An Introduction to Point Set Topology
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Lecture 35
Wallman Compactification – continued

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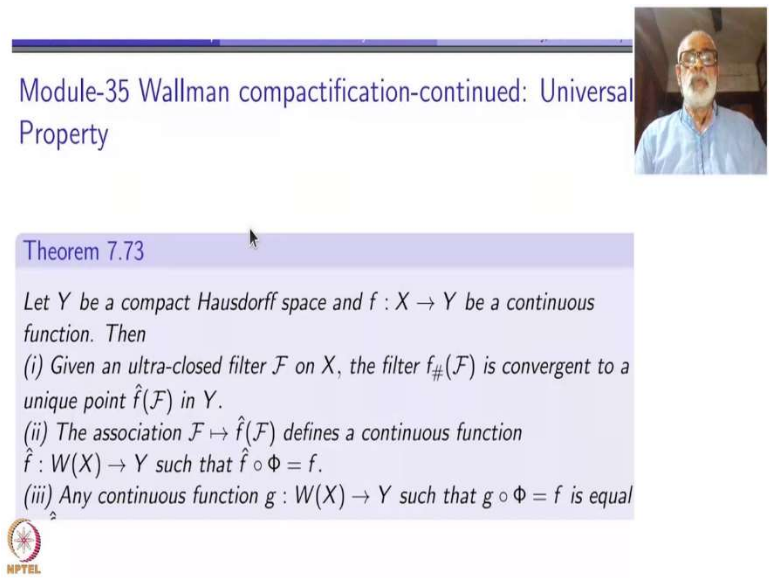

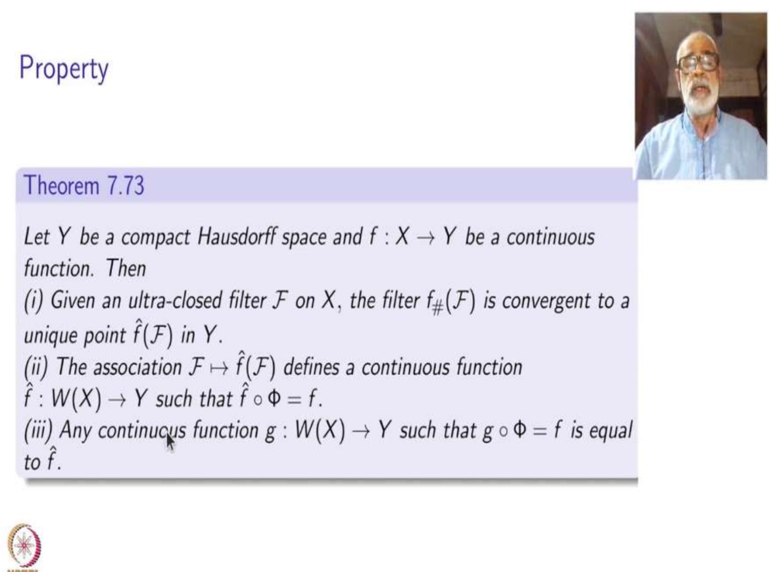

Hello. Welcome to Module 35, Wallman compactification-continued. This time we will study a universal property of Wallman compactification. So, here is a theorem. The statement is quite elaborate here.

Start with a compact Hausdorff space and a continuous function f from X to Y. Then

(i) given an ultra-closed filter $\mathcal{F}$ on X, the image filter $f_{\#}(\mathcal{F})$ is convergent to a unique point $\hat{f}(\mathcal{F})$ in Y.

It is convergent in Y and the point of convergence is unique. (That unique point is denoting by $\hat{f}(\mathcal{F})$. That is the meaning of the conclusion in (i). After all, the point depends upon both the filter $\mathcal{F}$ as well as the function f.)

(ii) The association $\mathcal{F}$ going to this $\hat{f}(\mathcal{F})$ which you have just defined in (i) is a continuous function from the space $W(X)$ of all ultra-closed filters, the Wallman copactiifcation of X to the given compact Hausdorff space Y, and it has the property that $\hat{f} \circ \Phi$ is identically f, the original function f.

(So, think of Φ as an inclusion map of X inside $W(X)$, then this $\hat{f}$ is nothing but the extension of f. That is the way I would like to think of f.)

(iii) The third statement tells you the uniqueness of $\hat{f}$ itself, namely, any continuous function g from $W(X)$ to Y such that $g \circ \Phi$ is f is equal to $\hat{f}$.

So, this is the universal property of the compactification that we are discussing now. So, let us go through the proof of this one.

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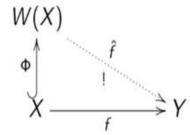

Proof: (i) Suppose $f_{\#}(\mathcal{F})$ is not convergent in Y. It means that for every $y \in Y$ there exists an onbd of U_y of y in Y which does not belong to $f_{\#}(\mathcal{F})$. But then $f(f^{-1}(U_y)) \subset U_y$ is also not in $f(\mathcal{F})$ which means $f^{-1}(U_y)$ is not in $\mathcal{F}$. Since Y is compact, we can choose finitely many y_i such that $Y = \cup_{i=1}^{n} U_{y_i}$ and hence $X = \cup_{i=1}^{n} f^{-1}(U_{y_i}) \in \mathcal{F}$. Since $\mathcal{F}$ is a ultraclosed filter, this contradicts cor 7.65. We conclude that $f_{\#}(\mathcal{F})$ is convergent. The uniqueness follows by the Hausdorffness of Y. This proves (i).

So, this is the picture that I have in mind of what I am going to do. Starting with a continuous function, of course, we have to assume Y is compact Hausdorff, I want to get a function $\hat{f}$ here. And I have already described how this function is got. Take an element here that means an ultra-closed filter on X, the image filter under f, that is the filter on Y, it may not be ultra-closed or anything, but it is convergent because Y is compact.

And that convergence is unique because Y is Hausdorff. So, that is the function $\hat{f}$.

We have to show that this $\hat{f}$ is continuous and this diagram is commutative. Such $\hat{f}$ is unique is the last part. The uniqueness here indicated by the sign !

Now, suppose $f_{\#}(\mathcal{F})$ is not convergent in Y. What does this mean? It means that for every y inside Y, there exists open neighbourhood U_y of y in Y which does not belong to $f_{\#}(\mathcal{F})$.

This is what we have seen earlier. Any filter does not converge means you have these neighborhoods not belonging to the filter. This is what we have used earlier also. But what is the meaning of this? $f^{-1}(U_y)$, U_y is a neighborhood of y in Y, so $f^{-1}(U_y)$ that is an open subset of X and f of that is not in $f(\mathcal{F})$, because U_y is not in F and U_y is a super set of $f(f^{-1}(U_y))$.

But this means that $f^{-1}(U_y)$ is not in $\mathcal{F}$.

Since Y is compact, we can choose finitely many y_i's such that Y is covered by the finitely many open sets U_{y_i}'s. But then what happens, if you take inverse images of this, they will cover X. So, X will be the union of these finitely many open sets. Again, we are using the last thing that we have proved last time and we used it.

Now, we are using it again here. So, what does this mean? This means that one of them is inside $\mathcal{F}$ because $\mathcal{F}$ is an ultra-closed filter on X. And that is a contradiction.

So, we conclude that $f_{\#}(\mathcal{F})$ is convergent. The point is that we are not proving $f_{\#}$ is an ultra-closed. And that may not be true also. I may not be eve a closed filter in general. That is needed. We go back to X and do the work there and conclude that the image filter must be convergent. And the convergence is uniqueness because Y is Hausdorff that is an extra assumption on Y. This proves (i) So, the function $\hat{f}$ is there now.

Automatically, if you start with a point $x \in X$ here and go to $\Phi(x)$, the atomic filter $\mathcal{F}_x$, what is $f_{\#}(\Phi(x))$? It is $f(\mathcal{F}_x)$. It is atomic filter generated by the singleton $f(x)$. And that filter will not converge to any other point than $f(x)$. Therefore, $\hat{f}(\Phi(x))$ must be equal to $f(x)$. So, if you come this way and go by f hat what you get is f. It is the conclusion here. So, that is what I have done here.

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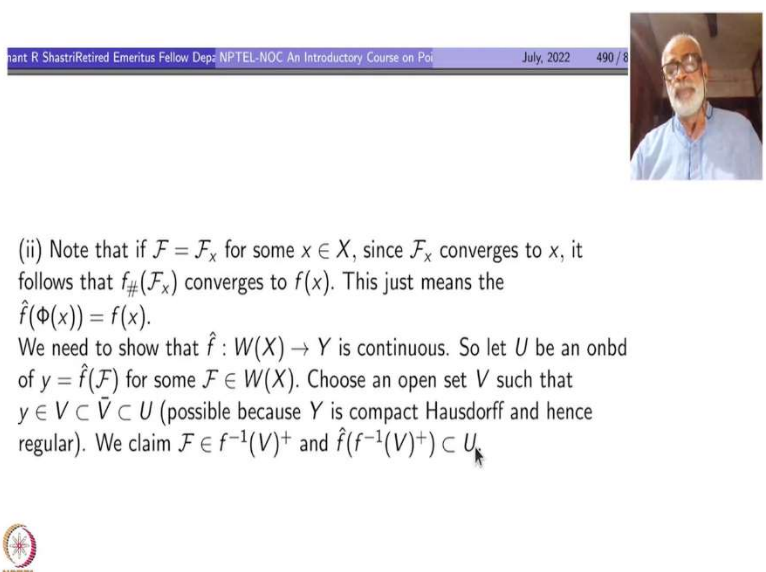

Now, I repeat it. Note that if $\mathcal{F}$ is $\mathcal{F}_x$ for some $x \in X$, (which is the meaning of that this $\mathcal{F}$ is $\Phi(x)$ that is the definition of Φ), $\mathcal{F}_x$ converges to x, it follows that $f_{\#}(\mathcal{F}_x)$ converges to f_x. This just means that $\hat{f}(\Phi(x))$ is $f(x)$. I saw it in a different way, because $f_{\#}(\mathcal{F}_x)$ is nothing but $\mathcal{F}_{f(x)}$, the atomic filter generate to $f(x)$.

We need to show that $\hat{f}$ from $W(X)$ to Y is continuous with the topology here being the Wallman compactification topology, whatever you have defined. So, start with a neighborhood of $y = \hat{f}(\mathcal{F})$. $\hat{f}(\mathcal{F})$ is some point inside Y for some $\mathcal{F}$ inside $W(X)$. So, I must produce a neighborhood of $\mathcal{F}$ such that that neighborhood goes inside U under $\hat{f}$.

Choose an open set V in Y such that y belongs to V contained inside $\bar{V}$ contained inside U. I start with an open subset U which is a neighborhood of y, So, how can you get this one? This is because Y is compact Hausdorff means it is also regular. Compact Hausdorff space is a T_4 space. So, in particular, it is regular also. And hence we get a neighborhood Y of y such that in $\bar{V}$ is contained inside U.

Now, we claim that this $\mathcal{F}$ is inside $f^{-1}(V)^+$. V is open in Y, $f^{-1}(V)$ is open in X, plus of that is an open subset in the Wallman compactification. I want to say that this $\mathcal{F}$ that we started with this filter here belong to this open set. Moreover, under $\hat{f}$ this $f^{-1}(V)^+$ goes inside U. That will prove the continuity of $\hat{f}$. So, once I state it is very clear that this happens, but let us verify this carefully.

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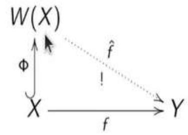
Since $y = \hat{f}(\mathcal{F}) = \lim f_{\#}(\mathcal{F}) \in V$, we have $V \in f_{\#}(\mathcal{F})$. Hence, there exists some $A \in \mathcal{F}$ such that $f(A) \subset V \Longrightarrow A \subset f^{-1}(V)$ and $A \in \mathcal{F} \Longrightarrow f^{-1}(V) \in \mathcal{F} \Longrightarrow \mathcal{F} \in (f^{-1}(V))^{+}$. Now let $\mathcal{F}' \in (f^{-1}(V))^{+}$. Then $f^{-1}(V) \in \mathcal{F}' \Longrightarrow f(f^{-1}(V)) \in f(\mathcal{F}') \subset f_{\#}(\mathcal{F}')$. Since $f(f^{-1}(V)) \subset V \subset \bar{V}$, this implies that $\bar{V} \in f_{\#}(\mathcal{F}')$. Now, from remark 7.40 (3), it follows that $\hat{f}(\mathcal{F}') = \lim f_{\#}(\mathcal{F}') \in \bar{V} \subset U$. This completes the proof of the continuity of $\hat{f}$ and hence that of (ii).
(iii) This is a consequence of the fact $\Phi(X)$ is dense in $W(X)$ and Y is Hausdorff. ♠

function. Then
(i) Given an ultra-closed filter $\mathcal{F}$ on X, the filter $f_{\#}(\mathcal{F})$ is convergent to a unique point $\hat{f}(\mathcal{F})$ in Y.
(ii) The association $\mathcal{F} \mapsto \hat{f}(\mathcal{F})$ defines a continuous function $\hat{f} : W(X) \to Y$ such that $\hat{f} \circ \Phi = f$.
(iii) Any continuous function $g : W(X) \to Y$ such that $g \circ \Phi = f$ is equal to $\hat{f}$.



Proof: (i) Suppose $f_{\#}(\mathcal{F})$ is not convergent in Y. It means that for every $y \in Y$ there exists an onbd of U_y of y in Y which does not belong to $f_{\#}(\mathcal{F})$. But then $f(f^{-1}(U_y)) \subset U_y$ is also not in $f(\mathcal{F})$ which means $f^{-1}(U_y)$ is not in $\mathcal{F}$. Since Y is compact, we can choose finitely many y_i such that $Y = \cup_{i=1}^{n} U_{y_i}$ and hence $X = \cup_{i=1}^{n} f^{-1}(U_{y_i}) \in \mathcal{F}$. Since $\mathcal{F}$ is a ultraclosed filter, this contradicts cor 7.65. We conclude that $f_{\#}(\mathcal{F})$ is convergent. The uniqueness follows by the Hausdorffness of Y. This proves (i).

Since y is $\hat{f}(\mathcal{F})$, that means what, it is the limit of $f_{\#}(\mathcal{F})$ and that limit is inside V. We have this neighborhood V is inside $f_{\#}(\mathcal{F})$, the whole of N_y is inside $f_{\#}(\mathcal{F})$. So, in particular V is here. Hence, there exists some A belonging to $\mathcal{F}$ such that this $f(A)$ must be contained inside V. That is the way $f_{\#}$ is defined. You take all image of members of $\mathcal{F}$ under f and then take all supersets that is the way $f_{\#}$ is defined.

So, there is some member here A inside $\mathcal{F}$, $f(A)$ is contained in V. So, that is the meaning of that V belongs to $f_{\#}(\mathcal{F})$. But this means A is contained in $f^{-1}(V)$ and this A is inside $\mathcal{F}$. That means $f^{-1}(V)$ must be inside $\mathcal{F}$, because $\mathcal{F}$ is a filter. That means this filter is in $f^{-1}(V)^{+}$. One part of the claim is over.

Now, I have to show that this $f^{-1}(V)^{+}$ is mapped inside U by $\hat{f}$. What does that mean? Take any ultra-closed filter belonging to the set $f^{-1}(V)^{+}$, look at the unique point to which it converges that convergent point must be inside U. that is the meaning of $\hat{f}$ of this set is contained in U.

So, start with $\mathcal{F}'$ inside $f^{-1}(V)^{+}$, which is just means $f^{-1}(V)$ belongs to $\mathcal{F}'$. What does this mean?

f of this one will be inside $f(\mathcal{F}')$. $f(\mathcal{F}')$ is contained in $f_{\#}(\mathcal{F}')$. This is a set here, but that is contained in each member here. So, in particular, this $f(f^{-1}(V))$ is a member of $f_{\#}(\mathcal{F}')$. Since $f(f^{-1}(V))$ is contained in V contained inside $\bar{V}$, $\bar{V}$ must be inside this one $f_{\#}(\mathcal{F}')$.

From the remark 7.43 which we have used a several times, it follows that the limit $\hat{f}(\mathcal{F}')$, that must be inside $\bar{V}$ and that $\bar{V}$ is inside U. This completes the proof of the continuity of $\hat{f}$ and hence that of statement (ii).

The statement (ii) says the association defines a continuous function and the commutativity of the diagram is already seen very easily.

Now, let us prove (iii). This is a consequence of the fact that $\Phi(x)$ is dense in $W(X)$ and Y is Hausdorff. Remember that if you have a Hausdorff space Y, suppose you have two continuous functions from one space to a Hausdorff space Y, then the set of points wherein the two functions agree that is a closed subset.

Now, apply that to this situation. You have two functions $W(X)$ to Y. What are they? They are both extensions of the same f or you may say that $\hat{f} \circ \Phi$ is f is equal to some $g \circ \hat{f}$. What does that mean here? If you think of $\Phi(X)$ as a subset of $W(X)$ both of them agree on $\Phi(X)$, but $\Phi(X)$ is dense in $W(X)$. If they agree on $\Phi(X)$, they must agree on $\overline{\Phi(X)}$, because $\overline{\Phi(X)}$ is the smallest closed set containing $\Phi(X)$. A closed set containing $\Phi(X)$ must contain $\overline{\Phi(X)}$. That $\overline{\Phi(X)}$ is the whole space. Those are the two functions agree on the whole space.

So, this is the meaning of universal property of $W(X)$. It would have been fantastic if we could prove such a thing without Y being Hausdorff, namely all T_1 spaces. Unfortunately we are not able to do that. That is one of the drawbacks of Wallman compactification I would say.

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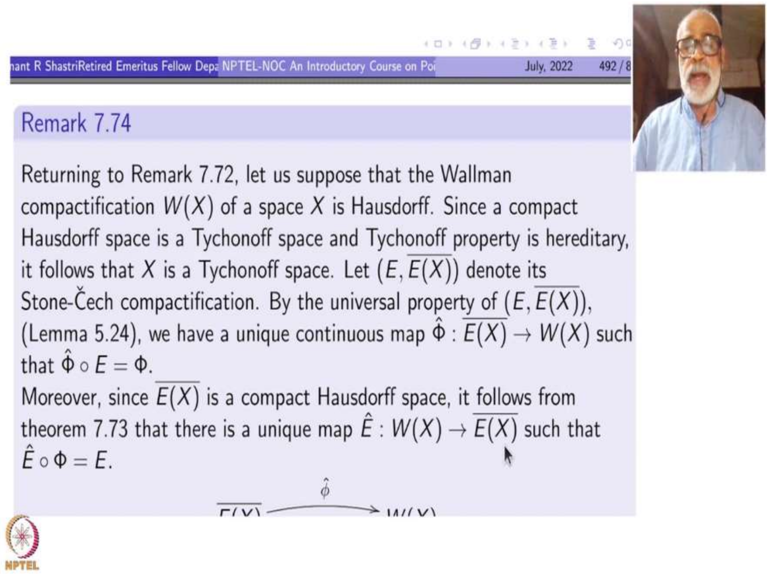

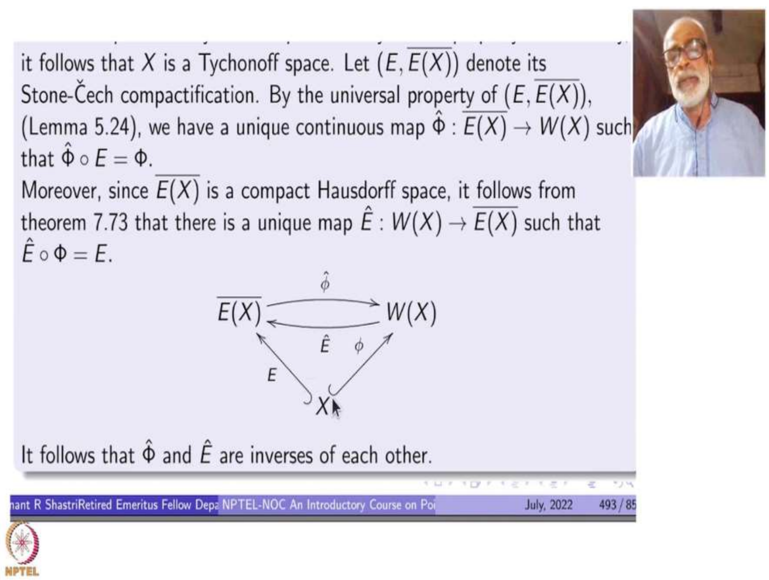

Returning to our remark 7.72, let us suppose that the Wallman compactification $W(X)$ of the space is Hausdorff. Take a special case, namely $W(X)$ is Hausdorff. Since a compact Hausdorff space is a Tychonoff's space and Tychonoff properties is hereditary, it follows that X itself is a Tychonoff space.

So, I said, it would have been wonderful and all that, but the very fact that the Wallman compactification is meant to get a compactification of a larger class of spaces, if you want it to be Hausdorff, you are forced to do it only for Tychonoff spaces. So, this is an inherent restriction. If you want to think of this as a weakness, then it is inherent in what you are trying to do. There is no other way. You cannot expect $W(X)$ to be Hausdorff all the time. That is not possible. That was not the intention of Wallman at all. So, that is the point of making this remark. So, we have to accept it.

Having done that, let us see what we can do when it is Hausdorff. So, X is already Tychonoff space, because $W(X)$ is Hausdorff.

The moment it is a Tychonoff's space, we can also get its Stone-Cech compactification $(E, \overline{E(X)})$. So this makes sense. So, what is the difference between these two compactifications is the natural question that we want to investigate now. So, this is answered by the two universal properties of both these compactification. So, let $(E, \overline{E(X)})$ denote the Stone-Cech compactification of X.

By the universal property of (($E, \overline{E(X)}$), we have a unique continuous function $\hat{\phi}$ from $\overline{E(X)}$ to $W(X)$ such that $\hat{\phi} \circ E$ is equal to ϕ. This is similar to what we have done for Wallman compactification just now. Moreover, since $\overline{E(X)}$ is a compact Hausdorff space, we can apply the previous theorem, just now that we have proved, to get a function $\hat{E}$ from $W(X)$ to $\overline{E(X)}$ such that $\hat{E} \circ \phi$ is equal to E.

So, we did the for any continuous function f from X to Y and then we called it $\hat{f}$. Now here f is E. So, I got $\hat{E}$ here. So, what we have got is here X is your space which is a Tychonoff space, it is sitting inside its Stone-Cech compactification here and Wallman compactification there. So, this is E is the Stone-Cech embedding here and this phi is the Wallman embedding, X going to the atomic filter $\mathcal{F}_x$. So, what happens, this function gets extended to $\hat{E}$.

See from here E is like this, so it gets extended to $\hat{E}$ like this. And this function gets extended to phi hat here. Now, what happens? $\hat{\phi} \circ \hat{E}$, that will be an extension of E inside $\overline{E(X)}$ itself, but the universal property of $\overline{E(X)}$ says that there cannot be two different extensions. Identity function from $\overline{E(X)}$ to $\overline{E(X)}$ is already an extension of E obviously, E composite identity is E.

So, there will be two of them, namely go by $\hat{\phi}$ and come back by $\hat{E}$ and the other one is the identity. So, there cannot be two. That means that this composite is the identity of $\overline{E(X)}$. Exactly same way $\hat{E}$ composite $\hat{\phi}$ will be identity of $W(X)$. What is the meaning of this? That these are homeomorphisms being inverses if each other. Not only that they are homeomorphisms, they are commuting with these embeddings.

The embedded object X goes to the embedded object. The homeomorphism, if you think of X as subspace of both of them, the homeomorphisms can be thought of as identity on the subspace. So, it is in this strong sense that we say that the Stone-Cech compactification and the Wallman compactification are the same for a Tychonoff space, whenever its Wallman compactification is Hausdorff. We have to assume that. That is the conclusion.

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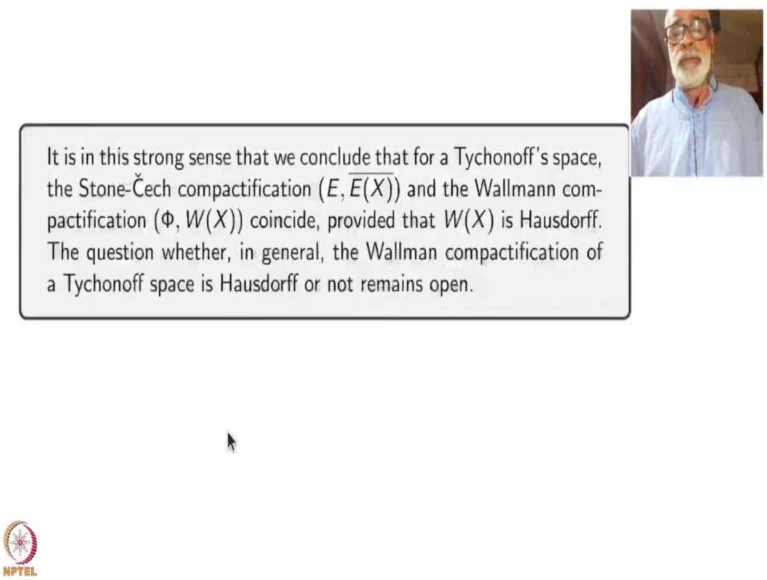

The general question is when is Wallman compactification is Hausdorff. Suppose we assume it is Tychonoff space, suppose you grant that, will you immediately say that $W(X)$ is Hausdorff. That is not a correct answer. You may need some more conditions. One does not know that. So, we will stop here. This is a good point to stop this study here. We cannot go on doing that. So, next time we will start a new topic. Thank you.