An Introduction to Point-Set-Topology
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Lecture 39
Separation of Sets- continued

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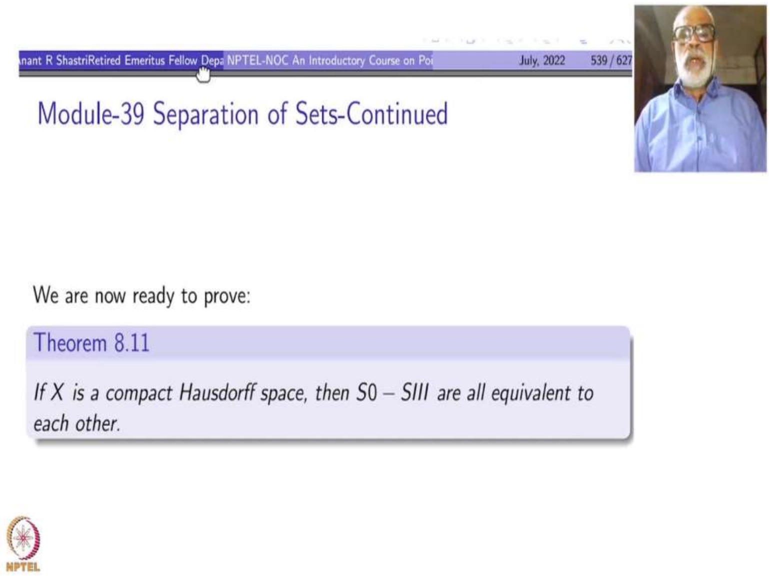

Hello, welcome to NPTEL NOC, an introductory course on Point-Set-Topology part II. Today, we shall continue with our study of the Separation of Sets module 39.

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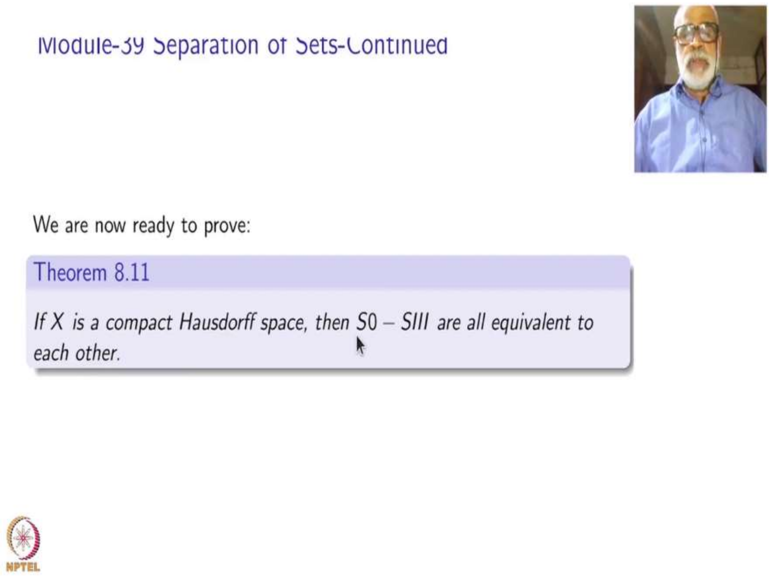

So, we have seen a number of implications and some non-implications also, but now for a compact Hausdroff space, you will see that all the four axioms are equivalent. So this is the statement. If X is a compact Hausdroff space, then S0, SI, SII, and SIII are all equivalent to each other.

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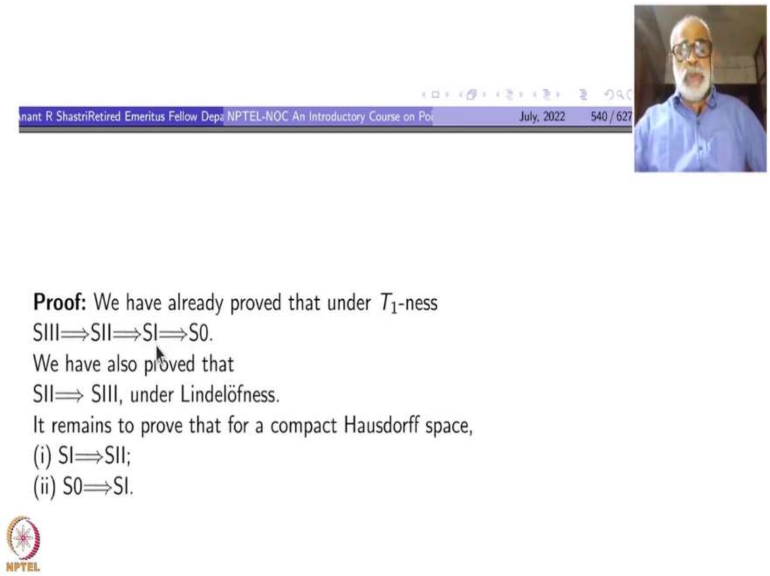

As I told you, we have already proved that under T_1-ness, SIII implies SII implies SI implies S0, just because every point is closed that is all we have to use. We have also proved that, under Lindelofness SII implies SIII, the reverse implication. So, it remains to prove that for a compact Hausdroff space, SI implies SII and S0 implies SI.

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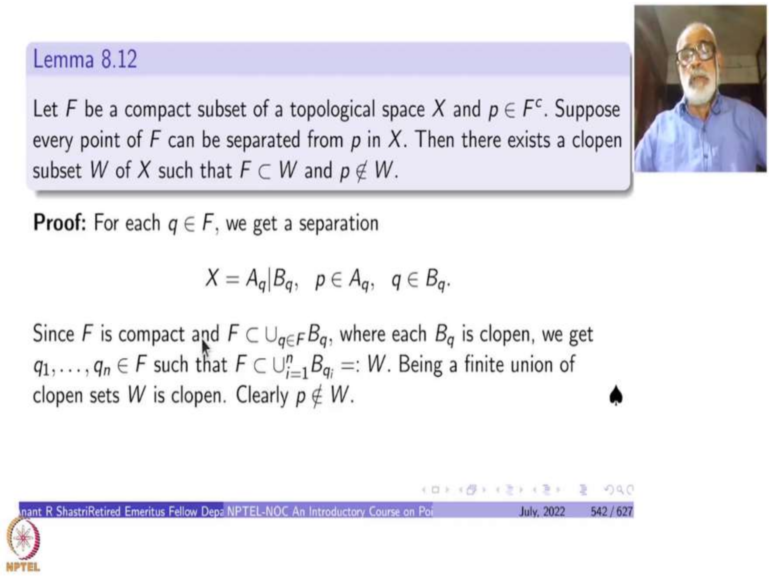

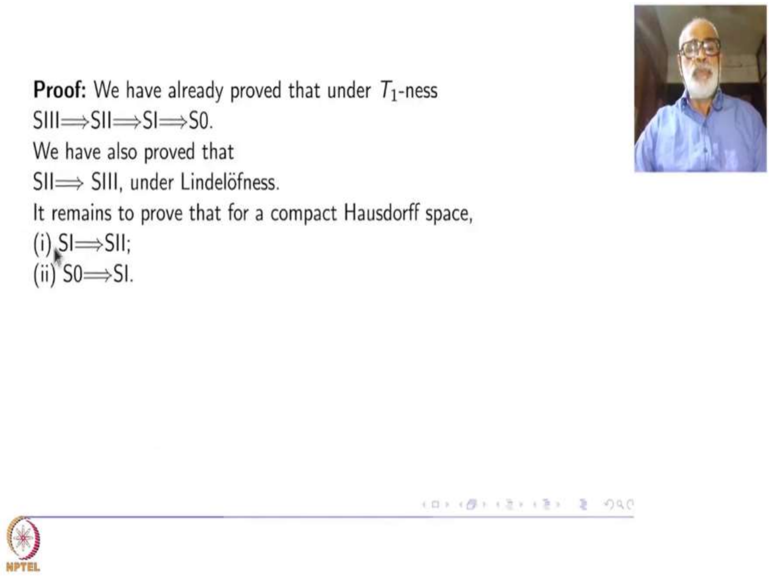

So, before proceeding further, I will state a lemma which can be used again and again and of course it will be used in proving these two statements. So, the lemma is:

Take a compact subset F of topological space X, take a point p in the F^c.

Suppose every point of F can be separated from $p \in X$. Then there exists a clopen subset W of X, such that it contains F and does not contain the point W.

So, from pointwise separation we have concluded global separation. Each point of F can be separated from p is the hypothesis, which is SII, but only for points of this compact subset F and the point p. So that a bit more general than assuming SII for all pairs of points, but this result will be useful for us.

So, what we do? To each point $q \in F$, we get a separation that is the hypothesis, X equal to $A_q|B_q$, with $p \in A_q$, and $q \in B_q$. Remember this just means that A_q and B_q are both open and closed and they are disjoint. Since F is compact and is contained in the union of B_q's, as q varies over F and since each B_q is open, so we get a finite cover, so I can write F subset of union of B_{q_i} for i ranging from 1 to n. Denote this union by W, which is obviously open as well as closed, being a finite union of clopen sets. Clearly, p is not a point of any of this B_{q_i}'s, so p is not inside W. Over.

So, compactness has helped us here just like in the case of Lindelofness, we have got SII implies SIII.

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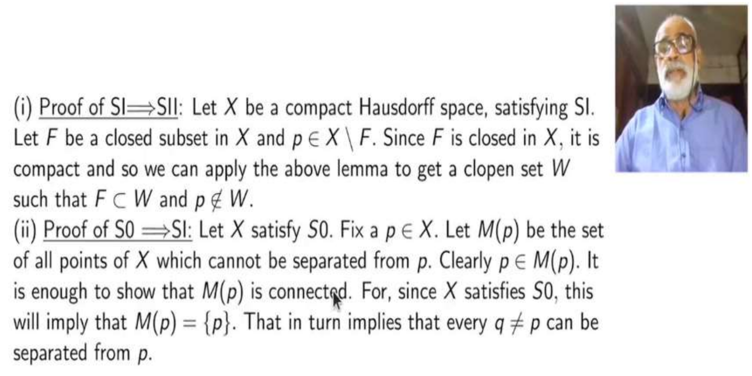

Now, we will use this lemma to prove both these implications SI implies SII as well as S0 implies SI. The first one comes very easily now.

Let X be a compact Hausdroff space satisfying SI. Let F be a closed subset of X and p belonging to $X \setminus F$. Since F is a closed in X it is compact.

So, we can apply that previous lemma to this F. So, what we get? We get a clopen set W such that F is contained inside W and p is not in W. So, this is almost a restatement of the lemma. Instead of assuming F is compact, all I have here is that F is a closed subset of a compact space X. So, I am getting that hypothesis.

Now, let us try to prove S0 implies SI. This will take a little more time.

So S0 implies SI means what? S0 means what? singletons are all components. From that I have to prove that distinct points can be separated.

So, let X satisfies S0. Fix a point p inside X and look at the set $M(p)$ of all the points q of X which cannot be separated from p in X. What we want to prove? We want to prove that every point other than p can be separated. Therefore, finally we have to prove that this $M(p) = \{p\}$

So, there are steps to prove that. First of all p itself is in $M(p)$, so it is enough to prove that $M(p)$ is connected, because the only connected subsets of X are singletons. That is what S0 means. So, what I will prove $M(p)$ is connected, then the proof is over.

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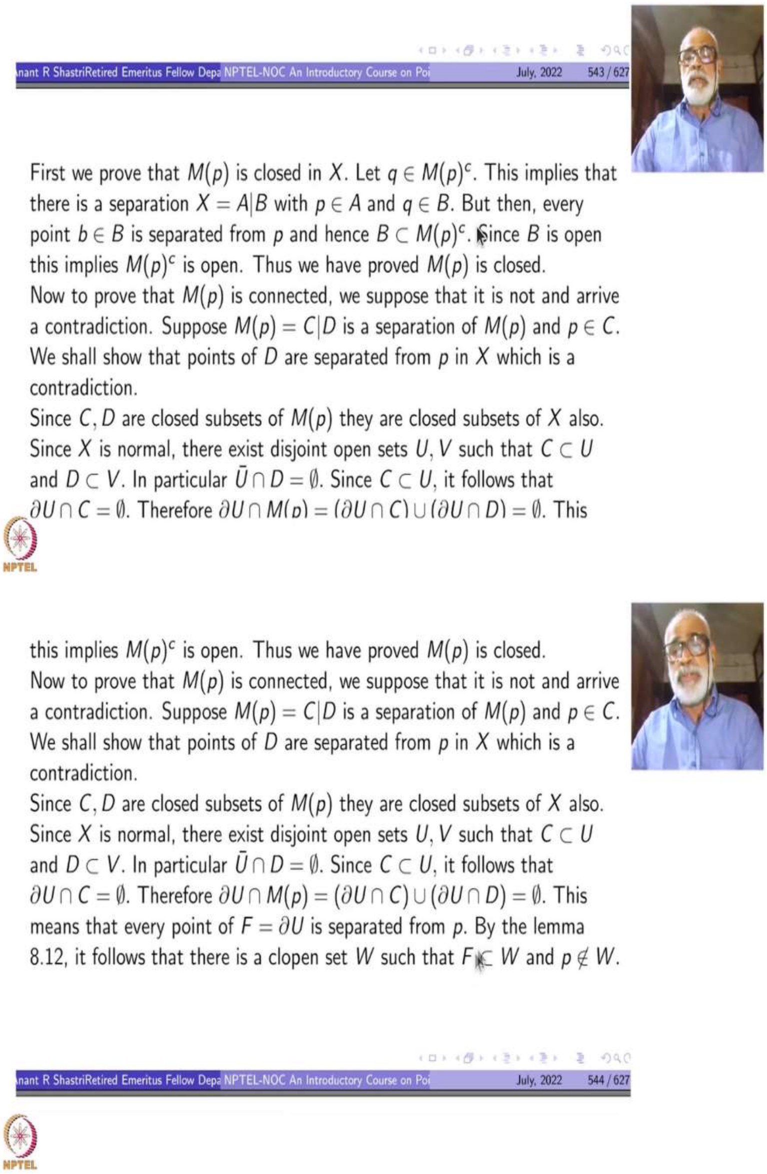

So, let us try to prove that $M(p)$ is connected,. The first step is to prove $M(p)$ is closed. (This is a strange thing, why do we need to prove that something is closed if we want to prove that is it is connected set? Wait.) So, let q be a point in the complement of $M(p)$. By the very definition, $M(p)$ is the set of all the points points which cannot be separated from p. This implies that there is a separation: $X = A|B$, p inside A and q inside B.

But then every point inside B is also separated from p. By the very definition, this just means that B is contained in $M(p)^c$. So, for each point inside $M(p)^c$, we have got an open subset B of X containing that point and contaiend in $M(p)^c$. That means this is an open subset by itself which just means $M(p)$ is a closed subset of X.

Now, we can complete the proof that $M(p)$ is connected. Suppose this is not connected. Then we will arrive at a contradiction. Not being connected means that there is a separation of $M(p)$ itself. It is not a separation of the whole space; it is separation of $M(p)$ because this not connected means there is there are two points which can be separated. So, there is a non trivial separation: $M(p) = C|D$.

So, I can assume p is either inside C or inside D. So let us assume p is inside C, by interchanging C and D if necessary. We shall show that points of D are separated from $p \in X$ itself. See this is separation of the subspace $M(p)$. In general, it does not imply that these two points can be separated in X itself. So that is what we want to prove that and that is the hardest part here.

So, once you prove that it is a contradiction, because these points of $M(p)$, they are points which cannot be separated from p.

So, how do you do that? C and D are close subsets of $M(p)$, so they are closed in X also. This is where we have used that $M(p)$ is closed. The passage from the subset to the whole. So, you have got disjoint closed subsets inside X.

Now, you use the hypothesis that X is compact Hausdroff. Therefore it is normal. Therefore there exist open subsets U and V such that C is contained in U, D contained in V, and $U \cap V$ is empty. In particular, it implies that U being open $U \cap D$ is empty so $\bar{U} \cap D$ will be empty.

Since C is inside U which is open, it follows that boundary of $U \cap C$ is empty, because for any open subset U, the boundary is $\bar{U} \setminus U$. It is always $\bar{U} \setminus U^0$ but U is open already and hence $int(U)$ is U. So boundary of $U \cap C$ is empty, because C is inside U. Therefore, boundary of $U \cap M(p) \ = \ C|D$ is the union boundary of $U \cap C$ and boundary of $U \cap D$. Both of them are empty the union is empty.

So, this means that for every point of boundary of U, let me denote it by F, is separated from p. Everything in $M(p)$ cannot be separated from p, so these points are separated from p. So, $F \cap M(p)$ is empty. F is a compact subset because it is a closed subset of a compact space.

So, now again I am using this lemma with which we started. It follows that there is a clopen subset W such that this F is contained inside W and p is not in W, F is boundary of U.

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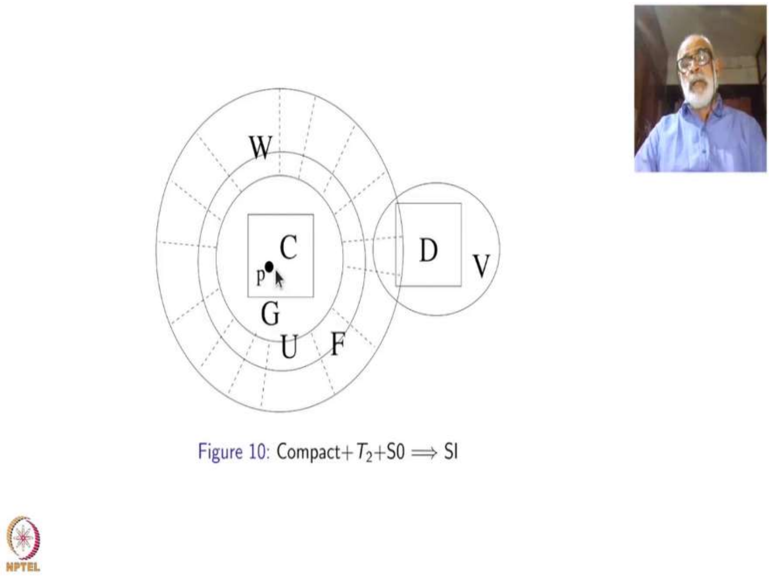

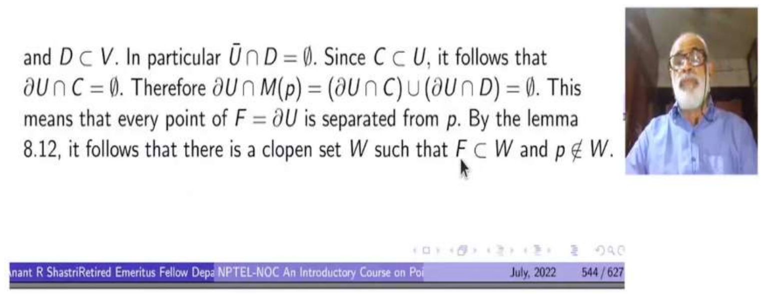

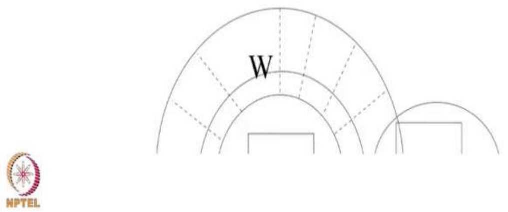

So, let me repeat what the kind of things we have done here. So, here is a picture. We started with a separation $C|D$, C here shown by a square, another square here D, so this is a separation of $M(p)$. After that using the normality, we found U and V disjoint open subsets containing C and D respectively. Then we look at the boundary of U i.e., F, F that does not intersect C nor D, that is what we proved.

Now, using the lemma, I can fatten this F to an clopen subset W, shown by this shaded part that is W, such that W is a neighbourhood of F and it does not contain this p, so this much we have done. So, how does this help? Now, we can complete the proof of (ii) as follows.

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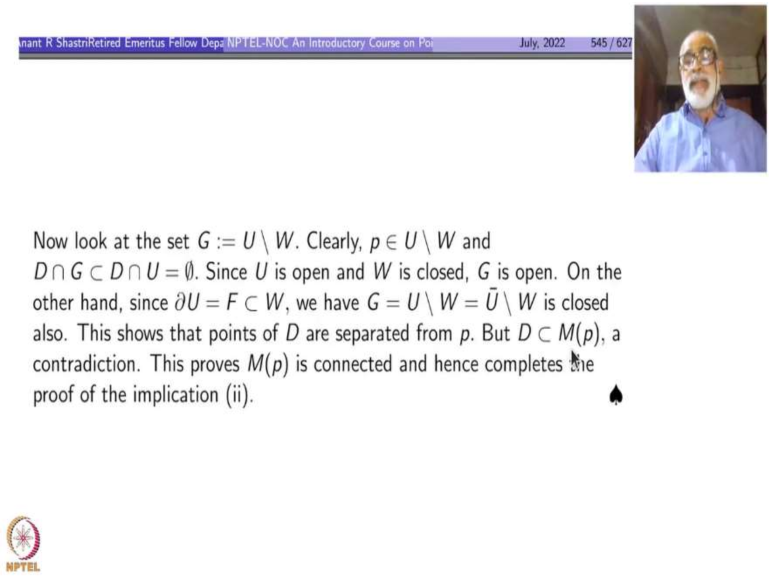

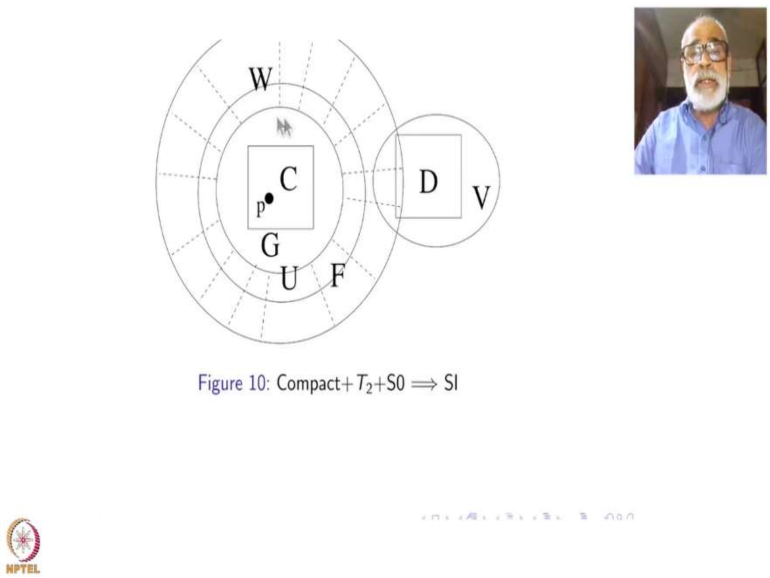

Look at G which is $U \setminus W$. So, this is U here and you have to throw away all the shaded part what you get is this G. unshaded part. Clearly that contains this entire of C. So, this G is $U \setminus W$ and p inside $U \setminus W$ which is G and this $D \cap G$ which is contained $D \cap U$ that is empty, see G is a subset of this U and $U \cap D$ is empty by choice of U and V, so $G \cap D$ is also empty.

Since U is open and W is clopen subset (remember clopen subset is what?) G is also open. On the other hand, boundary of U which is F is contained inside W, so we have G equal to $U \setminus W$ (this is by definition) is same as $\bar{U} \setminus W$, because the boundary is taken care here, boundary points are all contained inside W and I am subtracting that part.

So, G is closed also. So, we have found a clopen subset G containing the point p and not intersecting D. So points of are separated from p. But D is inside $M(p)$, we started with a non-trivial separation and that is a contradiction.

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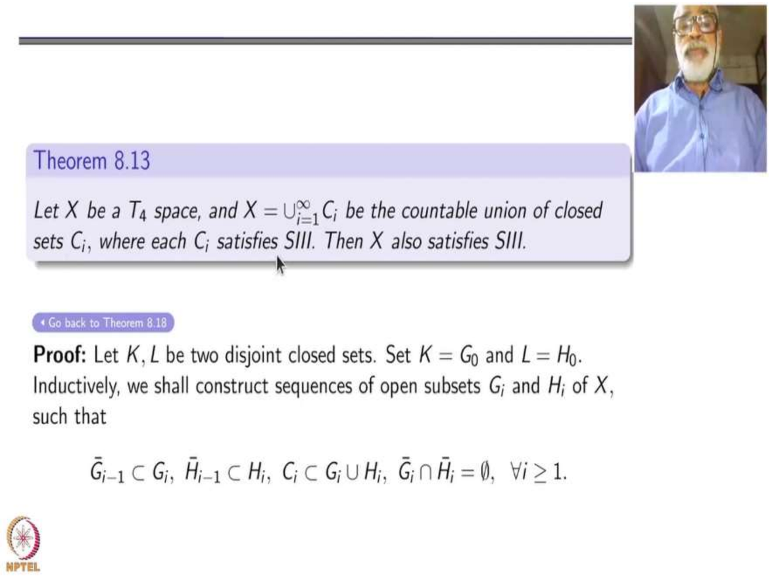

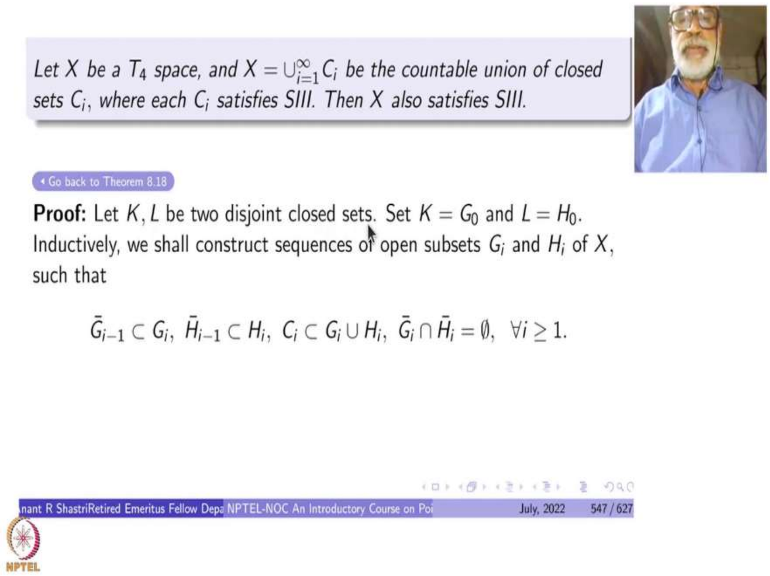

So, what we have in a compact plus Hausdroff space. all the 4 axioms are equivalent, so that is one of the reasons why we studied them first of all. However, life is not that easy, when we want to study them over a larger family of spaces such as non compact spaces, which are very much needed. That is why all these differences have to come there. Most probably we can assume Hausdroffness. Actually, finally in the dimension theory that we are going to develop, we are only taking metric spaces so, no problem. So, but we do not want to put compactness, so that is why all this.

So, we will continue to study of these things a little bit more now, which will be useful later on, in dimension theory. Take a T_4 space X which is written as a countable union of closed sets, each of them satisfying SIII. Then X also satisfies SIII.

So, let us prove this one. Let K and L be two disjoint closed sets inside X. Put K equal to G_0 and L equal to H_0. I am starting an inductive process here, construct sequences of open sets $\{G_i\}$ and $\{H_i\}$ of X such that $\overline{G_{i-1}}$ is inside G_i and $\overline{H_{i-1}}$ is inside H_i, each C_i is contained in the union of G_i and H_i, $\overline{G_i} \cap \overline{H_i}$ is empty, for each $i \geq 1$. Starting with K equal to G_0 and L equal to H_0. What are K and L? They have been given to be disjoint close sets. So, there you do not have to verify much, you have to verify only the last thing here.

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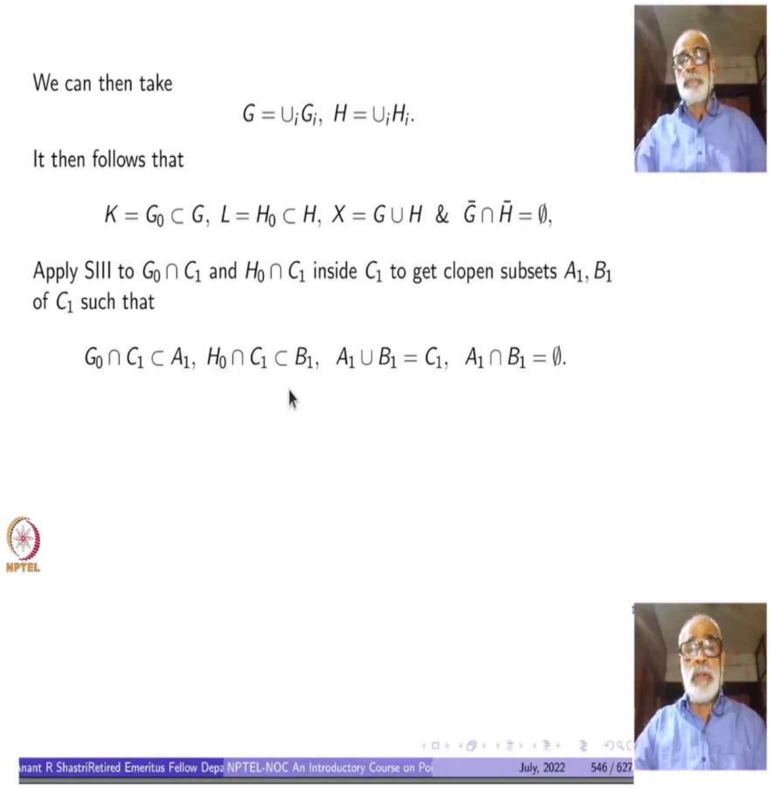

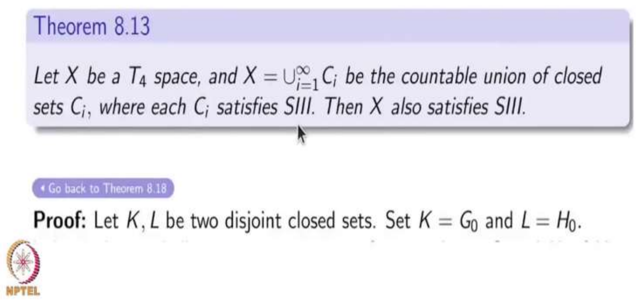

Once we have such a sequence, we can then take G equal to union of G_i's and H equal to union of H_i's. Then $G \cup H$ will contain the whole of X and hence is equal to X. So, starting with K which is G_0, that is also contained inside G, because G is union of all the G_i's. Similarly, L is contained inside H. Finally I want also to show that $\overline{G} \cap \overline{H}$ is empty. But that is easy because G is H^c and H is G^c and hence both are closed.

Once we have this sequence we will have proved something.

So, apply SIII to $G_0 \cap C_1$ and $H_0 \cap C_1$ inside C_1. C_1 satisfies SIII. So, I get a separation $C_1 = A_1 | B_1$, A_1 will contain the first set $G_0 \cap C_1$ and B_1 will contain the second set $H_0 \cap C_1$.

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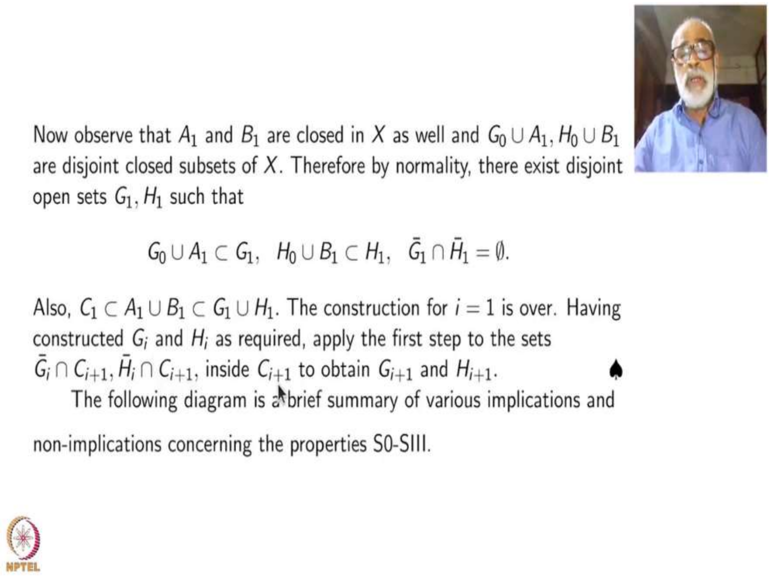

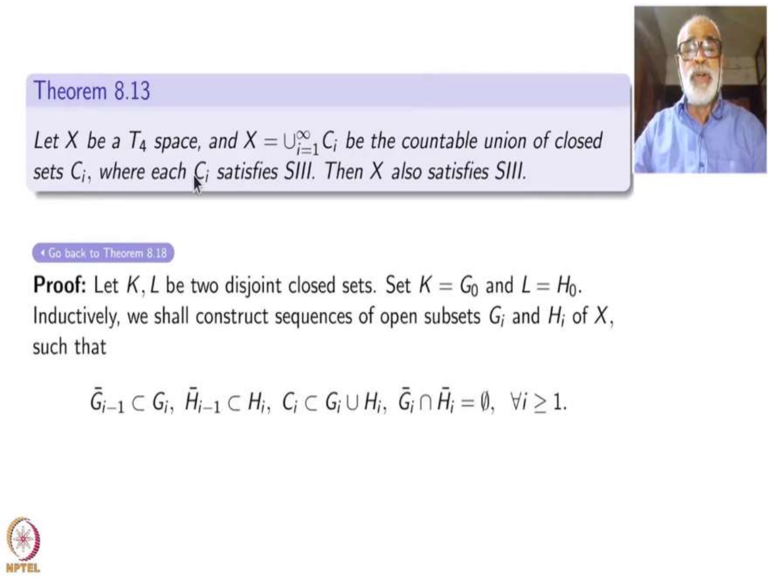

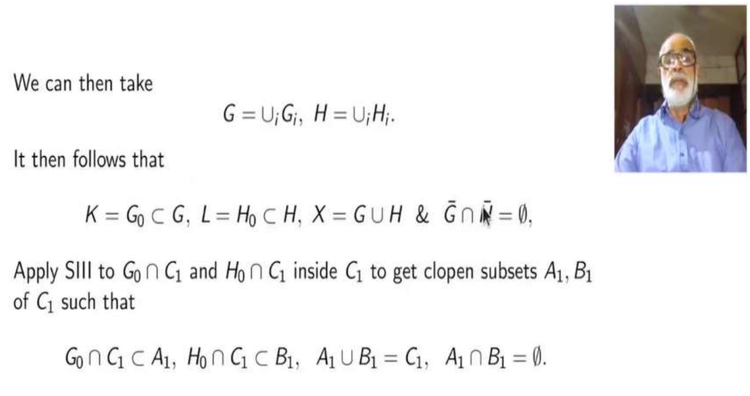

So, now observe that A_1 and B_1 are closed inside X also because C_1 is closed in X. So that is also hypothesis here each C_i is closed in X, union of countably many closed subsets. Therefore, $G_0 \cup A_1$ and $H_0 \cup B_1$ are disjoint close subsets G_0, and H_0 are closed anyway.

Therefore, I can apply T_4-ness of X now, I can fatten these things, there exist disjoint open subsets G_1 and H_1 such that this $G_0 \cup A_1$ is contained inside G_1 and $H_0 \cup B_1$ is contained inside H_1, $\overline{G_1} \cap \overline{H_1}$ is empty. Obviously, C_1 which is contained inside $A_1 \cup B_1$ is contained inside $G_1 \cup H_1$, so the construction of this sequence for $i = 1$ is over. From $i = 0$ to $i = 1$ whatever we have done, you repeat this step now by replacing 0 by 1, then replace 1 by 2 and so on. You will get the required sequence.

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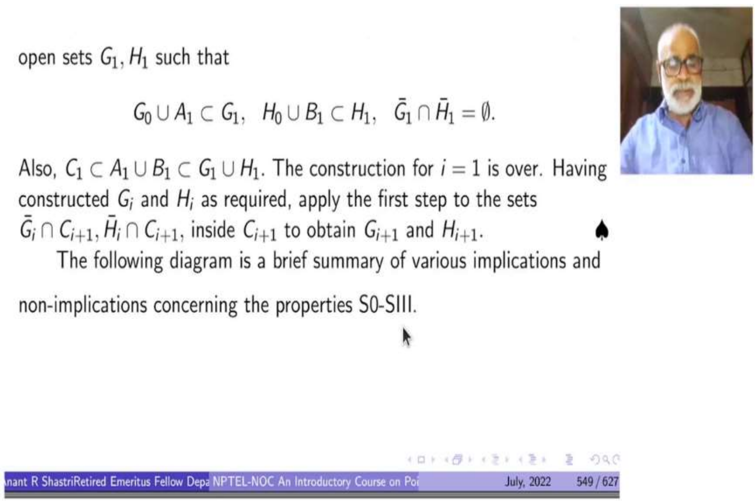

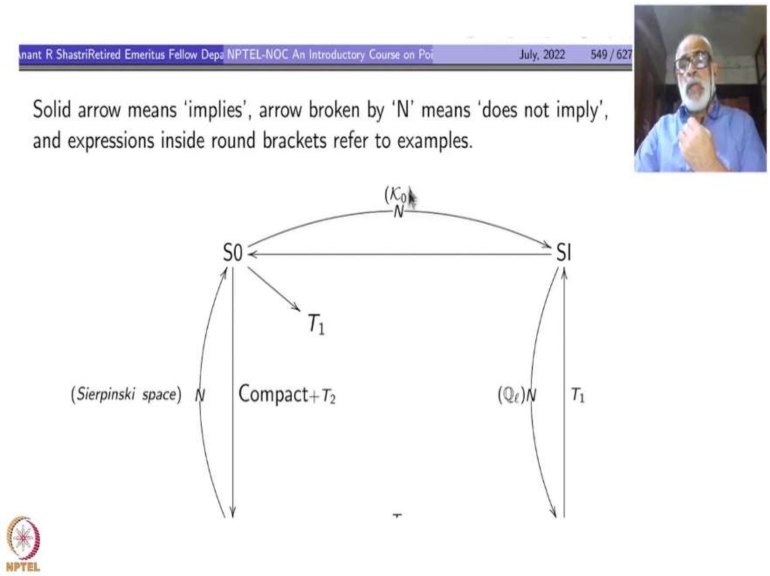

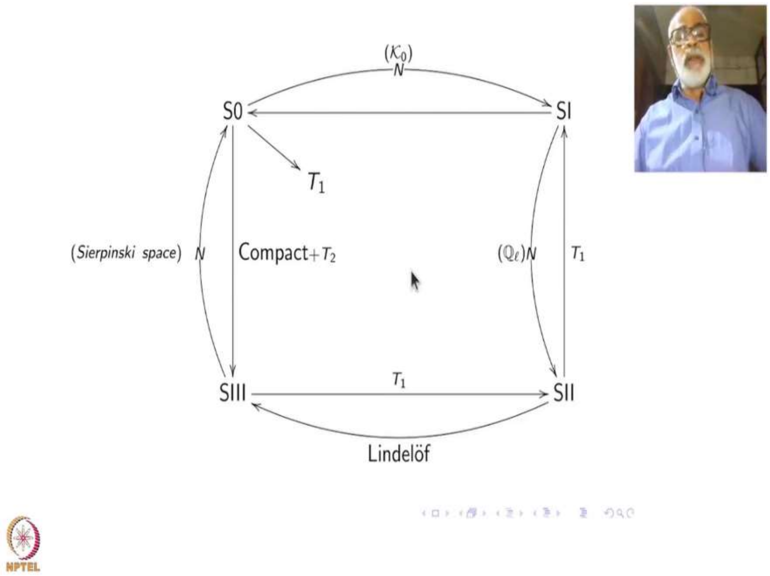

I would like to sum up a number of implications and non-implications, which we have proved, just that this could be like a ready-made reference, ready reckoner for you. So, this is the picture, in this picture the solid arrows like this, like this, they indicate implications. SI implies S0 like that which is always true. So I have put a broken arrow and labelled it by N to indicate non implications. S0 does not imply SI and what is the example? $\mathcal{K}_0$, which is our Kuratowski Knaster-Kuratowski example $\mathcal{K}$ setminus the apex point.

So, I have put it in bracket to remind you what gives you this example. So, like this, we have under T_1, SI implies S0. This is indicated by the solid arrow marked with T_1. Similarly, SIII implies SII under T_1. Under compact and Hausdroff, we proved S0 implies SIII. In fact all of them are equivalent anyway, once you have got here you can come back like this, so all of them are equivalent.

So, I have shown one arrow that is enough, then you can keep going, but we have also an elementary example to illustrate SIII may not imply S0, in general, the example is space the Sierpinski space. Remember Sierpinski space consisting of just two points one point is closed, other point is not a close set, one point is open other point is not an open set. There are no disjoint closed sets, no disjoint non-empty closed sets to be precise, therefore SIII is automatically satisfied. But you cannot separate the two distinct points. Infact, this space is a connected space. So this implication is not true.

Also, you have proved that under Lindelofness, SII implies SIII. If you have more things you can accommodate them in this diagram. You are welcome but I think this is enough. So by the way yeah this set $\mathbb{Q}_\ell$ of all points with all coordinates rational inside the ℓ_2-space gives an example of SI which is not SII, so this also you have proved. You can go and see where they are.

So, this is roughly the summary. There may be many other questions and example. For instance, I do not know, in general whether SIII (without T_1-ness) implies SII or not, whether SII without Lindelofness implies SIII etc. That may be due to lack of time also, lack of interest also. That does not mean that we have completed the whole thing, so that is not the whole idea, this is just an introductory course, not meant to be comprehensive. So, next time we will start the topological dimension theory, in genuine. So all these were more or less background preparations. Thank you.