An Introduction to Point Set Topology
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Lecture 34
Wallman Compactification

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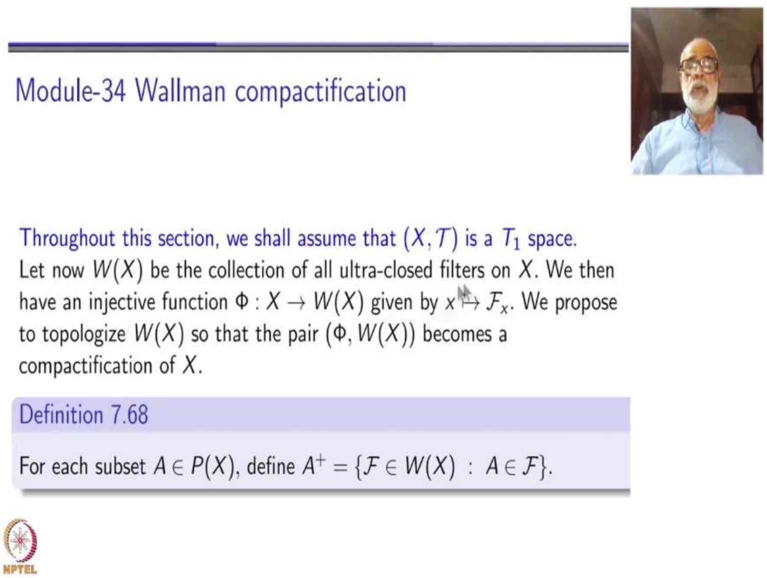

Hello. Welcome to NPTEL-NOC, an Introductory Course on Point-Set-Topology Part II. So, today we will study Wallman compactification, Module 34. As motivated last time, we shall make a blanket assumption that the space $(X, \mathcal{T})$ is a T_1 space. As before, let $W(X)$ denote the collection of all ultra-closed filters on X. We also have introduced this function phi from X to $W(X)$, namely, x going to the atomic filter $\mathcal{F}_x$ which we know is an ultra-closed filter, because X is a T_1 space.

We propose to topologize $W(X)$ so that the pair $(\phi, W(X))$ becomes a compactification of X. Recall, the compactification is actually an equivalence class, but we keep on saying that you pick up any representative that is also compactification. And the compactification satisfies that ϕ from X to whatever space we are taking is an embedding such that the image of that embedding is dense in the whole space $W(X)$, which is compact. So, these are the conditions for compactification. I am just recalling that is all.

So, going towards topologizing $W(X)$, let us introduce some notation here. For each subset A of X, define A^+ to be the set of all ultra-closed filters in X in which A is also a member. A must be an element of $\mathcal{F}$ for each $\mathcal{F}$ in A^+.

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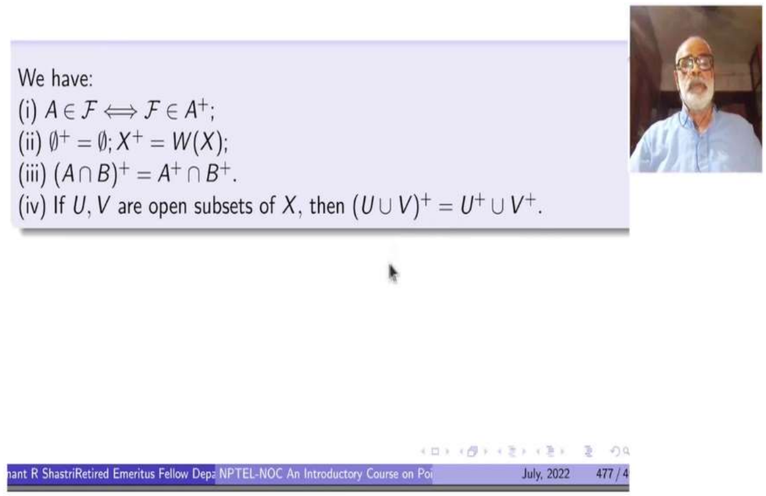

The following lemma on this notation is very simple, but very useful also for us.

(i) A is in $\mathcal{F}$ if and only if $\mathcal{F}$ is in A^+. So, this is the definition of A^+.

(ii) Empty set plus is empty. (Because no filter contains empty set.) And X^+ is the whole space $W(X)$. (Because all ultra-closed filters in particular, all filters contain X. So, these are straightforward.)

(iii) $(A \cap B)^+$ is equal to $A^+ \cap B^+$. So, this needs a little bit of explanation.

Take an ultra filter to which $A \cap B$ belongs. Then just by being a filter, it will contain both A as well as B. Therefore, that ultra-closed filter belongs to A^+ as well as B^+. Conversely, if the ultra-closed filter contains A as well as B then being a filter it contains the intersection also. Therefore, that filter belongs to $(A \cap B)^+$. So, there is no ultra-closed filter involved here; this is true for all filters.

But here when you come to open sets, we need to work with ultra-closedness itself. Namely,

(iv) If U and V are open subsets of X, then $(U \cup V)^+$ is $U^+ \cup V^+$. So, once again if $\mathcal{F}$ contains this element $U \cup V$, then we know that one of U or V must be inside $\mathcal{F}$ which is same to be saying $\mathcal{F}$ belongs to U^+ or V^+.

The converse part is easy. Once $\mathcal{F}$ contains U, it will contain $U \cup V$ which is a superset. Similarly, if $\mathcal{F}$ contains V its superset $U^+ \cup V^+$ will be there. So, this way it is easier.

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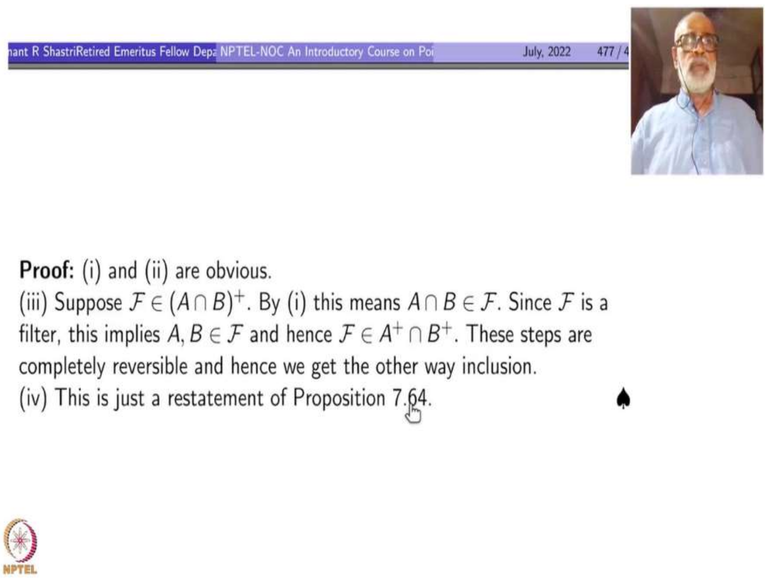

So, this is what I have written down here. So, let us go through. One and two are obvious. Just set theoretic definition. The third one is what I have seen, I will repeat it. $\mathcal{F}$ belongs to $(A \cap B)^+$. By one this means that $A \cap B$ is in $\mathcal{F}$. That is definition anyway. Since $\mathcal{F}$ is a filter, this implies A and B are inside $\mathcal{F}$ because they are supersets. Hence, $\mathcal{F}$ is inside $(A \cap B)^+$.

So, these steps are completely reversible because if A and B are in $\mathcal{F}$, their intersection is there. So, the fourth statement, as I told you, it is a consequence of that proposition which gives you that if U and V are open subsets and $U \cup V$ is there then one of them U or V must be there. So, this was proposition in 7.64 which was proved separately for ultra-closed filters.

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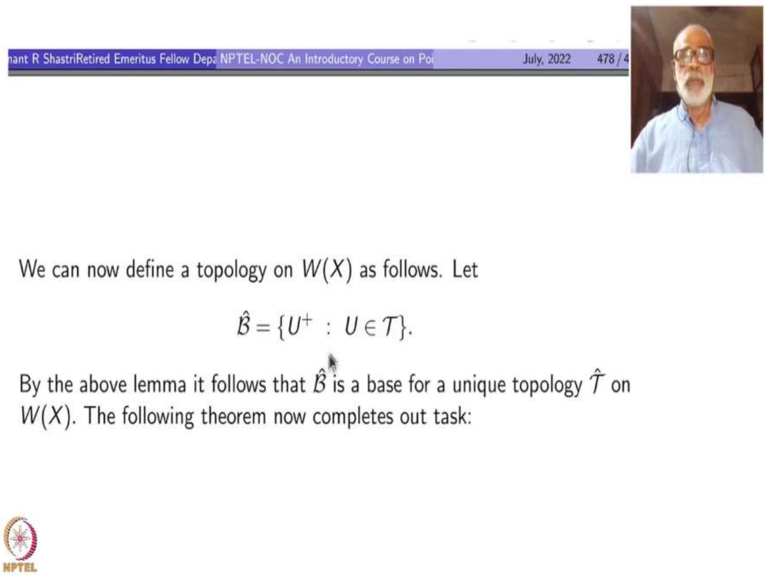

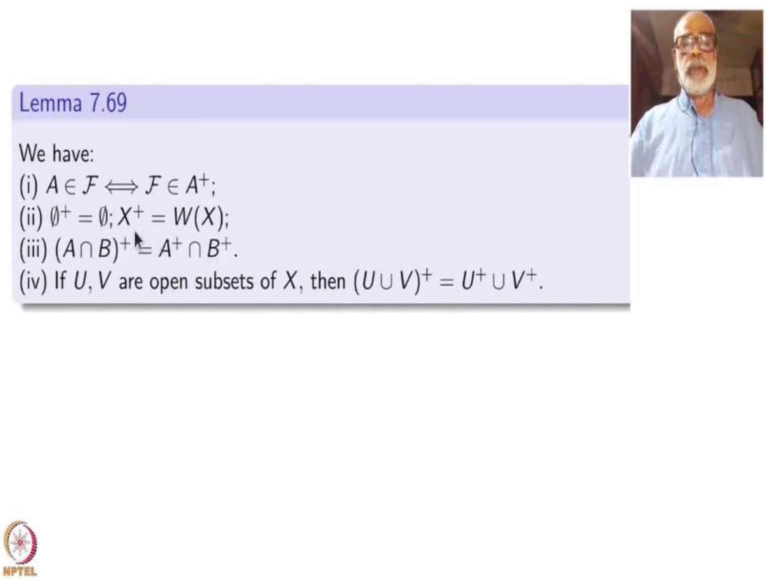

So, property (iii) especially, just means that the family of all these A^+'s, where A is an open subset of the topology on X, is closed under finite intersection. So, that is the important property for us. So, look at the family $\hat{\mathcal{B}}$ which is the collection of all U^+, where U is ranges over $\mathcal{T}$, now. (I am not taking arbitrary subsets. I am taking only open subsets in X.) Look at this collection. This collection is closed under finite intersection.

Therefore, it forms a base for a unique topology $\widehat{\mathcal{T}}$ on $W(X)$. Namely, all that I have to do is to take the collection of all arbitrary union of members of $\hat{\mathcal{B}}$, that will be the topology $\widehat{\mathcal{T}}$ on $W(X)$.

So, there are so many ways of giving a topology. We have to justify that this topology is the right one among all such. So that is the task now.

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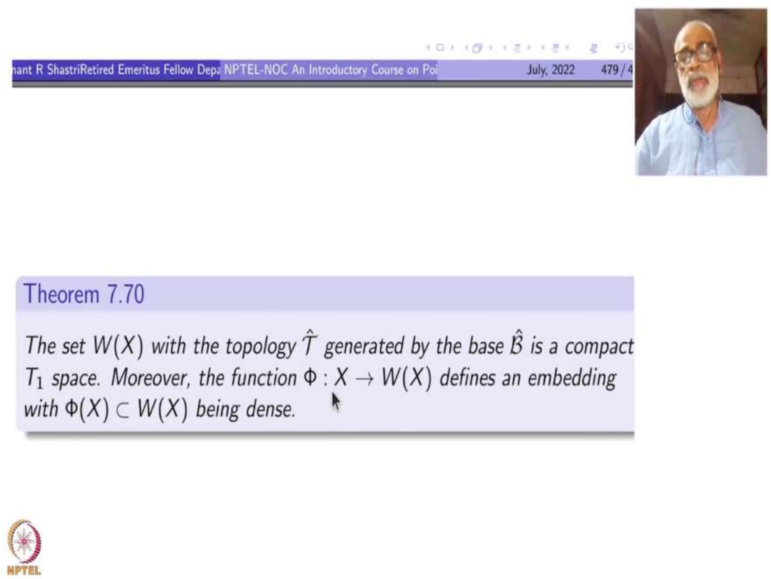

The set $W(X)$ with this topology $\widehat{\mathcal{T}}$ generated by the base $\hat{\mathcal{B}}$ is a compact T_1-space.

(We are not so much worried about T_1-ness, but compactness is the first thing we need.) Moreover, the function Φ from X to $W(X)$, (which we know is a set theoretic injection already) defines an embedding of X in $W(X)$. And the image $\Phi(X)$ is dense in $W(X)$.

So that will complete the statement that $(\Phi, W(X))$ is a compactification of X. Actually, this theorem says it is a T_1 compactification.

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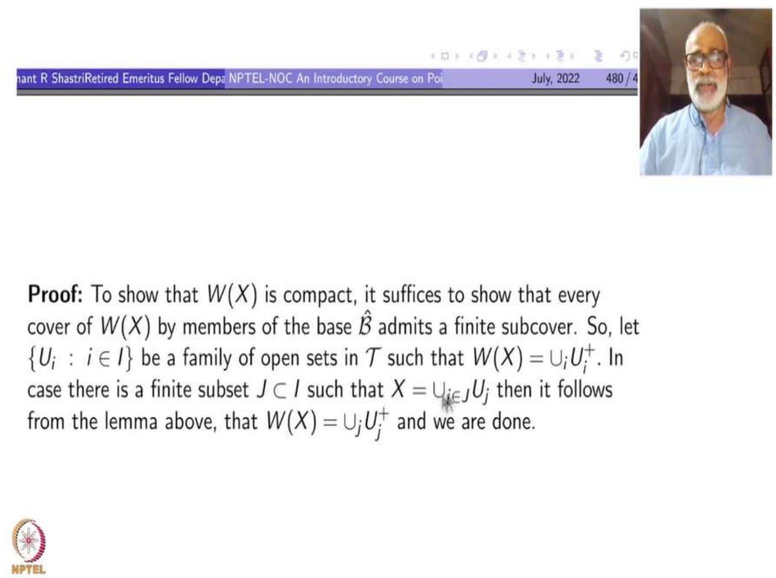

So, the first thing is to see that $W(X)$ is compact. For any space to be compact, it is enough to prove that every open cover from a given base, (you fix one base for topology, take any open cover from that base, members of that base) admits a finite sub-cover. This is a standard result that we are going to use now.

So, we are fixing this base, $\hat{\mathcal{B}}$. Take a subfamily of $\hat{\mathcal{B}}$ which covers $W(X)$ and let us see that it admits a finite sub-cover. So, let $\{U_i : i \in I\}$ be a family of open subsets in X, such that $W(X)$ is union of all U_i^+.

I am making a sub-case here. In case there is a finite subset J of I such that X itself is contained inside the corresponding U_j's, then it follows, by property (iv) and (ii), that $W(X)$ is equal to X^+ which is equal to the finite union of U_j^+'s. We have proved it for two at a time in the statement (iv) but that can be immediately generalized to any finite number of open sets. That is what it is. You take X^+, X^+ is $W(X)$. So, that will be equal to union U_j^+. Now, this is a finite cover. So, we are done.

The second case is, suppose the above is not the case. What is the meaning of that? There is no finite subfamily of $\{U_i\}$ which will cover X. Remember, the U_i^+'s are covering $W(X)$. Now, I am talking about U_i's being covering X. Why should they cover X and even if they do why should there be a finite subcover? There is no need. We are not assuming X is compact after all. So, suppose there is no such finite sub-cover for X. So, that case remains. So, that is the case we are going to address in a standard way. This kind of argument is used several times in topology especially when dealing with compact spaces.

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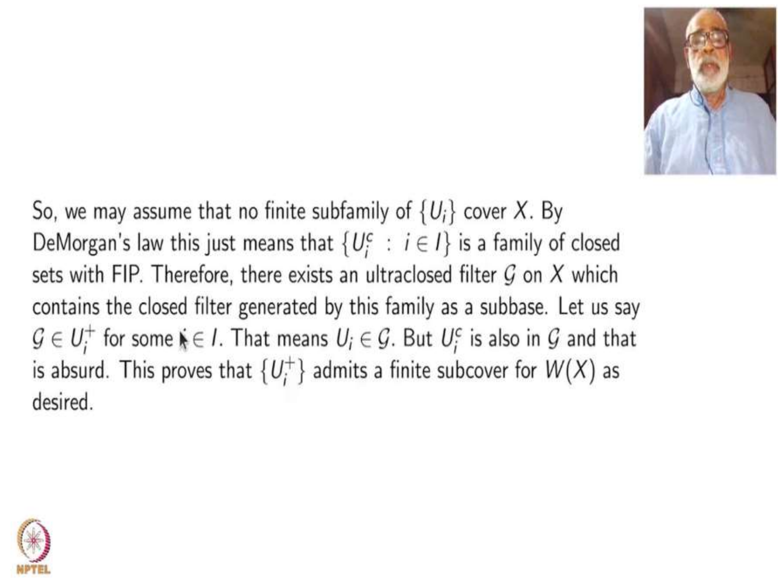

So, we are now assuming that no finite sub-family of $\{U_i\}$ covers X. By DeMorgan law, this just means that the complements of members of this family is a family of closed sets with finite intersection property. No finite intersection of these complements can be empty. So, it has finite intersection property.

Therefore, there exists an ultra-closed filter $\mathcal{G}$ on X which contains the closed filter generated by this family as a sub-base. Any subfamily of $P(X)$ which has finite intersection property will generate a filter on X. In this case, it is automatically a closed filter because these are closed subsets. Every closed filter is contained in an ultra-closed filter and that is what I am denoting by $\mathcal{G}$.

Let us say that this $\mathcal{G}$ after all is a member of $W(X)$, so it must be inside U_i^+ for some $i \in I$, because U_i^+ cover $W(X)$. What does this mean? By the very definition of U_i^+, this means this U_i is inside $\mathcal{G}$. Remember $\mathcal{G}$ is a larger filter which contains the filter generated by the family of finite intersections of U_j^c. Therefore, all the U_j^c are inside $\mathcal{G}$. That is absurd, because in a filter you cannot have both a set and its complement, U_i and U_i^c. So, compactness of $W(X)$ is proved.

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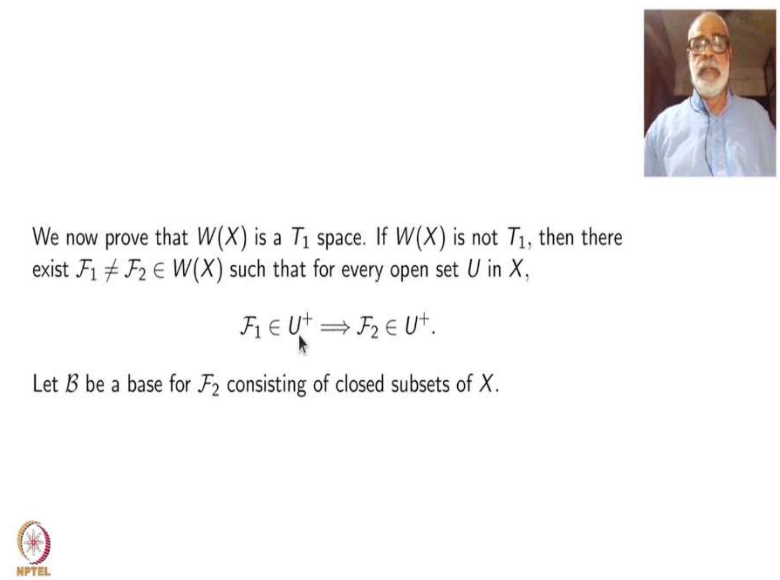

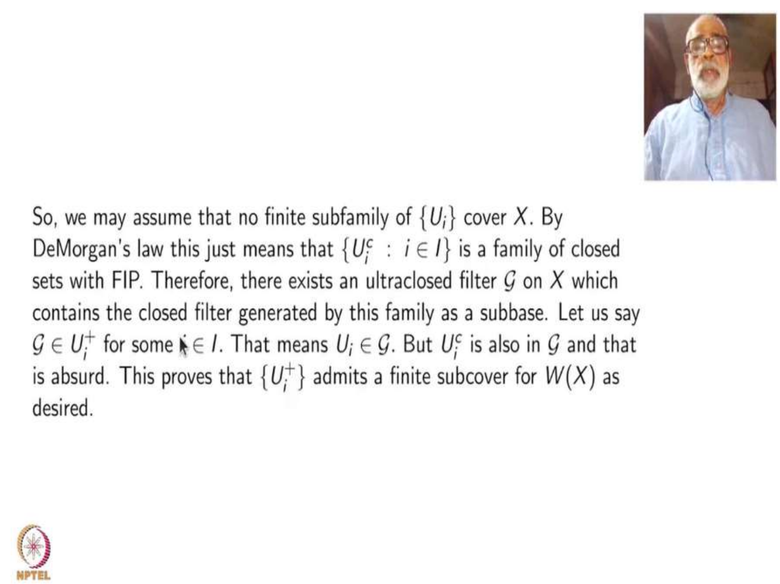

Let us now prove that $W(X)$ is a T_1 space. That comes very easily right now. If it is not a T_1 space, what does that mean? There exist a pair of points inside $W(X)$, they are now ultra-closed filters, distinct points means $\mathcal{F}_1$ is not equal to $\mathcal{F}_2$, such that for every open set U in X, if $\mathcal{F}_1$ is in U^+ then $\mathcal{F}_2$ is also in U^+. (As such I need to say consider open subsets of $W(X)$, but I can restrict myself to basic open sets and they will look like U^+).

Whenever you take a neighborhood of $\mathcal{F}_1$, it will be a neighborhood of $\mathcal{F}_2$ also. So, there will be such a pair. Starting with arbitrary pair $(\mathcal{F}_1, \mathcal{F}_2)$ it may not happen, just because it is not a T_1 space. But for some pair it is happening. That is the whole idea.

So, $\mathcal{F}_1$ is in U^+ implies $\mathcal{F}_2$ is in U^+. Now, you take $\mathcal{B}$ as a base for $\mathcal{F}_2$ consisting of closed subsets of X, possible because, by definition, every closed filter must have a base consisting of closed subsets.

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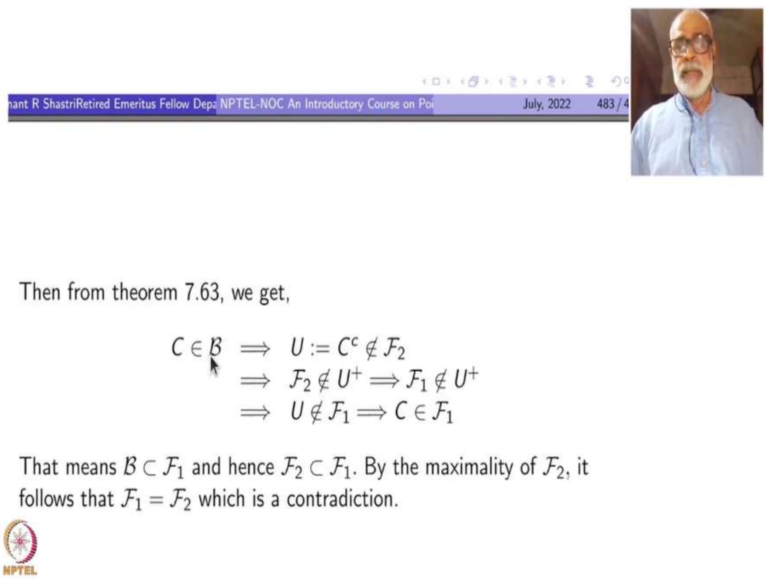

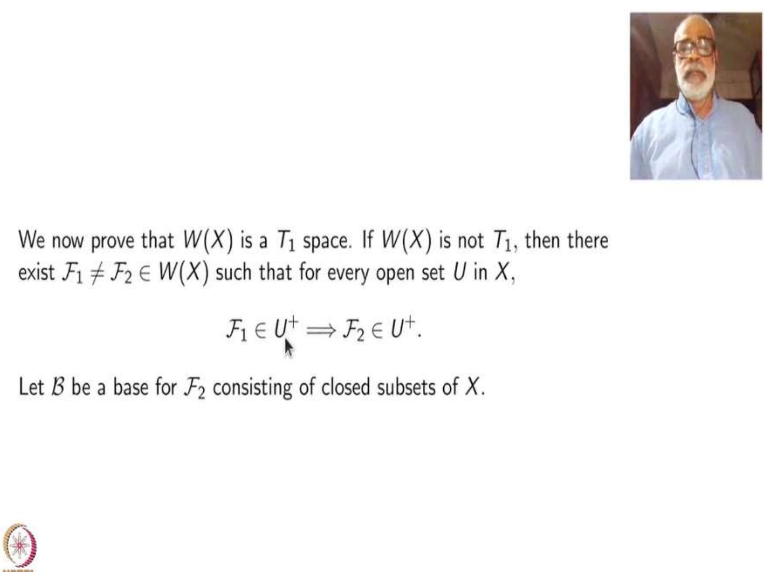

From an earlier theorem, if we take a member C of this base $\mathcal{B}$, then the complement which is an open subset, cannot be in $\mathcal{F}_2$ because the member C is in $\mathcal{F}_2$. So, $\mathcal{F}_2$ is not in U^+ is the same thing as saying that U is not in $\mathcal{F}_2$. If $\mathcal{F}_2$ is not in U^+ by this choice $\mathcal{F}_1$ cannot be in U^+ and $\mathcal{F}_1$ is not in U^+ just means that U is not in $\mathcal{F}_1$.

If U is not in $\mathcal{F}_1$, this is where you have to use this criterion, given any open set U, U or U^c must be inside any ultra-closed filter. So, U is not in $\mathcal{F}_1$ implies U^c which is C is inside $\mathcal{F}_1$. So, what we have proved here? We have proved that this base $\mathcal{B}$ for $\mathcal{F}_2$ is completely contained inside $\mathcal{F}_1$. Therefore, the filter generated by $\mathcal{B}$ viz., $\mathcal{F}_2$ will have to be contained inside $\mathcal{F}_1$, because $\mathcal{F}_1$ is a filter.

We started with two ultra-closed filters. So, one cannot be contained inside other or they are equal. So, whichever way, you have a contradiction, because you have started with the assumption that tau hat is not T_1.

Watch out these arguments carefully. If you are just doing these things with just ultra-filters, could you complete this? Can you get a contradiction. You keep examining these kinds of things, to understand really the role of ultra-closed filters.

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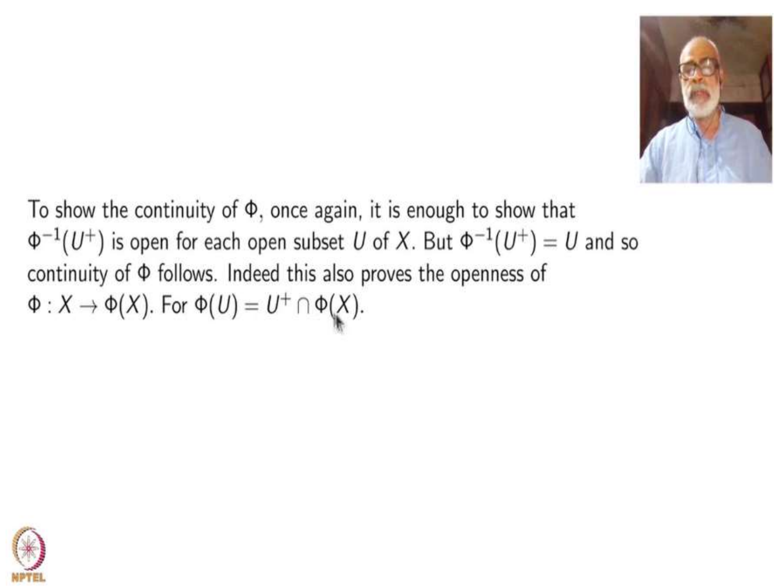

To show the continuity of Φ is the next task and then there will be one more, namely, to show that Φ is a closed map or an open map onto its image. That will complete the embedding. And finally, we have show that the image is dense also.

So, to show that Φ is continuous, it is enough to show that inverse image of basic open sets is open. So, if you take a basic open set, it will look like U^+, where U is an open subset of X.

What is $\Phi^{-1}(U^+)$? Remember, what is Φ? Φ take x to $\mathcal{F}_x$, the atomic filter. So, start with an atomic filter, it comes from some point in X, because there is one, one mapping. So, $\Phi^{-1}(U^+)$ will consist of all those points $x \in X$ such that $\mathcal{F}_x$ belong to U^+ which is the same as saying U belongs to $\mathcal{F}_x$ which is, in turn the same as saying x belongs to U.

So, $\Phi^{-1}(U^+)$ is just equal the original set U. So, U is open. So, continuity of Φ follows.

Not only that, you observe that this also means that $U^+ \cap \Phi(X)$ is precisely $\Phi(U)$ because inside $\Phi(X)$ we only have the atomic filters, the atomic filter containing the whole of U just means that the point x is inside U. So, it is $\Phi(U)$. $\Phi(U)$ is equal to $U^+ \cap \Phi(X)$ is stronger than just saying Φ is continuous. It implies that Φ is an open mapping onto its image and hence is an embedding. (I can use this equation again.)

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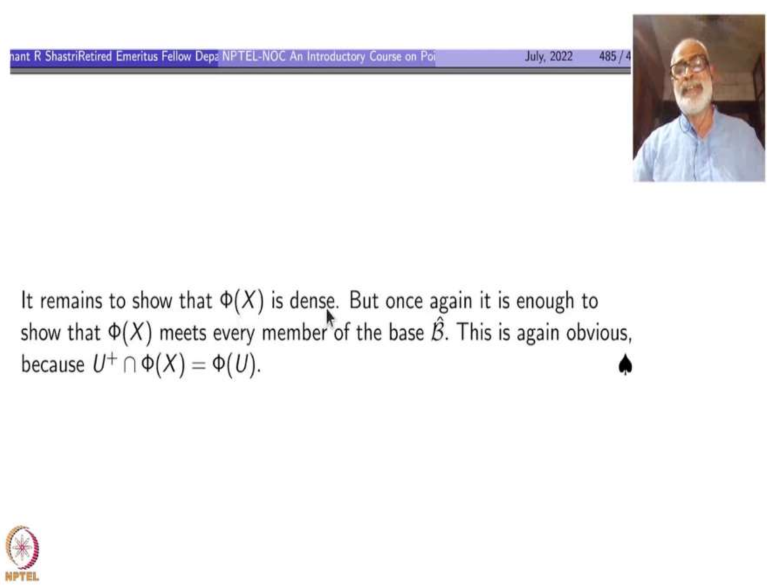

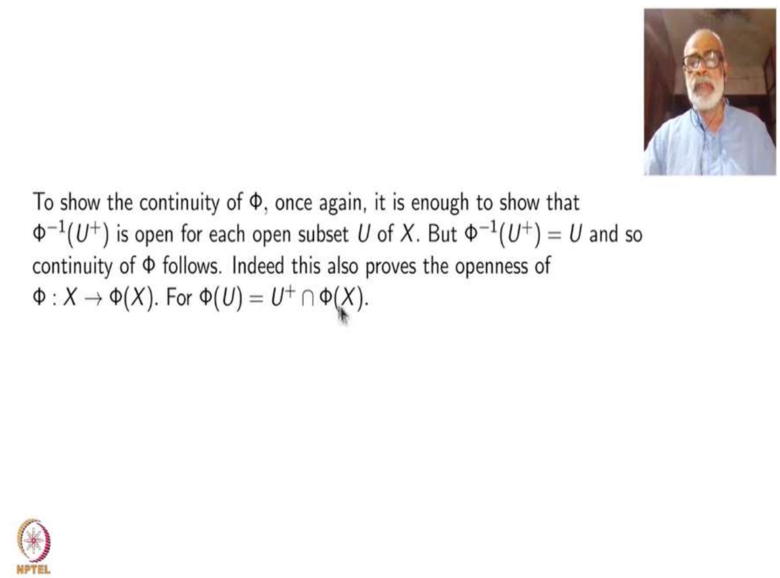

It remains to show that $\Phi(X)$ is dense, but once again it is enough to show that $\Phi(X)$ meets every member of the base $\hat{\mathcal{B}}$. $\Phi(X)$ is dense means it should meet every non-empty open set. But inside a non-empty open set, there will be some member of $\hat{\mathcal{B}}$, the base. This follows from the above equation again.

Take any basic open set U^+. $\Phi(X) \cap U^+$ is $\Phi(U)$, so it is non-empty, over. So, you see that just this one equation proves density, continuity as well as openness.

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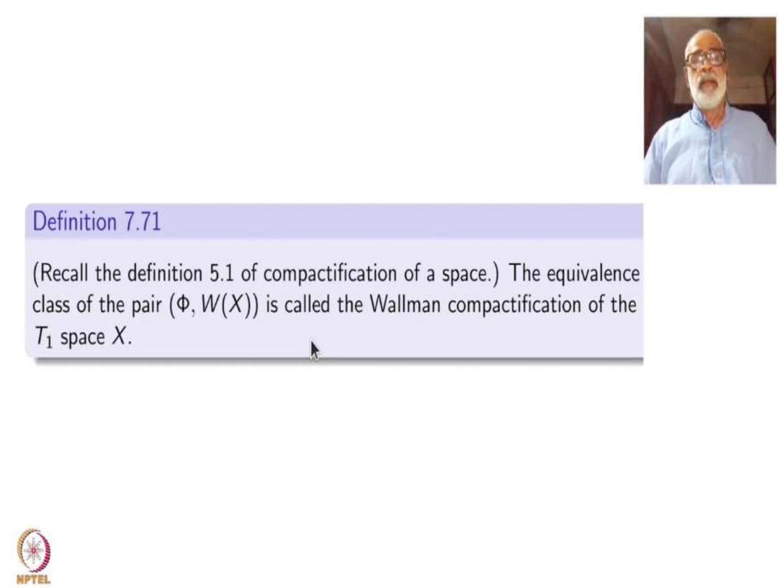

So, let us complete now the definition of Wallman compactification. As I told you earlier, a compactification is a certain equivalence class. But here we are usually taking one particular representative all the time. Anything which is equivalent to this one will be also called Wallman compactification, in general.

So, here we are, first of all, we are starting with any T_1 space X, we construct this $W(X)$ which is just the collection of all ultra-closed filters on X with the very specific topology with the base consisting of U^+'s, where U is oan pen subset of X. So, that compact space along with the embedding phi of X inside that, this is called Wallman compactification of the T_1 space X. You can remember that $W(X)$ is automatically T_1.

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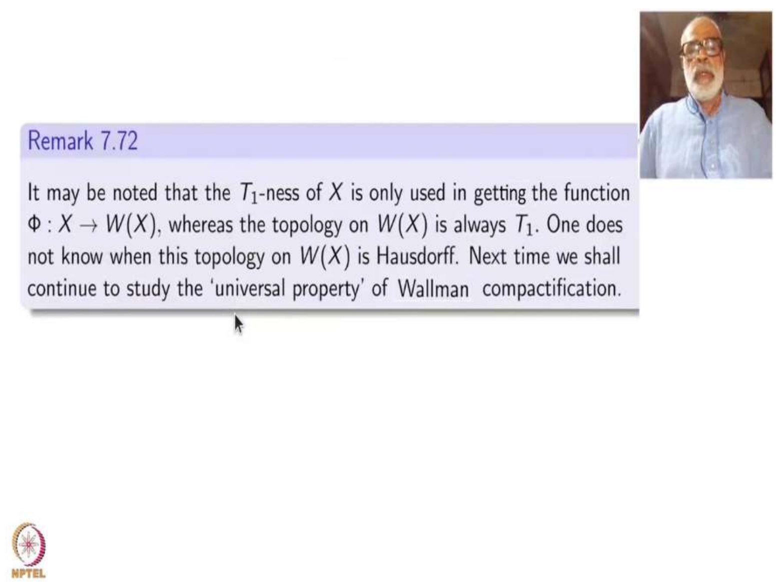

So, here are a few remarks. The T_1-ness of X is only used in getting the function Φ from X to $W(X)$. Why, because I start with X and then I construct $\mathcal{F}_x$. But why should $\mathcal{F}_x$ be an ultra-closed filter? It is an ultra-filter. Why it is closed? That is where I use the T_1-ness of X. So, singleton $\{x\}$ is closed. So, singleton $\{x\}$is a closed base for the whole of $\mathcal{F}_x$. Any atomic filter has singleton set as a base. So, that is one immediate way to get an embedding of X into $W(X)$ that we have used.

Automatically Φ is injective. There is no problem. Whereas, the topology on $W(X)$ is always T_1. The T_1-ness of X is not used in this one. One does not know when this topology on $W(X)$

becomes Hausdorff. Why I am making this comment? Usually, one would like to know, because we are in some sense obsessed with Hausdorffness. So, would $W(X)$ be Hausdorff under some suitable nice condition on the topology of X? If it is freely T_1 by putting a little extra condition maybe you will get this to be Hausdorff.

But here I am saying that one does not know when this topology on $W(X)$ is Hausdorff. So, next time, we shall continue to study the universal property of Wallman compactification. Remember, the universal property of Stone-Cech compactification and that is a universal property or even some kind of universal property of one point compactification also. This will be somewhat similar to that. Of course, in each case, it will be slightly different. Thank you.