An introduction to Point-Set-Topology Part-II
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Lecture 37
More examples

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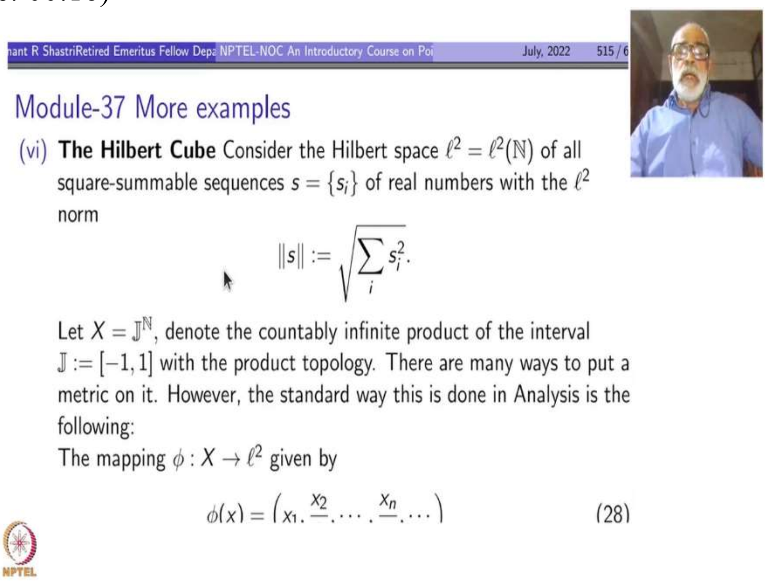

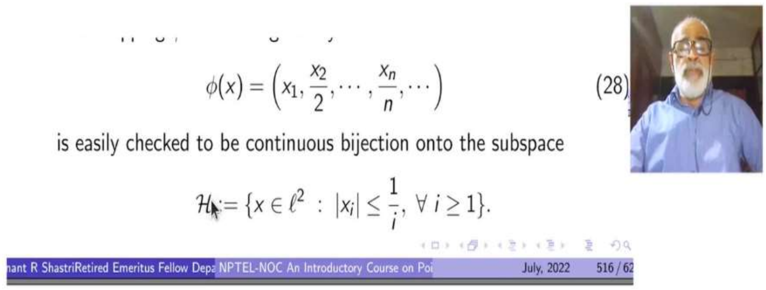

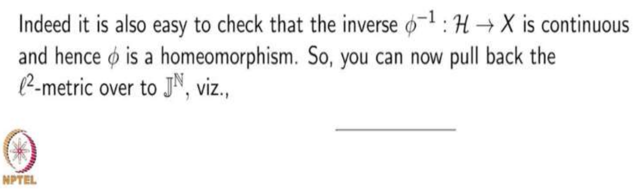

Indeed it is also easy to check that the inverse $\phi^{-1} : \mathcal{H} \to X$ is continuous and hence ϕ is a homeomorphism. So, you can now pull back the ℓ^2-metric over to $\mathbb{J}^{\mathbb{N}}$, viz.,

$$d((x_n),(y_n)) = \sqrt{\sum_n \frac{(x_n - y_n)^2}{n^2}}.$$

Any space which is homeomorphic to $(\mathbb{J}^{\mathbb{N}}, \mathcal{T}(d))$ is called a Hilbert cube. The model $\mathcal{H}$ is the most popular one for the Hilbert cube, though there is no standard notation for it. Often it is convenient to use the notation $\mathbb{J}^{\mathbb{N}}$ with the metric defined as above for $\mathcal{H}$.

Welcome to NPTEL NOC an introductory course on Point-Set-Topology Part II. Today, we will study some more examples of this S0, SI, SII and so on. Actually, we started seeing examples last time, they were all SII. So, we will continue that. Here is one of the important examples, the Hilbert Cube. Consider the Hilbert space ℓ_2 over a countable set of points, namely, you can take the natural numbers.

So, recall that the space ℓ_2 consists of square summable sequences of real number under the ℓ_2 norm which is nothing but take the sum of all the squares and then take the square root. For topological reasons, we will have a different look at this space. Indeed, we are now going to consider a smaller set here, namely, let us start with the product of infinite countably many copies of the closed interval $J = [-1, 1]$.

So, let us denote it by X. X is $\mathbb{J}^{\mathbb{N}}$. There are many ways to put a metric on it. So, what we will do? We will use this ℓ_2 space and the metric given by it, we will take a standard embedding of X (standard means it is used by a lot of people especially in analysis) in ℓ_2 and take the induced metric. Take a point $x \in X$, it is like a sequence (x_i) where each x_i is between -1 and 1, we will send it to point x_1, the second coordinate is $x_2/2$, the third coordinate will be divided by 3 and so on, the n-th coordinate will divide by n, and so on. So, this is an arbitrary sequence of real numbers between -1 and 1. So, this is another sequence. Now, what happens if you take the sum of squares that is convergent, sum is convergent.

So, this will be an element of ℓ_2. Note that x itself is not an element of ℓ_2. It is easily checked that this map is a continuous bijection onto its image in ℓ_2.

What is that the image? It is the set of all points whose the i-th coordinate is in modulus less than or equal to $1/i$. Because look at the image here, x_n is already between -1 and 1, therefore $|x_n|/n$ is less than equal to $1/n$. Conversely, on a point of ℓ_2, if you put this condition, after multiplying the n-th coordinate by n, it will be still inside $[-1, 1]$. Therefore, that will come will be a point of X.

So, that will show you that this is a bijection. Continuity of this one is obvious. So, it is a continuous bijection.

But actually, the way I have defined this phi, I am defining the i-th coordinate, I am just multiplying by the integer i. So, even the inverse function is also continuous there is no problem. Which just means that this map ϕ is a homeomorphism of $\mathbb{J}^{\mathbb{N}}$ into a subspace of the Hilbert space.

That subspace we are denoting by this $\mathcal{H}$ and calling it the Hilbert cube. The notation may be different, but it is commonly agreed that this should be the space named Hilbert cube. The name is quite standard, but not the notation. So, now without reference to $\mathcal{H}$ all the time, we can just write down the metric on $X = \mathbb{J}^n$ itself, which is induced by this ϕ namely given any two point x and y inside X, $d(x, y)$, distance between x and y can be defined by taking $x_n - y_n$, take the square, divide by n^2, take the sum over n and then take the square root. So, this is the metric. I do not want to write the norm here because X is not a vector space. Any space which is homeomorphic to $(\mathbb{J}^{\mathbb{N}}, \mathcal{T}_d)$ namely, the metric induced topology here, that is called a Hilbert cube.

After all we are studying topology, so, anything which is homeomorphic to this may be called a Hilbert cube, we are not very much particularly interested in the actual metric here, but only on the topology, and now, we know that this topology is nothing but the product topology on $\mathbb{J}^n$. So, quite often it is convenient to think of this one as just $\mathbb{J}^n$, and $\mathcal{H}$ can be thought of as $\mathbb{J}^n$ with the product topology. This is all about convention and notation.

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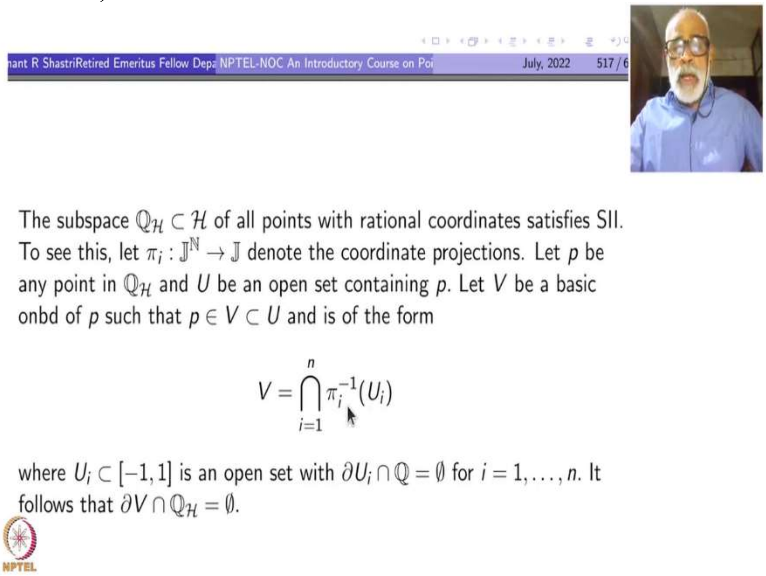

The subspace $\mathbb{Q}_{\mathcal{H}}$ of $\mathcal{H}$ of all points with rational coordinates satisfies SII. So, this is what we want to say. First of all, it is metric space. Now, inside that it satisfies the SII is what I want to say. To see this, let $\mathbb{J}^{\mathbb{N}}$ to $\mathbb{J}$ be the i-th coordinate projection. Let p be any point in $\mathbb{Q}_{\mathcal{H}}$, (remember all the coordinates of all points here will be rational). Take any point p in $\mathbb{Q}_{\mathcal{H}}$. Take an open subset containing p.

Let V be a basic open set containing p, that is open neighbourhood, such that p belongs to V contained inside U, after all every open nbd will be containing a basic neighborhood around that point. So, this basic neighborhood will be of the form by the very definition of the product topology, intersection of finitely many $\pi_i^{-1}(U_i)$, where each U_i, is an open subset of $[-1, 1]$. Finitely many of i, so there will be a maximum say n, and I can include all of them up to n, no problem. But here I am going to choose this V such that each U_i has this property namely the boundary of U_i, which will consist of at most two points depending upon where the interval is taken), both the points must be irrational.

That means boundary of $U_i \cap \mathbb{Q}$ is empty for all $i = 1, 2, \ldots, n$. Then you take the inverse image of U_i under π_i, take the finite intersection, that is V. Automatically it will imply that boundary of $V \cap \mathbb{Q}_{\mathcal{H}}$ is empty. See this V is a subset of the entire $\mathbb{J}^{\mathbb{N}}$ or you may say $\mathcal{H}$, Intersection with $\mathbb{Q}_{\mathcal{H}}$ is empty because the boundary points have at least one coordinate irrational. Here you have to use the elementary fact for subspaces of a product space, namely, if you have $X \times Y$, and a subset A of X, and B of Y, boundary of $A \times B$ is equal to boundary of $A \times B$ union $A \times \partial B$. Use it again and again finitely many times, finite intersection case, you will get this result. So, that is all.

For each point you can choose a neighborhood with their boundaries empty means what? That is the property SII. So, we have verified. We have got another example, namely, subset of all points with all coordinates rational inside the Hilbert cube is also SII.

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(vii) Exactly similarly, we can show that the subspace $\mathcal{I}_{\mathcal{H}} \subset \mathcal{H}$ of all points with all coordinates irrational also satisfies SII.

(viii) In contrast, the subspace $\mathbb{Q}_\ell$ of all points whose coordinates are all rational in the Hilbert space ℓ^2 does not satisfy SII. It suffices to prove that the boundary of any bounded open nbd U of 0 in $\mathbb{Q}_\ell$ is nonempty.

Exactly similarly, we can show that $\mathcal{I}_{\mathcal{H}}$ inside $\mathcal{H}$ is also SII. What is $\mathcal{I}_{\mathcal{H}}$? All the coordinates of all the points are irrational. So, you can reverse the role here, that is fine.

In contrast, the subspace $\mathcal{Q}_\ell$ of all points whose coordinates are all rational in the entire Hilbert space ℓ_2 does not satisfy SII. So, this is what we want to prove. Inside the smaller set, namely, inside the compact subspace $\mathcal{H}$ this is working but if you go to the entire ℓ_2, the unbounded thing, there SII is not satisfied.

So, we will have examples of spaces which does not satisfy SII. So, once again how do you show that something is not SII? It suffices to prove that the boundary of any bounded open neighborhood of 0 in $\mathbb{Q}_\ell$ is nonempty. So, at 0, we can prove that property SII fails.

So, take a bounded open neighborhood U of 0, then its boundary is nonempty; inside ℓ_2, it is obvious but we claim that inside $\mathbb{Q}_\ell$ it is nonempty. So, in other words you take the entire boundary intersect it with $\mathbb{Q}_\ell$ and show that it is nonempty. So, this is done as follows, So, let me show you the picture here first.

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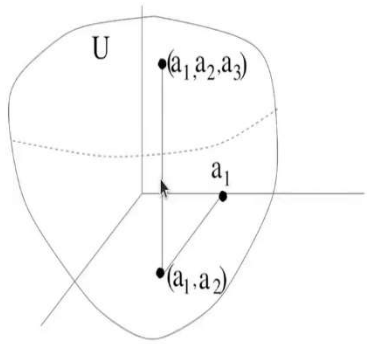

Figure 8: Boundary of a bounded open set in $\mathbb{Q}_\ell$ is non empty

So, this is origin. Of course, my picture cannot contain infinitely many coordinates. So, you have to think of this one as a bunch of infinitely many coordinates. So, this is the origin. This is U is a bounded neighborhood of the origin. I start with a point $p_1 = (a_1, 0, 0, \dots)$ on the x_1 -axis inside U.

Be sure in the inductive process, I want to do something, so be sure that this a_1 is not far away from the complement of U. It is not far away from the complement of U. That means what? Let us say distance between a_1 and the U^c, which is a closed set, let us say, it is less than 1. This distance is less than 1. Why is this possible? The x_1-coordinate axis intersects both U and U^c.

Next, I keep this first coordinate of a_1 as it is and move the second coordinate.

And choose $p_2 = (a_1, a_2, 0, 0, \dots)$ inside U_2 so that this point is again closer to the boundary say, this distance is less than $1/2$. Having chosen $p_1, , p_2, \dots, p_n,$ p_{n+1} will be chosen, keeping the first n-cooridnates the same but changing the $(n+1)$-th coordinate so that the distance between p_{n+1} and complement of U is less than $1/(n+1)$.

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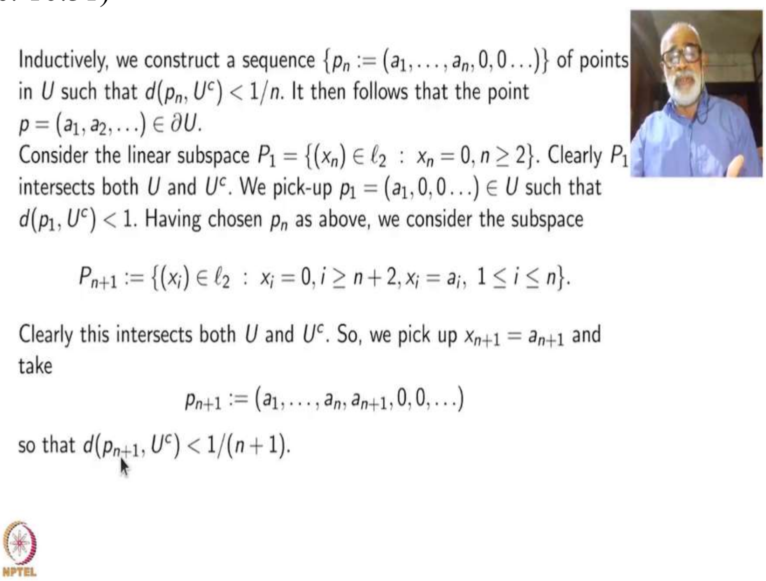

Thus, Inductively, we construct a sequence $\{p_n\}$ points $p_n = (a_1, a_2, \ldots, a_n, 0, 0, \ldots)$ inside U such that distance between p_n and U^c is less than $1/n$. It then follows that the limit sequence namely $p = (a_1, a_2, a_n, \ldots)$ is actually on the boundary of U. Because first of all, all these things are inside U, so this will be inside $\overline{U}$. But the distance between this p and U^c is 0 because it is less than $1/n$ for all n. Therefore, this will be in U^c as well. And $(\overline{U})^c$ so it is inside boundary of U.

So, how it is done? I have already explained to you. So, let me read that one. Consider the linear space P_1 equal to the set of all (x_n) belongs to ℓ_2 such that the n-th coordinate is zero for n beyond one. That is the x_1-axis you can say. Clearly, P_1 intersects both U and U^c. See, because U contains the origin, this line has to intersect U. And because U is bounded, the line has to intersect the complement. So, this is the property I am using. So, on that subspace, there will be a point inside U very close to U^c. The process I have to do later as well. So, pick up $p_1 = (a_1, 0, 0, \ldots) \in U$ such that its distance from U^c is less than 1.

Having chosen p_n, look at P_{n+1} equal to the set of all (x_i) in ℓ_2 such that the coordinates are all 0 beyond $n + 2$, and up to n coordinates, x_i must be a_i as chosen before. This subspace is again a line. You see if I fix up all n-th coordinates n coordinates and also all coordinates beyond $n + 2$ also, and vary only the $(n + 1)$-th coordinates, that is line which will pass through p_n and will go put of U because U is bounded.

So, we pick up $(n+1)$-th coordinate such that the point is at a distance less than $1/(n+1)$ from the complement of U. So, each time work inside the real line to get such a point that is all.

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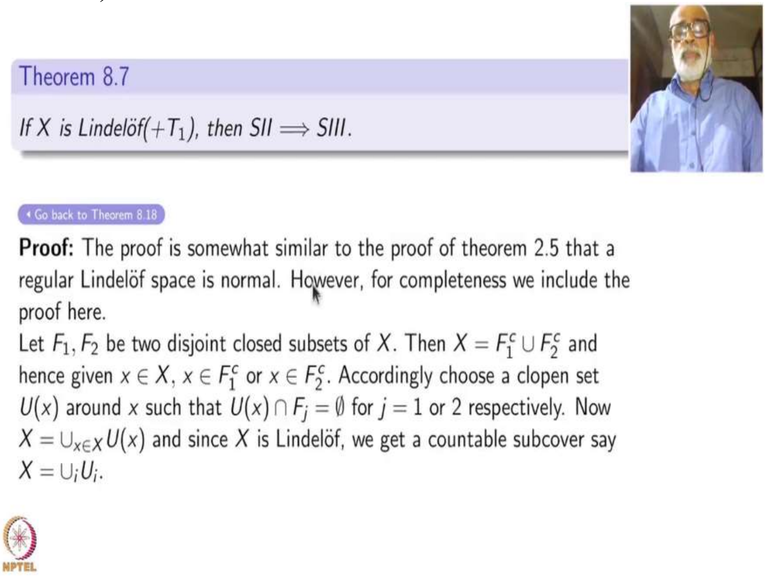

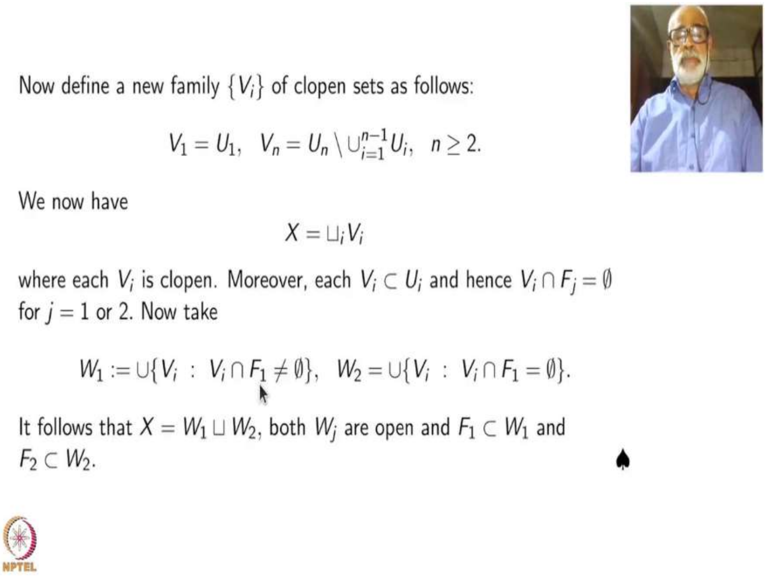

So, now, we will have some theorems. So, what we have done, we have shown one example which does not satisfy SII. It is a subspace of a Hilbert space. $\mathbb{Q}_\ell$ does not satisfy SII. Now, here is a theorem.

If X is Lindelof, then SII implies SIII.

Remember that we have assumed that all our spaces are T_1 right from the first lecture on this topic.

But just to remind you, I have put T_1 in the bracket here. If X is Lindelof, then SII implies SIII.

The proof is somewhat similar to what we had done long back maybe theorem 2.5, that a regular Lindelof space is normal. However, for the sake of completeness we include the proof here. Because we are not exactly doing normality here nor we are exactly using regularities, both are stronger hypotheses, SII stronger than regularity, SIII stronger than normality.

So, let us go through this proof carefully. Take F_1, F_2 two disjoint closed subsets of X. Then X is union of their complements because they are disjoint. Therefore, given x inside X either x is inside F_1^c or x is inside F_2^c, maybe it is in both, does not matter. One of them will be true. Accordingly choose a clopen set U_x around x such that this $U_x \cap F_j$ is empty. Accordingly I have to do. If x is in first one, i.e, x is F_1^c, I have $U_x \cap F_1$ empty.

If x is inside F_2^c, I will have $U_x \cap F_2$ is empty. So, this is by SII. The property SII assures you that such a clopen subset U_x exists. For each point x, I have chosen a U_x, therefore, this X is union of all the U_x's and each of them is open, X is Lindelof implies we have a countable subcover here. Now, we define a new family of clopen subsets. Remember they are both open as well as closed also.

So, how do I do? These kind of steps are same as what we have done in proving regular Lindelof implies normality, this step is same thing. Take $V_1 = U_1$, V_2 obtained by subtracting V_1 from U_2, so, it is $U_2 \setminus V_1$. Like that V_n will be U_n setminus whatever you have taken earlier, namely union of all the V_i, i ranging from 1 to $n-1$.

Now, what we have is each V_i is disjoint from other V_j's So, it is a disjoint family. Each V_i is clopen, why? Because finite union of clopen sets is clopen and a clopen minus a clopen is clopen. And we have written X as disjoint union of countably many open subsets.
Moreover, each V_i remember, is contained inside U_i and hence $V_i \cap F_j$ will be empty for $j = 1$ or 2.
The entire family $\{V_i\}$ is divided into two families accordingly, viz, according to whether $V_i \cap F_1$ is empty or non empty.

Now, take W_1 to be union of all those V_i's such that $V_i \cap F_1$ is nonempty. And W_2 is the union of other V_i's namely, all those V_i's such that $V_i \cap F_1$ is empty. So, I have taken all the V_i's here. Some of them are here, some of them are there and they are disjoint families. It follows that X is a union of W_1 and W_2, $W_1 \cap W_2$ is empty. Being unions of clopen sets W_1 and W_2 are both open that is enough. (In fact, they are closed also.) And F_1 will be inside W_1, and $W_1 \cap F_2$ is empty and hence by very definition, F_2 has to be inside W_2. So, what we have done is writing X as disjoint union of two open set, this is a separation now, with F_1 inside W_1 and F_2 inside W_2. That is the property SIII.

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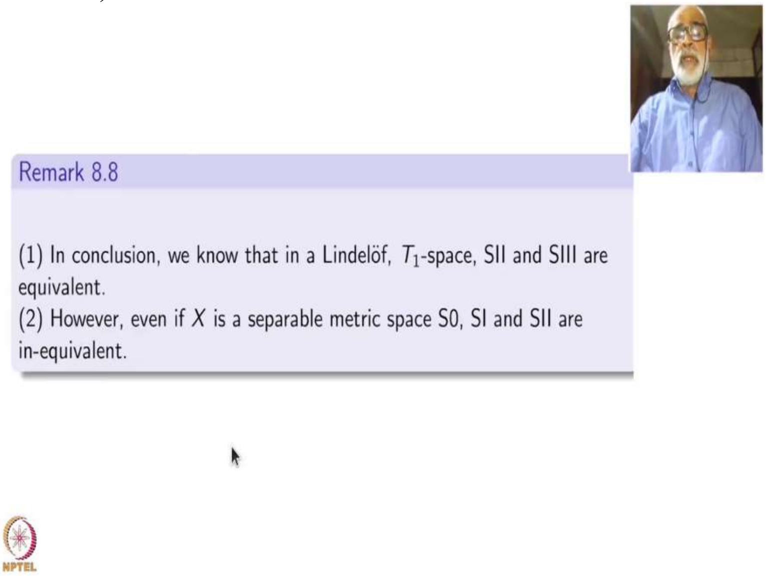

In conclusion, we know that in a Lindelof T_1 space SII implies SIII. But we have already proved SIII implies SII as soon as X is a T_1 space. So, these two are equivalent.

However, even if X is a separable metric space, then the first three of them here, viz., S0, SI and SII are in-equivalent. Proof is not all that easy. In a T_1 space, we have seen that SII implies SI implies S0. So, assume that X is separable metric space even then you cannot go back in these arrows that is the meaning of this one. So, we will have to produce examples finally.

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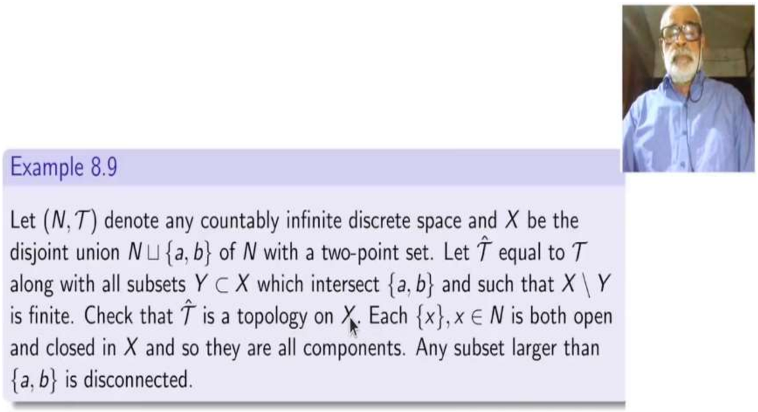

So, here is one example which is not a metric space. But it is an easy example. Start with a countably infinite set with a discrete topology that is my $(\mathbb{N}, \mathcal{T})$. Now, take two extra points a and b. So, $\mathbb{N}$ disjoint union $\{a, b\}$, that is my X. Now, I want to put a topology $\widehat{\mathcal{T}}$ on this X. This will consist of all the members of $\mathcal{T}$ along with all subsets Y of X which intersect $\{a, b\}$ and such that $X \setminus Y$ is finite. Intersect $\{a, b\}$ means they should contain a or b or both.

Also the complement of that Y inside X must be finite. Check that this definition makes $\widehat{\mathcal{T}}$ a topology on X. Not very difficult. This kind of thing you have done several times now. For each x in X, if x is inside $\mathbb{N}$, then it is discrete topology. So, $\{x\}$ is both open and closed in X . Because all the open subsets in tau they are inside $\widehat{\mathcal{T}}$ also and the complement of $X \setminus \{x\}$ is just $\{x\}$ and so finite. So, $\{x\}$ they are all components. If a singleton set is both open and closed that must be a connected component.

Next, we claim that any subset of X larger than $\{a, b\}$ is disconnected. Can you see that? Take $\{a, b\}$ itself as a subspace of X, what is the subspace topology? It is discrete. Therefore, $\{a, b\}$ itself is disconnected. Therefore, anything larger than $\{a, b\}$ will be also disconnected. So, this shows that this topology $\widehat{\mathcal{T}}$ is totally disconnected. All the singletons are components. However, this is not a T_1 space at all. Why? Because take any neighborhood of $a \in X$ not in $\{a, b\}$ if you take a subspace that is discrete, but any neighborhood of a in $\widehat{\mathcal{T}}$, the complement is finite, any neighborhood of b is has complement is finite. But $\mathbb{N}$ is infinite. So, the two neighbourhoods will intersect. (So, that is similar to this co-finite topology.)

Therefore, $(X, \widehat{\mathcal{T}})$ is not a T_1 space. So, these extra points a, b have been added just to make it non T_1 space. But this is totally disconnected space.

You do not want to study such anomalous examples that is reason why we are assume T_1-ness right in the beginning.

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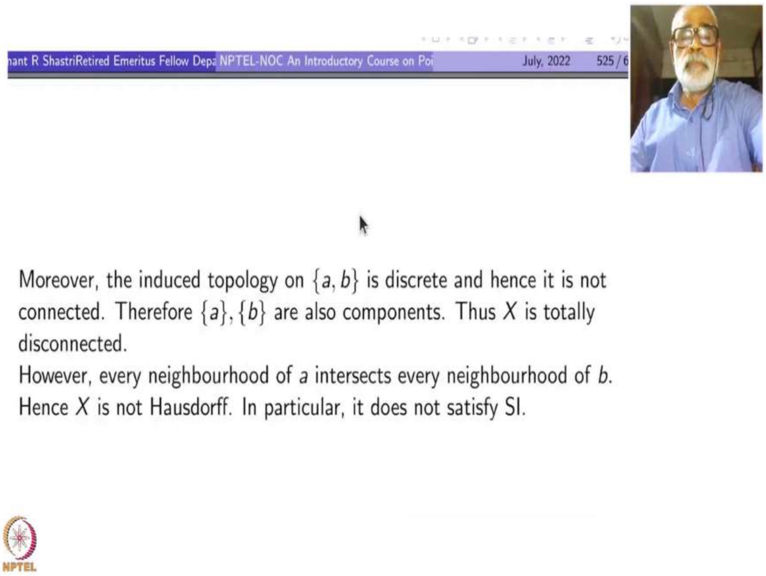

So, I have repeated it here. So, the same thing will tell you that of course, it is not Hausdorff either because neighborhoods have a and b are not disjoint. In particular it does not satisfy....

Oh! sorry, sorry. This space is T_1 but it is not T_2. I want a space which is not even T_1 and this example is not good for that. That is the whole idea. What we have proved is that X is not Hausdorff. Because open subsets around a and open subsets around b they always intersect. If a space is not Hausdorff, it cannot be satisfy SI because SI is stronger than Hausdorffness. So, this is a totally disconnected space which is not SI.

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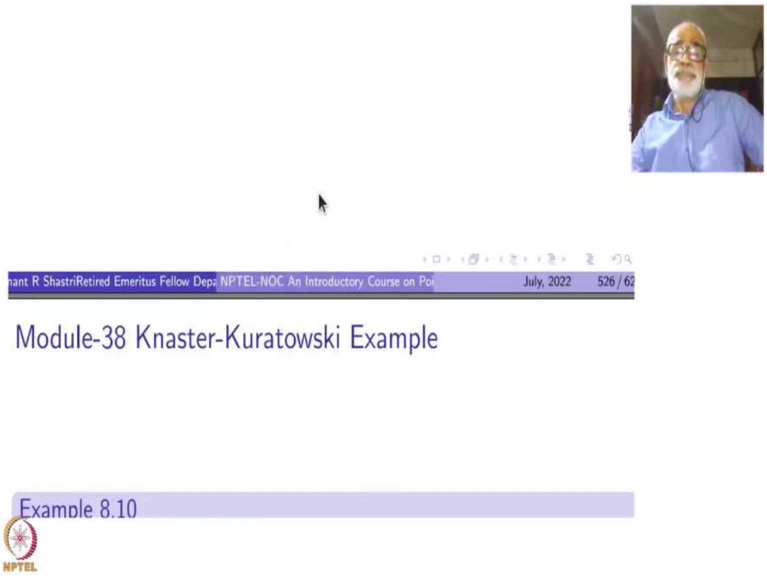

Next time we will have within the metric spaces such examples but for that we have to work very hard, but this is a very well-known example in topology Knaster-Kuratowski example. So, that we will study next time.