = w - f(g(w))$. Therefore $w = f(g(w))$, also $w_0 = f(v_0)$ (by the uniqueness). Moreover, g itself is continuous and is differentiable at w_0.

So, what is the meaning of this? This just means that $f \circ g$ is identity on M'. So, that is the meaning of this one. Moreover, part (a), already tells you that this map g from M' to $\bar{B}_\rho$ is a unique one and part (b) says it is continuous on M' and differentiable at w_0.

So, this is all the implicit function theorem apply to this special case. So, are we through? Not yet. So, we have to see what is happening here.

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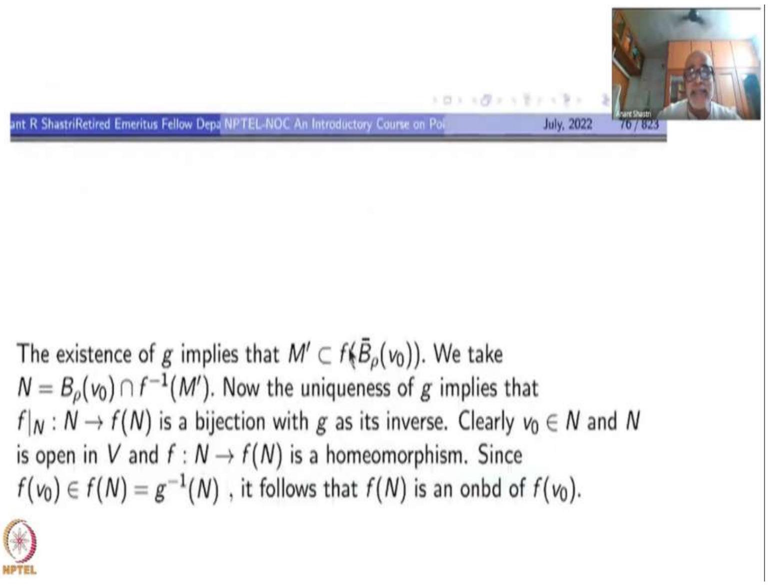

The existence of g implies this M' is inside the image of f, indeed inside f of the closed ball. See what it means to say that for $w \in M'$, we have point $g(w)$ belonging to the closed ball such that $w = f(g(w)$. So M' is contained in the closed ball. That is the meaning of the existence. Therefore, what we take is N to be this open ball intersection with $f^{-1}(M')$. M' is an open set already. So, $f^{-1}(M')$ is an open set, intersect it with the open ball that is an open subset of V. Clearly it is a neighbourhood of v_0 because v_0 is inside $f^{-1}(M')$, $f(v_0) = y_0$.

The uniqueness of g implies that this map f restricted to N from N to $f(N)$ is a bijection. Because $f \circ g$ is an identity, we told you that g is the left inverse of f or which is the same as saying that f is the right inverse of g. But now, g is unique. So, f must be injective also. So, f is a bijection with g as its inverse.

So, I have already told you that v_0 is inside N and N is open inside V. Also, f from N to $f(N)$ is a homeomorphism. But why $f(N)$ is open in W? Because $f(N) = g^{-1}(N)$ because $g \circ f$ is identity. So, $f(N)$ is also an open subset, it is a neighborhood of $f(v_0)$.

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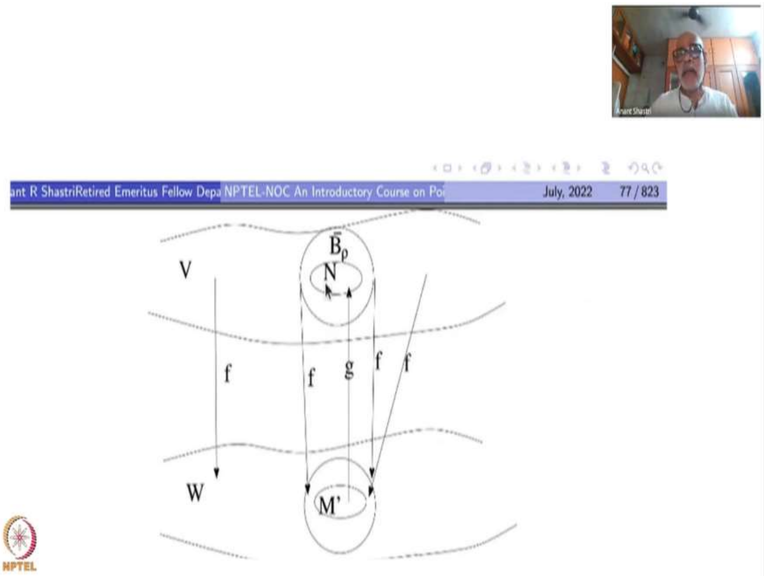

So, here is the picture I have drawn V to W this f could be a many-to-one function. It is not assumed to be one one map or anything like that. There is no need for such assumptions. So, what we started? We started with a neighborhood M' of v_0 and $\bar{B}_r ho$ here a neighbourhood of w_0.

For each point in M', there is a unique g inside this ball. What is gf of that is back to M' here is w. So, go by g and come back by f that is identity. This means that M' is covered by the image of f. This is some larger thing. If you take f of this, it could be larger. It covers M'.

But for points of M' inside this one, there is only one point which here coming to that, that is the uniqueness part of g. If there are other points here coming here, the uniqueness will fail, right? But some point here may come here, some point may here may come here I do not care. Inside this open ball, there is only one g.

So, therefore when you take this M' and inverse of that inside this one, there may be some other point, I am intersecting it with the open ball that is what I am calling N. On N to M', f is a bijection now and its inverse is g. And what we know is that g is differentiable at $f(v_0)$, here this is w_0. g is continuous and f is continuous, they are inverse of each other, so they are homeomorphisms. N is open. M' is open. But I am not taking the whole of M' here what I am taking is $f(N)$. $f(N)$ is open. $f(N)$ is also a neighbourhood of $f(v_0)$.

So, only thing that remains to see is: why this g is differentiable on the whole of $f(N)$. We know this only at one point. What is that point? To begin with y_0 was an arbitrary point of an open subset U of Y on which the derivative of f is invertible. This hypothesis is true for all points of N now. Remember that was the starting point of our choice of the neighborhood N here, by cutting down the neighborhood U itself such that $D(f)$ is invertible at all points of N. So, that hypothesis is there.

Therefore, for every point $v' \in N$, here I can apply the above conclusion and say that g is differentiable at $f(v')$, though the choices of the neighbourhoods for homemorphism etc may be different but do not matter. Therefore, g is differentiable at all the points of $f(N)$.

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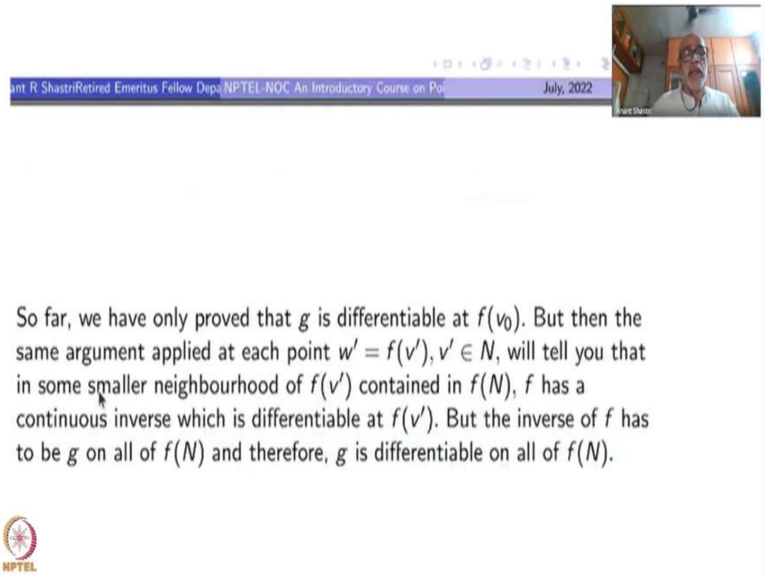

So, this is the last thing I repeat here. So, far we had only proved that g is differentiable at $f(v_0)$. But then the same argument applied at each point w' is $f(v')$ where v' range over N will tell you that in some smaller neighborhood all that is there in the background we can ignore them.

But $f(v')$ contained inside $f(N)$ everything. f is a continuous inverse which is differentiable at $f(v')$. f as a continuous inverse that. But continuous inverse itself is the same g now. There is no other because f is already one one map. But the inverse of f has to be g on all of $f(N)$. Therefore, g is differentiable at $f(N)$ on the whole of $f(N)$.

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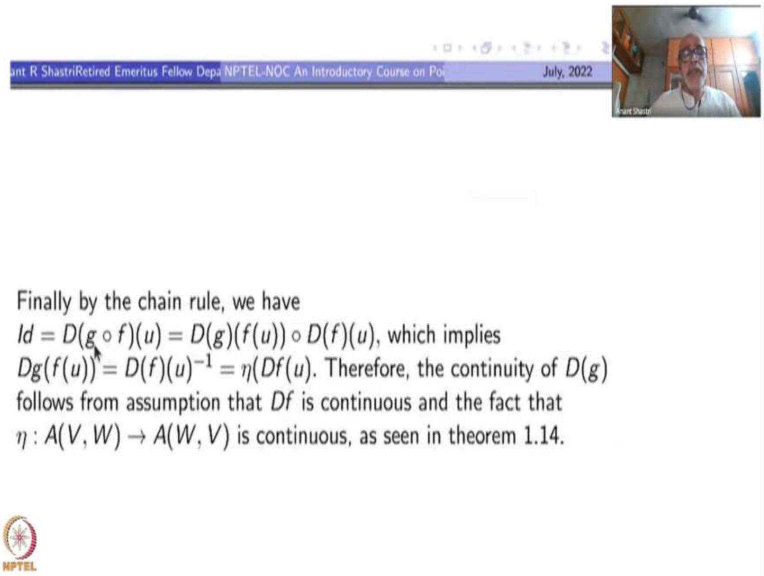

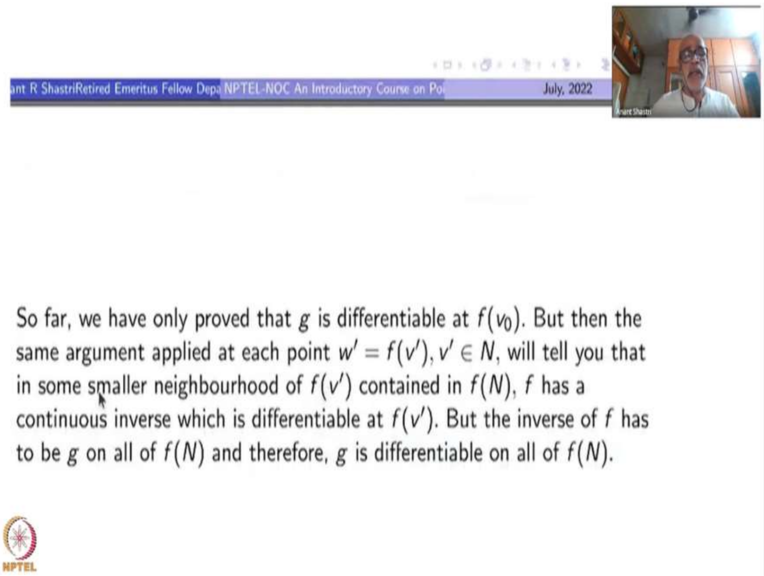

So, final thing is that since $g \circ f$ is identity, by the chain rule, $Id = D(Id)(u) = Dg(f(u)) \circ D(f)(u)$. Similarly for $f \circ g$. Therefore these tow derivatives are inverses of each other: $D(g)(f(u)) = (D(f)(u))^{-1} = \eta(D(f)(u))$.

Therefore, the continuity of this $D(g)$ follows from the assumption that $D(f)$ is continuous and the fact that η is continuous, as seen in theorem 1.14. So, that completes the argument, completes all the proofs of, all the assertions of the inverse function theorem. So, theorem 1.14 is proved. So, that is all today. So, let us stop here.