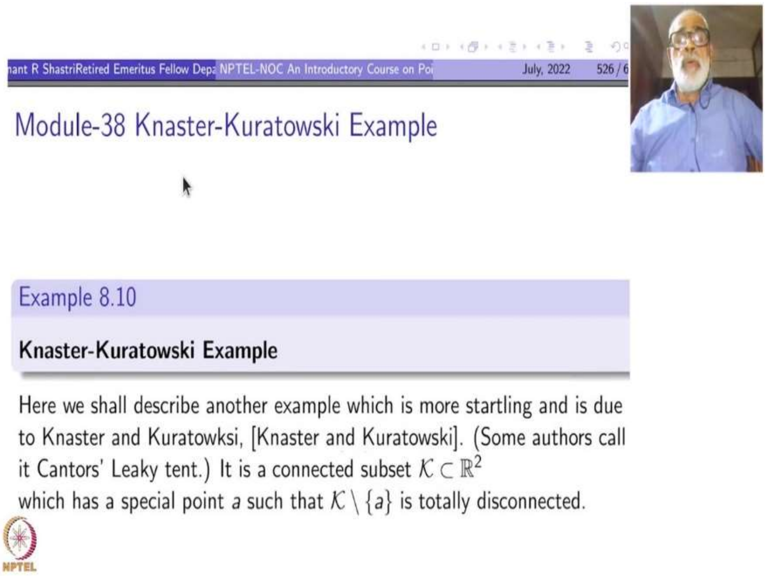
An introduction to Point-Set-Topology Part-II

Professor Anant R Shastri
Department of mathematics
Indian Institute of Technology, Bombay

Lecture 38

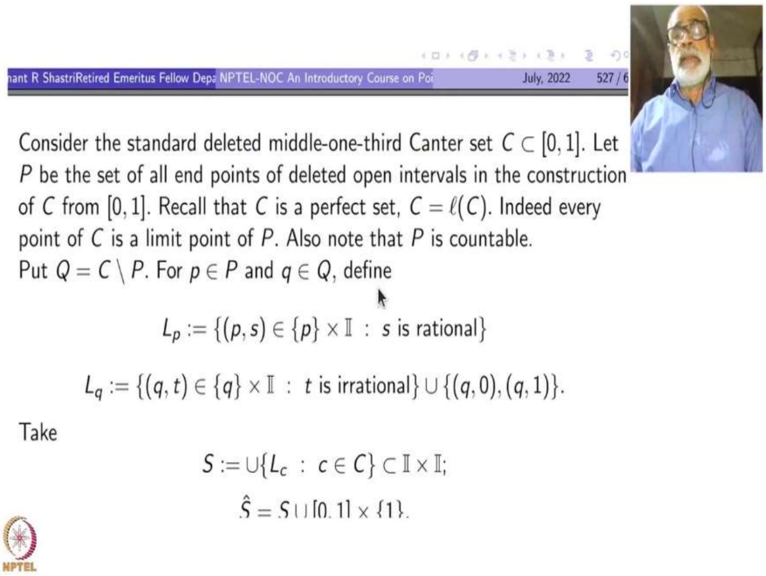
Knaster-Kuratowski Example

(Refer Slide Time: 00:17)



Welcome to NPTEL NOC an introductory course on Point Set Topology Part 2. We continue our study of separation properties S0, SI, SII, SIII and so on. So, today we will just study one example known after Knaster-Kuratowski, Kuratowski you must be familiar with, has done a lot of mathematics. You might have learned some of them. I am not very sure whether you know anything about Knaster however. The example that we are going to give is named after Canter also, it is called Cantor's leaky tent by some authors. This name is quite descriptive. It tells you how it looks like. Later on I will give you a picture of it. So, the point of this example is that it is a compact connected subset of $\mathbb{R}^2$. So, it is everything that you want to have, yet what is happening is that if you delete just one point from it, it is totally disconnected. So, that is why it is very startling when it was produced, it was a sensation.

(Refer Slide Time: 02:06)



So, let us start the construction, but it will take some time. Right now, we are going to do something in between. We start with the Canter, Canter's name is there any way. We start with the deleted middle one third Canter set C contained inside the closed interval $[0,1]$. The only property of this Canter set that we are going to use is that it is totally disconnected and it has a countable subset which is dense in it. So if you do not remember anything else you should just remember this much; it is uncountable, it is a complete metric space, because it is a closed subspace of $[0,1]$, it is totally disconnected and there is a countable subset which is dense in it. So, we are going to use only this much actually.

So, now let P be the set of all endpoints of deleted open intervals in the construction of the Canter set C from $[0,1]$. That P is a countable set and it is dense in C. Also C is a perfect set that means every point of C is a limit point of C. Indeed every point of C is a limit point of P itself, that is P is dense.

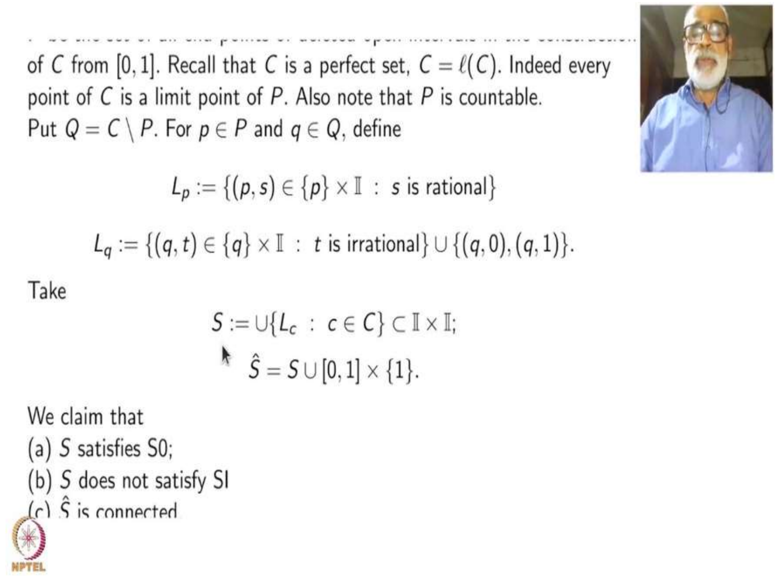
Also note that P is countable. So, you can forget about what P is. I have produced such a P namely, the endpoints of the deleted middle one thirds. Remember starting with $[0,1]$ you are deducting the open interval one third to two third that is the first step, the endpoints are one third and two third. They will be there of course in C. To begin with, it had only the end point so $[0,1]$, then one third two third will be there. Then $1/9, 2/9$, etc. all those things will make up the set P here.

Now take Q to be the complement of P in C. For each $p \in P$, and $q \in Q$, I am going to define certain subsets of $\mathbb{I} \times \mathbb{I}$. Closed interval $[0,1] \times [0,1]$. So what are they? The space L_p consists of points whose first coordinate is p, and the second coordinate s lies between 0 and 1

of course, but s must be rational. So, you can easily remember this countable set. Next L_q is the set (q, t) of all points whose first coordinate is q and the second coordinate t irrational and lies between 0 and 1. Except that we include the two end points viz, $(q, 0)$ and $(q, 1)$.

So, for each point c in the Canter set C, I have defined subsets of the vertical line segments subsets L_c of $\{c\} \times \mathbb{I}$ in two different rules, when p is inside P or inside Q.

(Refer Slide Time: 06:20)



Now, take S to be the union of all these L_c as c ranges over C. So, S is actually a subset of $C \times \mathbb{I}$. Now, I make a small modification in S, namely $\hat{S}$ is the union of S along with the line segment parallel to the x-axis, viz., $[0, 1] \times \{1\}$, the top face of $\mathbb{I} \times \mathbb{I}$. So, the construction of S and $\hat{S}$ is over. The topologies are subspace topologies coming from $\mathbb{I} \times \mathbb{I}$. We are not going to change the topology.

So, we claim the following three things.

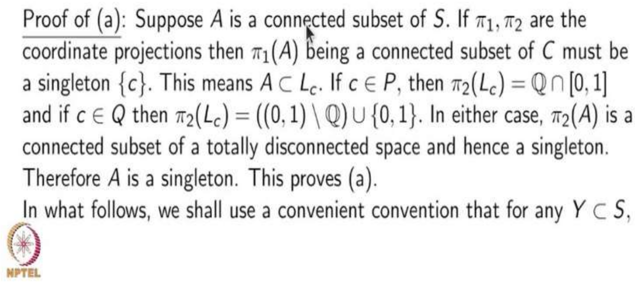
(a) S satisfies S0, i.e., totally disconnected; that is the meaning of this one.

(b) S does not satisfy SI, though this is a Hausdroff space.

(c) $\hat{S}$ is connected. Why this space in connected? It should be just because we have put this entire interval at the top. That will take care of connectivity. However, we have to prove it. It is not that easy.

So, these three things we are going to prove now. Hope the definitions of S and L_c are clear. For p inside this countable P, the points of L_p have the second coordinates rational. In the complement Q, the second coordinate is irrational, except the endpoints $(q, 0)$ and $(q, 1)$, they are allowed that is all. And $\hat{S}$ has more points than S viz., namely the entire line segment $[0,1] \times 1$. So, let us proceed.

(Refer Slide Time: 08:46)



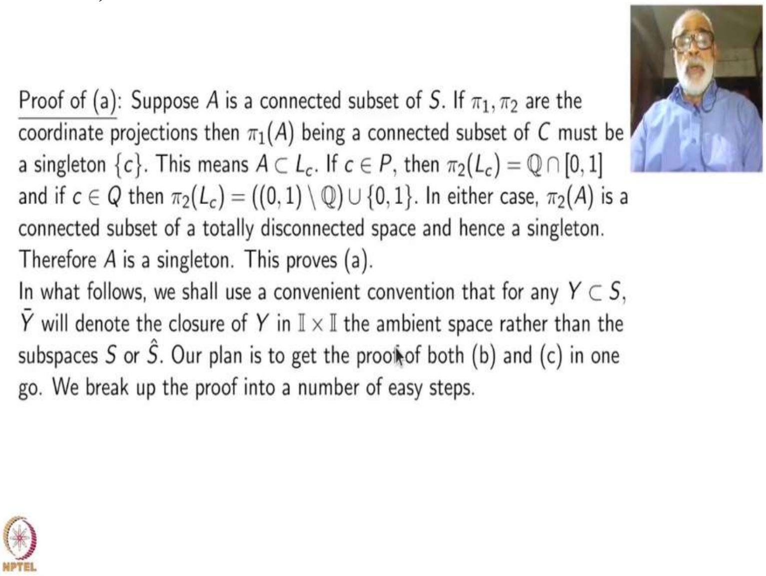
The first part (a) is not so difficult to that S is totally disconnected, viz., each point is a component. Suppose A is a connected subset of S. I want to show that A is a singleton. Look at the two projection maps π_1 and π_2, restricted to S into $[0,1]$. Where are the images contained in? That is what you want. If you take the first projection, the image will be inside C. Because the line segments are taken only based on the points of C. So, $\pi_1(A)$ being a connected subset of C must be a singleton. We are using the fact that C is totally disconnected. So, $\pi_1(A)$ is a singleton what does it mean? A is contained in the line $\{c\} \times \mathbb{I}$ for some $c \in C$.

Now, look at $\pi_2(A)$. If c is inside P, then $\pi_2(L_c)$ is equal to in $\mathbb{Q} \cap [0,1]$. See, all the second coordinates are rational. Therefore it is totally disconnected. And if c is inside Q, the $\pi_2(L_c)$ will be contained $[0,1] \setminus \mathbb{Q}$, all the irrational numbers, and two extra points 0 and 1. It does not matter.

In either case $\pi_2(L_c)$, they are totally disconnected.

$\mathbb{Q}$ is totally disconnected. Also its complement, the set of all irrational numbers is disconnected. Adding two extra points does not change the total disconnectedness. So $\pi_2(A)$ is totally disconnected as well as connected. Therefore $\pi_2(A)$ is a singleton. It follows that A is a singleton. Thus (a) is proved.

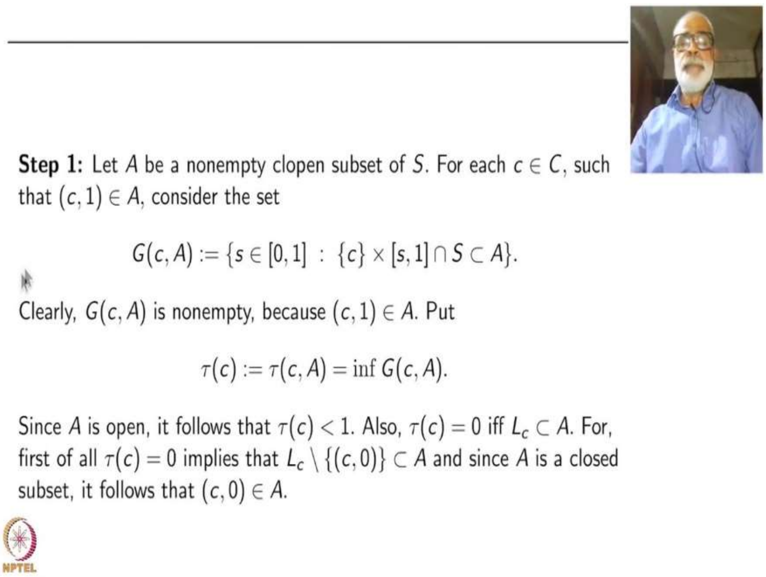
(Refer Slide Time: 12:08)



Now, for the next (b) and (c) what I am going to do? First I will have a convention: for any subset Y of S, $\bar{Y}$ will denote the closure of Y in the ambient space $\mathbb{I} \times \mathbb{I}$, rather than the subspace S or $\hat{S}$. So, be careful that the closure notation is taken in the larger space $\mathbb{I} \times \mathbb{I}$, that is all. This is just a convenient notation; otherwise, each time I have to say closure in S and closure in $\hat{S}$, closure in $\mathbb{I} \times \mathbb{I}$ and so on. The closures of a subset in different spaces could be different. So, that is why you have to be careful here.

Our plan is to get the proof of both (b) and (c) in one go. However, we have to break up the proof into a number of easy steps. Let us see how easy they are.

(Refer Slide Time: 13:23)



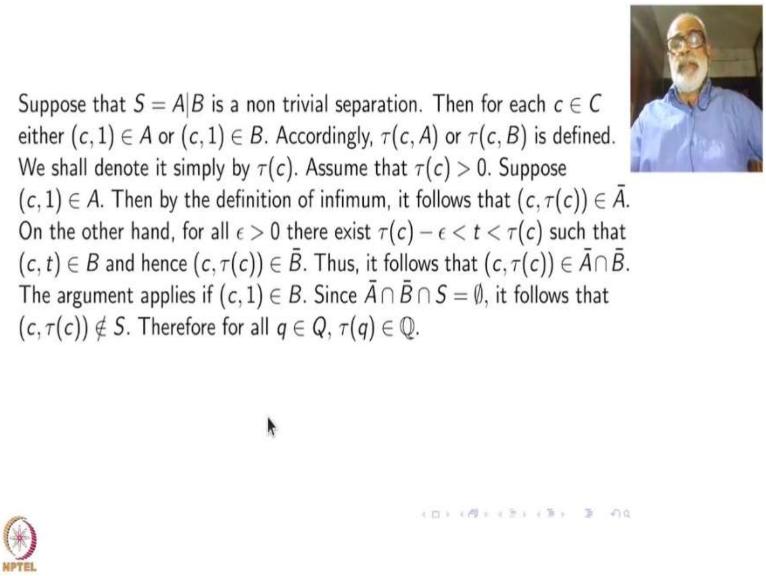
Step 1 is: Let A be a nonempty clopen subset of S. For each $c \in C$, such that $(c, 1)$ is inside A (this is hypothesis: the subset A of S is clopen and $c \in C$ is chosen such that $(c, 1)$ is in A; only then I am going to define this subset), let $G(c, A)$, which depends upon c as well as A, is the set of all s in $[0, 1]$ such that singleton $\{c\}$ cross the entire line segment $[s, 1]$ intersected with S must be contained inside A. After all A is a subset of S and so we have to take the intersection.

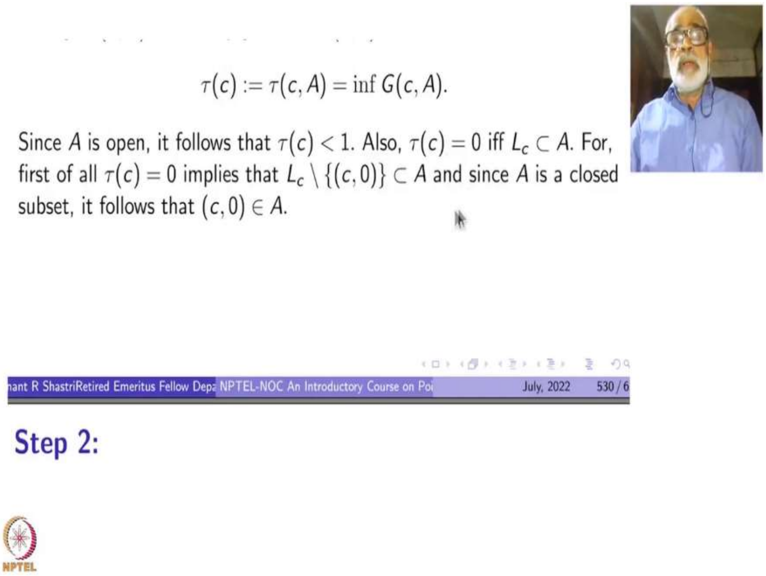
Look at the set, this is a subset of $[0, 1]$ and is nonempty. So, infimum makes sense the infimum will also between 0 and 1. Now use the fact that A is open. We started with A as a clopen set. It follows that $\tau(c)$ must be less than 1. Why? Because, once it is open and $(c, 1)$ is there, a small neighborhood of that point is in A, which means some line segment $\big(c \times (\epsilon, 1]\big) \cap S$ is inside A. Then the infimum will be less than that ϵ which is less than 1. So, $\tau(c)$, though as such is between 0 and 1 it is actually strictly less than 1, because S open.

Next. Can it be 0? Yes it can be 0, but what happens if $\tau(c) = 0$? Then L_c is contained inside A. And conversely. If the entire line segment intersected with S is inside A then obviously, what happens 0 will be inside $G(c, A)$. Therefore, the infimum will be 0. Conversely, if this infimum is 0, that means the open interval $\big(c \times (0, 1)\big) \cap S$ is contained inside A. Therefore, this entire L_c except perhaps the point $(c, 0)$ is inside A.

Just because the infimum is 0. But then $(c, 0)$ will be a limit point and A. See now I am using the hypothesis that A is closed. So, $(c, 0)$ is also inside A. That just means that entire L_c is inside A. So, remember this criterion now : $\tau(c) = 0$ if and only if L_c is contained inside A. Hypothesis on A is that A is clopen and $(c, 1)$ is inside A to begin with. So, this is our first step.

(Refer Slide Time: 18:08)





The second step is: Start with a proper clopen subset of S. We are taking a non trivial separation S now. There may be many separations. Actually there are many, namely, you take a separation $C = A|B$, of the Canter set, then raise the whole thing above. Union of all the L_c's where c ranges over A (respectively B), that will be automatically a separation of S. (Not a separation of $\hat{S}$. $\hat{S}$ is finally shown to be actually connected. So, there are no separations. So, that is what we are going to prove but S has many separations.

So, we now study one fixed separation of S deeply, $S = A|B$. I am going to claim something:

For each $c \in C$, this $(c,1)$ will be either inside A or inside B, because $A \cup B$ is the whole of S . Accordingly $\tau(c,A)$ or $\tau(c,B)$ will be defined, which one you do not know, it will depend upon whether this point is inside A or inside B. To treat both of them together, I will just a notation $\tau(c)$. There is a lot of symmetry here, because A and B together define a separation of S.

So, I am coming back into step 1 here. If $\tau(c) = 0$, I have concluded something. Now suppose $\tau(c)$ is positive then what happens? Note that $\tau(c)$ is always less than 1, that much we know. Suppose $(c, 1)$ is inside A. There are two cases. So, just for definiteness, suppose $(c, 1)$ is inside A. Then by the definition of infimum, it follows that $(c, \tau(c))$ is in the closure of A (this bar refers to the closure inside $\mathbb{I} \times \mathbb{I}$).

It is limit point of the set and therefore, is inside $\bar{A}$. On the other hand, for every epsilon positive, you can find a t between $\tau(c) - \epsilon$ and $\tau(c)$ such that (c, t) is inside the complement of A, namely inside B. That is because this $\tau(c)$ is the infimum, the moment you take something less than infimum, there will be a point which is not in the set A. If it is not in A, it must be inside B, because $A \cup B$ is the whole of S.

Therefore, $(c, \tau(c))$ is in the closure of B also. So, what we have proved is if $\tau(c)$ is positive then $(c, \tau(c))$ is in $\bar{A} \cap \bar{B}$, assuming that $(c, 1)$ is inside A.

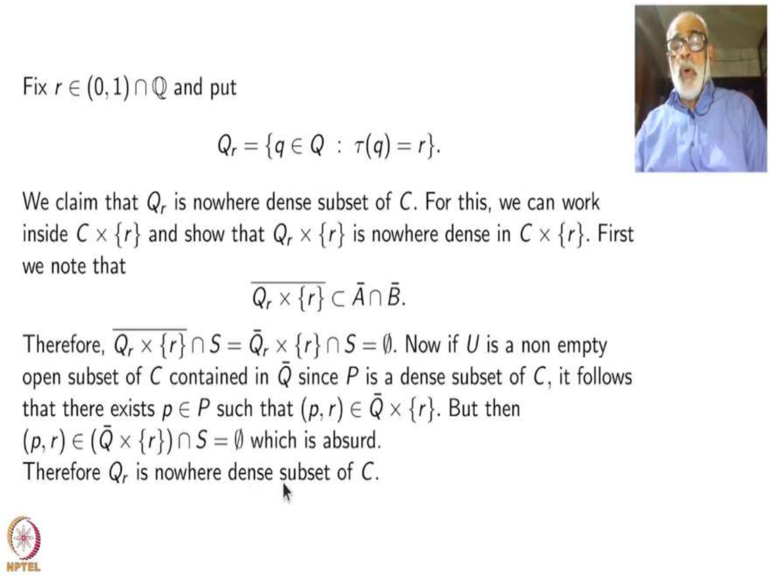
But the argument is exactly similar if $(c, 1)$ is inside B. Therefore in either case, $(c, \tau(c))$ is inside $\bar{A} \cap \bar{B}$. But what is $(\bar{A} \cap \bar{B}) \cap S$? That is $(\bar{A} \cap S) \cap (\bar{B} \cap S)$. Remember A is a closed subset of S. So, $\bar{A} \cap S$ is A. Similarly, $\bar{B} \cap S$ is B. So, this is empty.

It follows that $(c, \tau(c))$ is not in S. Look at the definition of $(c, \tau(c))$.

Suppose c is inside Q. We have assumed that $\tau(c)$ is positive, we have also shown that it is less than 1. Therefore, if at all $(c, \tau(c))$ is in S if and only if $\tau(c)$ is irrational. Therefore, $(c, \tau(c))$ is not in S means that $\tau(c)$ must be rational.

So, we have understood the two cases, when $\tau(c)$ is 0 or positive. In particular, for all $q \in Q$, $\tau(q)$ is rational.

(Refer Slide Time: 23:51)



Now, we exploit this one. Fix a rational number $r \in (0,1)$ and define (this can be done for r irrational also but it is not our concern) Q_r equal to the set of all those points $q \in Q$ such that corresponding $\tau(q) = r$.

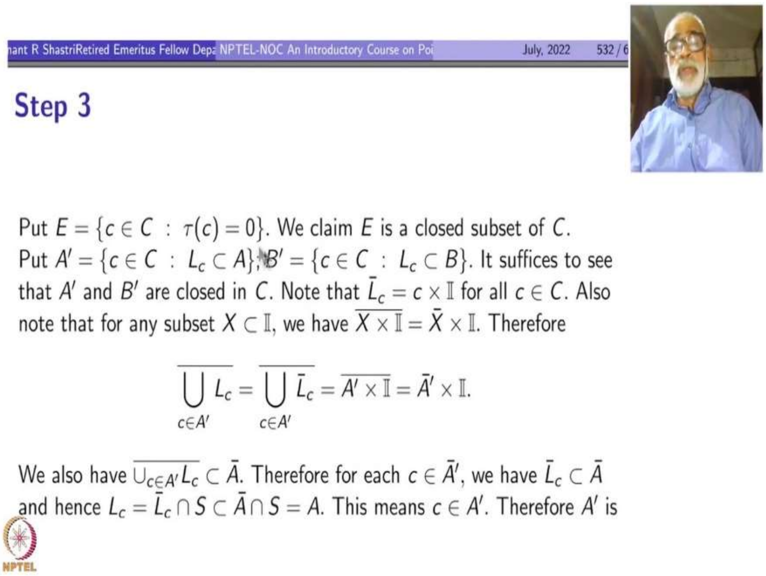
Note that $\tau(q)$ is defined for all $q \in Q$, that is the whole idea; I am still working with the given separation $A|B$ of S. That is all fixed.

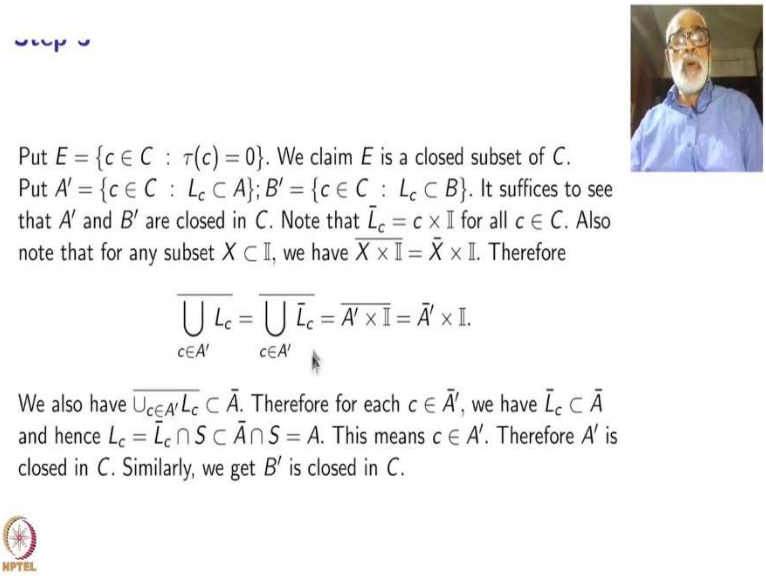
So, the set of all $(q, \tau(q))$ is being split up horizontally. We claim that this Q_r is nowhere dense subset of C. Q_r is a subset of Q and hence that of C as well. For this we can work, instead of in the bottom line $C \times 0$, with a copy of C, namely $C \times r$, to show that $Q_r \times \{r\}$, which the copy of Q_r is nowhere dense in $C \times r$ first. We note that $Q_r \times \{r\}$ is nothing but the set of points $(q, \tau(q))$ where $\tau(q) = r$. Therefore, its closure is contained in inside $\bar{A} \cap \bar{B}$. That is what we have shown. Therefore, $(\overline{Q_r \times \{r\}}) \cap S$ is empty.

I want to show that $Q_r \times \{r\}$ is nowhere dense, that is, $\overline{Q_r \times \{r\}}$ has no interior. So, suppose you have a nonempty open subset U of C contained in $\overline{Q_r}$, since P is a dense subset of C, it follows that there exist $p \in P$ such that p is is $\overline{Q_r}$, i,e., (p, r) is in $\overline{Q_r} \times \{r\}$. If you have any open subset of C there will be a point of P in it, that is all I am using. But if this point is in $\overline{Q_r}$, then (p, r) will be in $(\overline{Q_r} \times \{r\}) \cap S$; and the latter set is an empty set?

This contradiction shows that no nonempty open set is contained inside $\overline{Q_r}$. Therefore, Q_r is nowhere dense subset of C.

(Refer Slide Time: 28:15)





Now, we are coming more or more closer to the conclusion.

Step 3 has the more topology now. I define another set E, the set of all points c of the Canter set such that $\tau(c) = 0$. We claim that E is a closed subset of C. Put A' equal to all those $c \in C$ such that L_c is a subset of A' and B' equal to all those $c \in C$ such that L_c inside B. Remember $\tau(c) = 0$ implies L_c is inside A or inside B depending upon whether $(c, 1)$ is inside A or inside B. So, that was our first step. Therefore, this E is nothing but A' union B'.

So, it suffices to show that A' and B' are closed subsets of C. Then E will be closed. Now look at $\overline{L_c}$. It is nothing but c cross the whole of the interval $\mathbb{I}$. Because, L_c has either have all the irrational points or all the rational points of $c \times \mathbb{I}$ in either case, the closure is c cross the whole interval. Also note that for any subset X of $\mathbb{I}$, we have $X \times \mathbb{I}$ closure is nothing but $\bar{X} \times \mathbb{I}$. This is a general property of the product spaces, nothing to do with the interval $\mathbb{I}$. (If you take closure of $X \times Y$, in space $Z \times Y$, where Y and Z are any arbitrary spaces, and X is a subset of Z, then it is always $\bar{X} \times Y$. So, that is what I am using for the product topology here.)

Now look at the closure of the union of L_c, where c ranges over A', namely, all those c for which L_c is inside A. It contains L_c and hence it contains $\overline{L_c}$. This is true for all c in A' and therefore it contains the union of all $\overline{L_c}$'s where c ranges over A'. is the same as the closure of the union of the closures of L_c for $c \in A'$ which is equal closure of $A' \times \mathbb{I}$ and therefore equal to $\bar{A}' \times \mathbb{I}$.

On the other hand each L_c is contained inside its closure and so I can take the union and then take the closure. Therefore there is equality here. I have to say this one because this union is an arbitrary union, it is not a finite union. So, these two are the same.

But now $\overline{L_c}$ is what? it is equal to $\{c\} \times I$ and c ranges over A'. So, I can write this as the closure of $A' \times \mathbb{I}$, which is nothing but $\bar{A}' \times \mathbb{I}$.

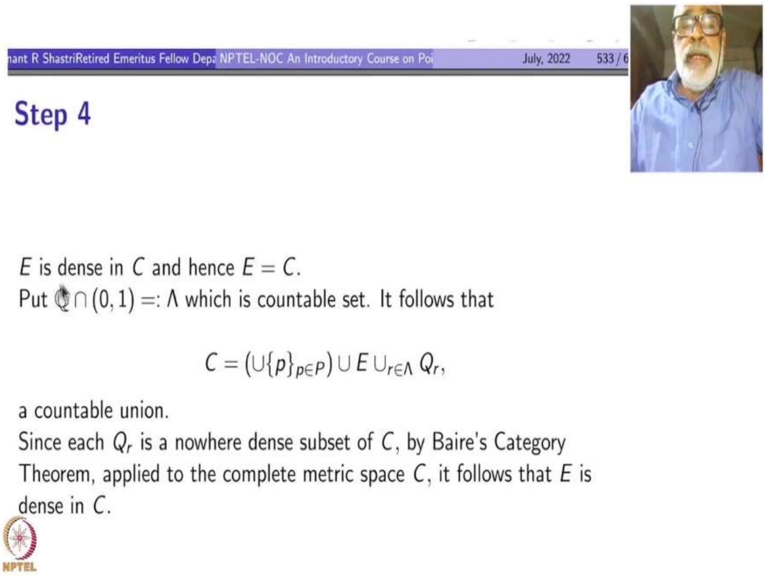
Moreover, each L_c, for $c \in A'$, by definition, is inside A. Therefore, this left hand side is contained in $\bar{A}$. That means $\bar{A}' \times \mathbb{I}$ is contained in $\bar{A}$.

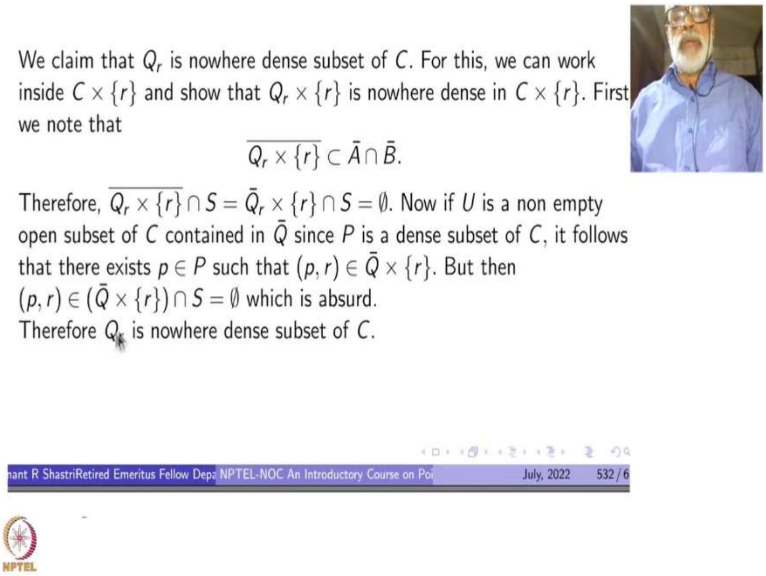
Therefore, if you start with a point $x \in \bar{A}'$ here, and if you take $x \times \mathbb{I}$, it is contained in $\bar{A}' \times \mathbb{I}$ and hence in $\bar{A}$. Therefore, L_x which is equal to $\overline{L_x} \cap S$ is contained in $\overline{A} \cap S$ which is equal to A, because A is a closed subset of S. That just means that x is in A'. See I started with x belonging to $\overline{A'}$ and shown that it is in A', which just means that A' is closed. Similarly, B' is also closed.

Therefore, E the set of points $c \in C$ for which $\tau(c) = 0$ is a closed subset.

So far, we have proved so much of topology on this one. We just do not know whether this is nonempty. For all that matters, your discussion, this may be on an empty set. We have not proved that it is nonempty. We have not proved that any of these L_c is contained inside A or B. May be each of them has got separated by the separation $A|B$. Even after having done so much of topology, we will still do not know that. The next step is precisely that.

(Refer Slide Time: 35:01)





Step 4: E is dense in C and hence E is actually equal to the whole of C. The strange thing is you try to prove E is just nonempty, there is no other way than actually proving that it is equal to C. This is the typical way wherever you apply Baire's category theorem and that is what we are going to do, we are going to apply Baire's category theorem here we are not just proving E is nonempty. We actually prove that this is a dense subset.

So, another notation here, just temporary notation: set of rationals in the open interval $(0,1)$, let us denote it by Λ. The only thing that I am concentrating on this one is that Λ is a countable set. This is going to be an indexing set now. It follows that the elements of C are the following types. First, those which are inside P. So, I am writing that part as union of $\{p\}$ when p ranges over P. Then those which are in E (this E maybe empty, I do not know) and finally those which are in the sets indexed by Λ what are these sets, Q_r where r ranges over Λ , we have proved each Q_r is nowhere dense.

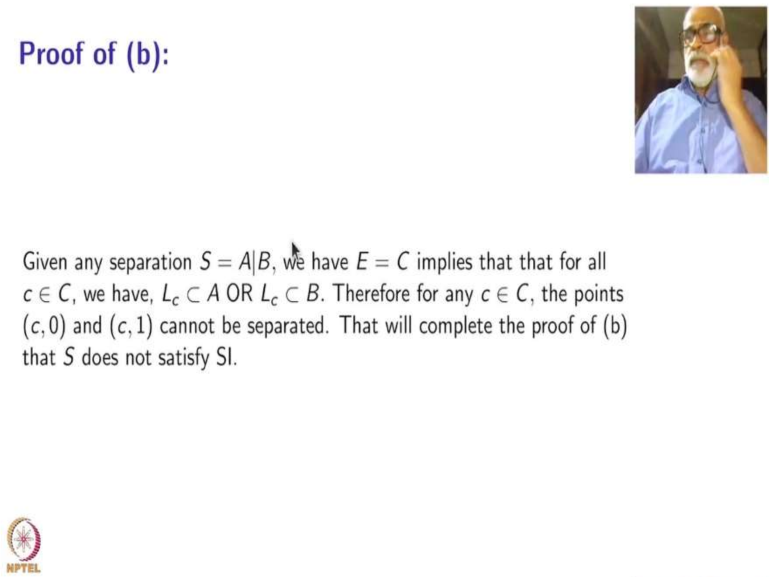
So, how many of them? there countably many. So, C is written as a countable union, because P is also countable. Of these the first type and the last type are nowhere dense.

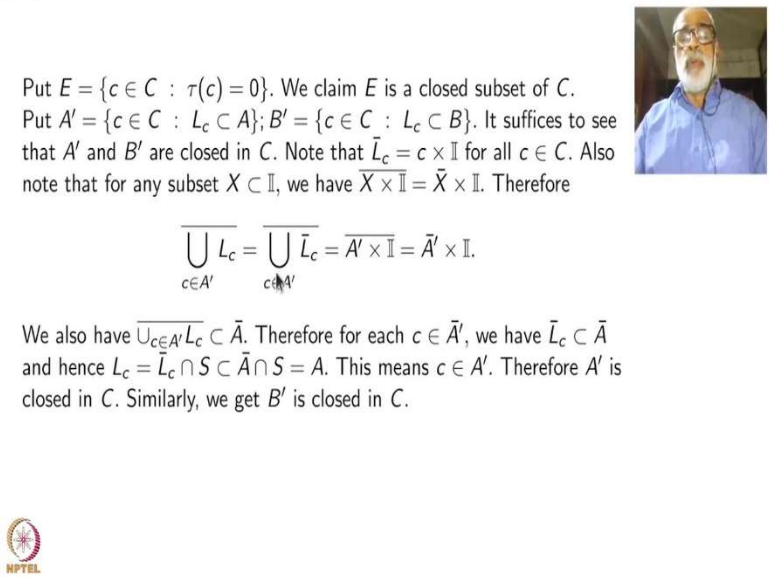
If E is also nowhere dense what happens? You will get a contradiction to Baire's category theorem, because C is a complete metric space. Indeed, once you write like this countable union, the stronger statement of Baire's category theorem says that one of the sets must be dense. So, it follows that E is dense.

This proof turned out to be easy for us. But without Baire's category theorem, it is just impossible to do anything like this. However, to apply Baire's theorem, we have to do all these preparation.

We can now complete the proof of both (b) and (c) very easily.

(Refer Slide Time: 38:55)





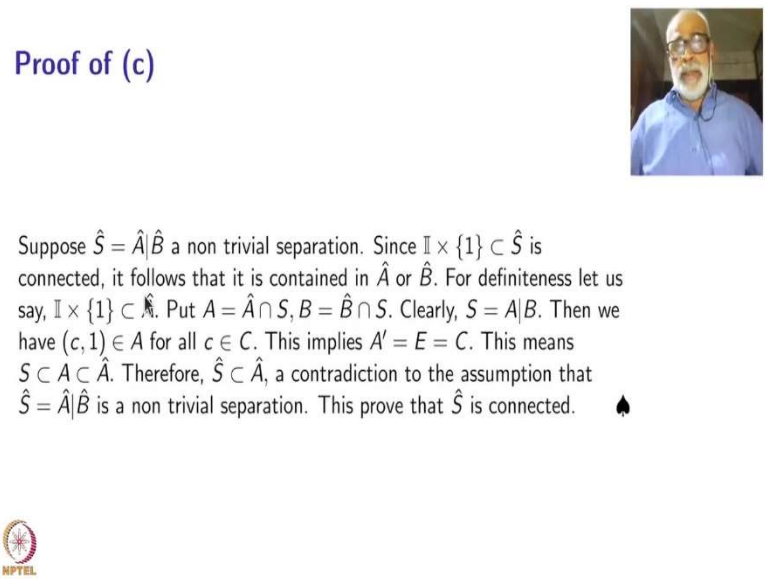
So, what is (b)? It says that S does not satisfy SI. Does not satisfy SI means I have to produce two distinct points in S, x and y which cannot be separated. So, what do I do? Given any separation S, $S = A|B$, we have the corresponding E is equal to C. What does this imply? This implies that for c inside C, L_c is either inside A or inside B.

Therefore, for any c inside C, the points $(c, 0)$ and $(c, 1)$ cannot be separated, because are in the same L_c. Over.

I have produced plenty of pairs of points which cannot be separated. Essentially we have observed that there are vertical separations what this says is that two points lying on the same vertical line, they cannot be separated. So, there is nothing like horizontal separation. There is nothing even slanted separations and so on only vertical separations are there, plenty of them because C itself is what? totally disconnected.

If you remember like this it will be easy, and then you can reproduce the detailed proof, there is no problem.

(Refer Slide Time: 41:16)



So, now, proof of (c). Remember that $\hat{S}$ has one extra thing in it, namely, the entire line segment $[0, 1]$. So, suppose, $\hat{S}$ has separation $\hat{S} = \hat{A}|\hat{B}$. (We want to prove $\hat{S}$ is connected. Any disconnected space must be having non trivial separation. Non trivial separation means what? $\hat{A}$ and $\hat{B}$ are nonempty, both $\hat{A}$ and $\hat{B}$ are closed and they are disjoint and the union is $\hat{S}$. That is all. So, I am just here recalling what is the meaning of non trivial separation).

But $\mathbb{I} \times \{1\}$ is a connected set and hence contained inside the left hand side. Therefore, it must be contained inside $\hat{A}$ or $\hat{B}$. What does this imply? For definiteness let us say that $\mathbb{I} \times 1$ is inside $\hat{A}$. Put A equal to $\hat{A} \cap S$ and $\hat{B}$ equal to $\hat{B} \cap S$. From $\hat{S}$ come to the subspace S. Then S itself will be separated namely S equal to $A|B$. A and B will be closed subsets of S and they are disjoint obviously, because $\hat{A}$ and $\hat{B}$ are themselves disjoint in S hat.

So, this S is equal to A separated B. In the proof of (b) we have seen that $(c, 1)$ is either inside A or inside B. but right now $(c, 1)$ are all inside A, by this assumption that the entire $\mathbb{I} \times 1$ is inside $\hat{A}$. $\hat{A} \cap S$ is A. So, all $(c, 1)$ are inside A, for all $c \in C$, which just means that A' is the whole of C and B' is empty. So, this is A' equal to C. This means the entire set S is inside A. But A is a subset of $\hat{A}$. Therefore, $\hat{S}$ itself is in $\hat{A}$ because $\hat{S}$ is nothing but the closure of S, and $\hat{A}$ is a closed subset. That is a contradiction to the assumption that $\hat{S}$ equal to $\hat{A}$ separation $\hat{B}$ is a non trivial separation. So, we have proved that $\hat{S}$ is connected.

Now I come to the final construction of the Knaster-Kuratowski example.

(Refer Slide Time: 44:35)

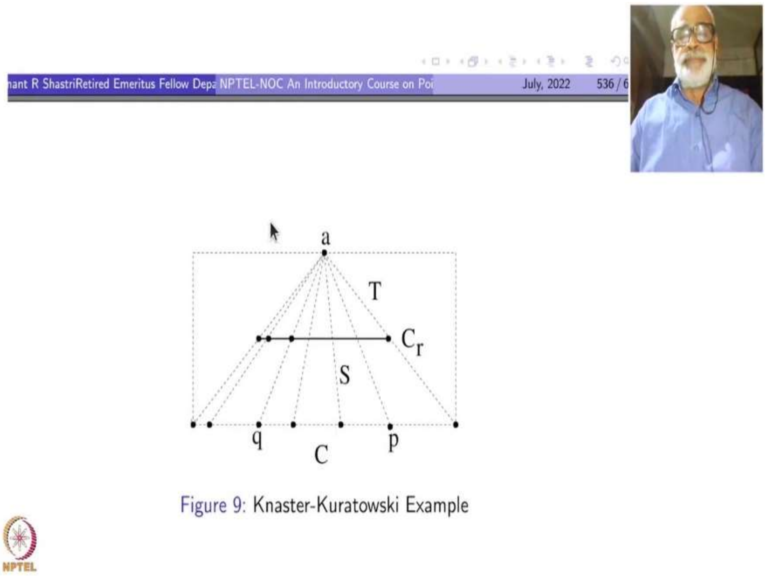


Figure 9: Knaster-Kuratowski Example

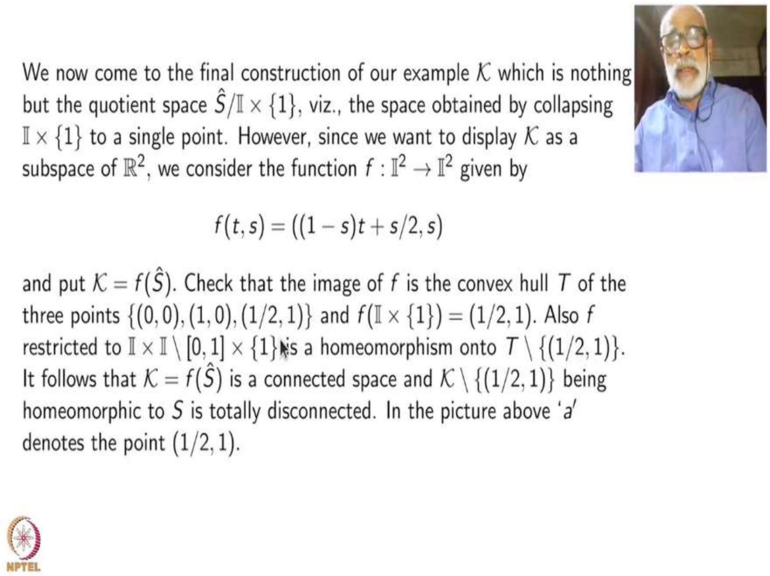
So, this is the picture. Starting with these dot, dot, dots, all these form the Canter set. Then what we did, we took lines vertical lines like this for $p \in P$ all the second coordinates are rational. For points in the complement, all the second coordinates are irrational. So, that was my S. Then we added this line to get $\hat{S}$.

But now what we want to do is you just bring the entire line segment here to a single point here, collapse the line. How do you collapse it? Along the horizontal lines horizontal keep moving. So, this line segment from this 0 to 1 will be moved to the line from this point to this point, How much to move? Depends upon its y coordinate. Always, the second coordinate remains the same. Move them along horizontal lines.

So, this line has become one single point. In other words if you delete this point and if you delete the entire line here, whatever remaining thing here will be the open subset it will be homeomorphic to a subspace of S. Namely, all the points $(c, 1)$ are removed from $\hat{S}$. Being a subspace of a totally disconnected space S, that will be also totally disconnected. So, what happens is that a is the apex point of this tent. That is why it is called Canter's tent. If you remove that point, it becomes a leaky tent that leaky tent is totally disconnected. Why this is connected, this is just obtained by as a quotient of $\hat{S}$ where we have identified this entire line to a single point that is all.

I have written down the formula for this quotient map from $\hat{S}$ to this tent, this I denote by $\mathcal{K}$. So, how to get this one from $\hat{S}$? So I will show you that one now.

(Refer Slide Time: 47:23)



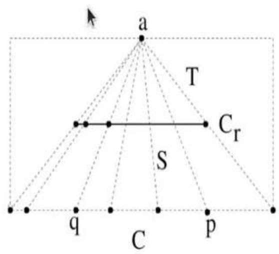


Figure 9: Knaster-Kuratowski Example

Here is the example. So, we construct final construction of this $\mathcal{K}$, which is nothing but the quotient of $\hat{S}$, wherein $\mathbb{I} \times \{1\}$ is collapsed to a single point. So, this is a notation. However, since we want to display $\mathcal{K}$ as a subspace of $\mathbb{R}^2$, we consider the function f from $\mathbb{I}^2$ to $\mathbb{I}^2$ given by $f(t, s) =$ (something, s). You see the second coordinate does not change at all. Everything is happening along the horizontal lines. I am going to for the x-coordinate? I will take $(1 - s)t + s/2$. When s is 0, what is this point, it is (t, s), the identity map. When s is 1, what happens, this will be 0, and this will be $1/2$ so $f(t, s)$ is $(1/2, 1)$.

So, all the points $(t, 1)$ are going to a single point. That means$\mathbb{I} \times 1$ is going to a single point. If s is less than 1, this is a bijection, check that. So, put $\kappa(\mathcal{K}) = f(\hat{S})$, the image of S under f. Check that the entire image of f is the full triangle T based on the $\mathbb{I} \times \{0\}$ and with apex point $a = (1/2, 1)$. That is $f(\mathbb{I}^2)$. The base of this triangle will be this closed interval then this will be the apex. So, this κ is going to be a subspace of this triangle T.

The same kind of description is there all the time. Restrictions on the y-coordinate of those points on the line joining a point $(c, 0)$ to the point a. Throw away the top line $\mathbb{I} \times 1$ from from $\mathbb{I} \times \mathbb{I}$ and throw away the point a from T, then f is a homeomorphism. Therefore, it follows that κ which is equal to $f(\hat{S})$ is a connected space, being the quotient image of a connected space under f, and $\kappa \setminus \{a\}$, being homeomorphic to a subspace of S is totally disconnected. In the picture the point a is nothing but $(1/2, 1)$. So, everything is proved.

(Refer Slide Time: 51:00)

It follows that $\mathcal{K}$ satisfies neither S0 nor SI. We shall see later that adding an 'extra point' does not destroy SIII. Therefore we conclude that $\mathcal{K} \setminus \{(1/2, 1)\}$ neither satisfies SII nor SIII.

So, let us stop here with the final remark that $\mathcal{K}$ satisfies neither S0 nor SI. Why? Because it is connected, over. We shall see later that adding an extra point does not destroy SIII. Therefore, we conclude that $\mathcal{K} \setminus \{a\}$ satifies neither SII nor SIII. Let us stop here. Next time we will study a little more about S0 SI, SII, SIII, etc, and introduce what are called 0-dimensional spaces. Slowly we will start the study of dimension theory. Thank you.