An Introduction to Point Set Topology
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Lecture 32
Ultra-filters and Tychonoff's theorem

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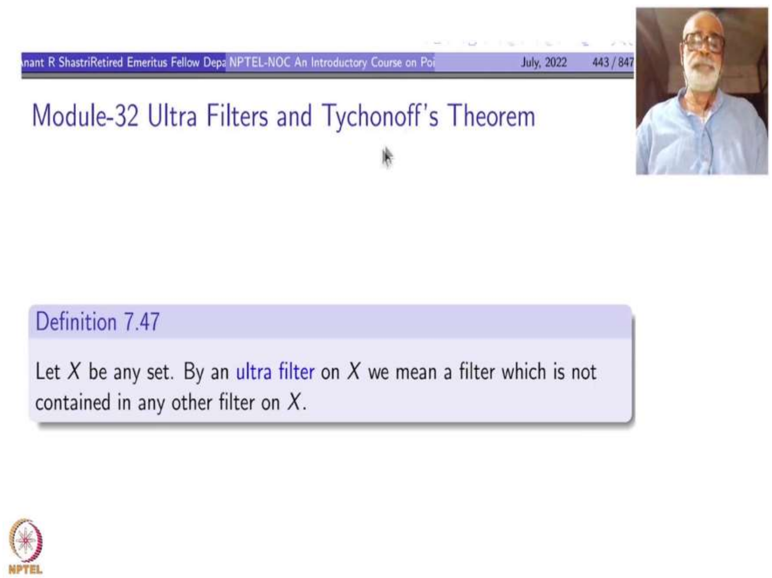

Hello. Welcome to NPTEL-NOC An Introductory Course on Point Set Topology. Today, we will continue with our study of filters, especially ultra-filters today in Module-32. As an easy consequence, today we are going to derive another proof of Tychonoff's theorem.

Start with any set X. By an ultra-filter on X, we mean a filter which is not contained in any other filter. In other words, in the family of all filters partially ordered by the standard inclusion, ultra-filters are maximal elements.

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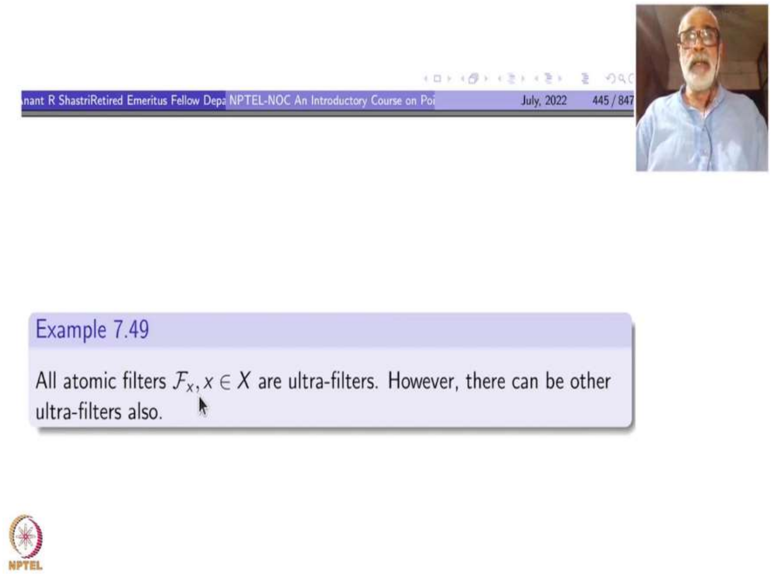

All atomic filters $\mathcal{F}_x$, for $x \in X$, are ultra-filters. However, there can be other ultra-filters also, you never know. Remember, this $\mathcal{F}_x$ means all subsets of X which contain the x. If we have one more member there, that means that member does not contain x, and then singleton x and that member would have intersection empty. That is not allowed. Therefore, $\mathcal{F}_x$ is maximal. That is the idea.

So, these atomic filters do play an important role.

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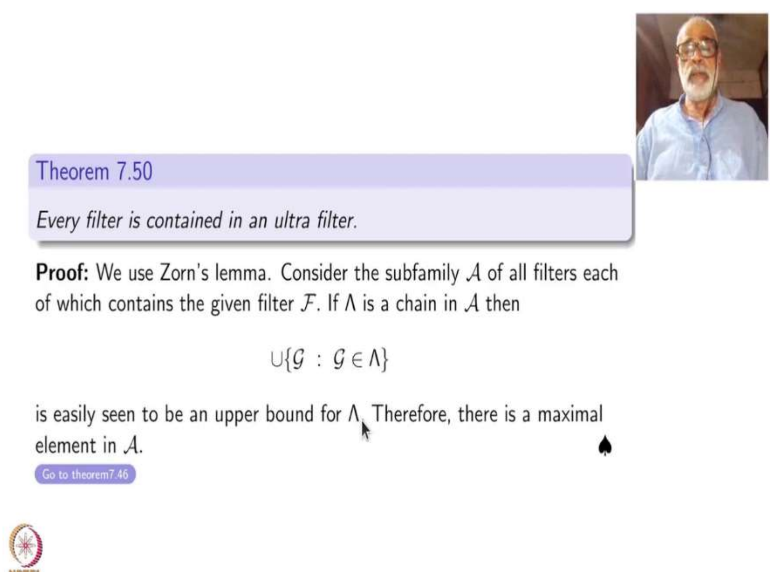

Every filter is contained in an ultra-filter. So, this kind of result we are now very familiar with follows from using Zorn's lemma. What we do? Start with a filter $\mathcal{F}$, look at the family $\mathcal{A}$ of all filters on X which contain the given filter $\mathcal{F}$. This family is partially ordered as usual, by inclusion. If Λ is a chain in $\mathcal{A}$, then the standard arguments will give you that union of all members of this chain is again a filter which will contain all of members of the chain. Therefore, that will be an upper bound for this chain. Now, you can apply Zorn's lemma, to conclude that there must be a maximal element in $\mathcal{A}$.

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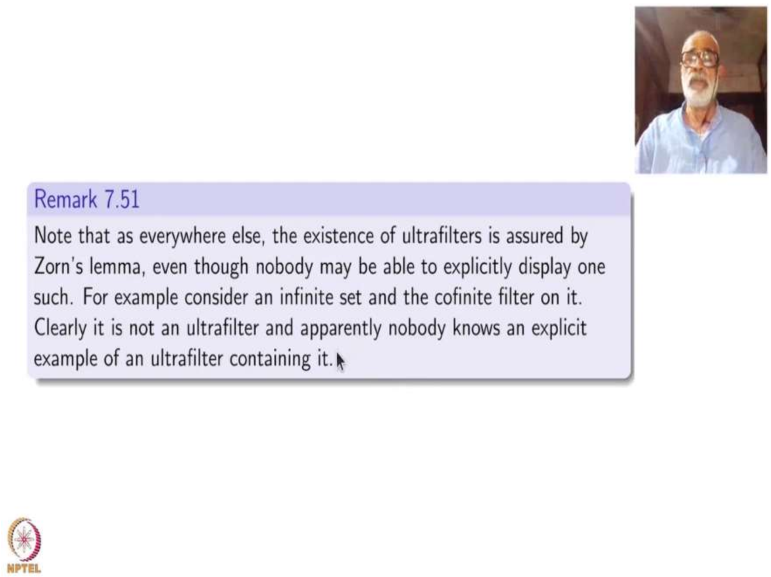

As everywhere else, the existence of ultra-filters is assured by Zorn's lemma, but Zorn's lemma is a very strange thing. Even though nobody may be able to explicitly display one such, we are guaranteed that they exist. This is not a joke. For example, consider an infinite set and the cofinite filter that you have considered, namely, the collection of all non empty subsets such that their complements are finite. (You leave out the empty anyway.)

So, that is a filter. Clearly, it is not an ultra-filter. It is not maximal, you can put extra elements there. Of course, you have to be careful. But apparently nobody knows an explicit example of an ultra-filter containing it. Zorn's lemma says that there is one. In fact, there will be many. But you know you would to construct one explicitly!

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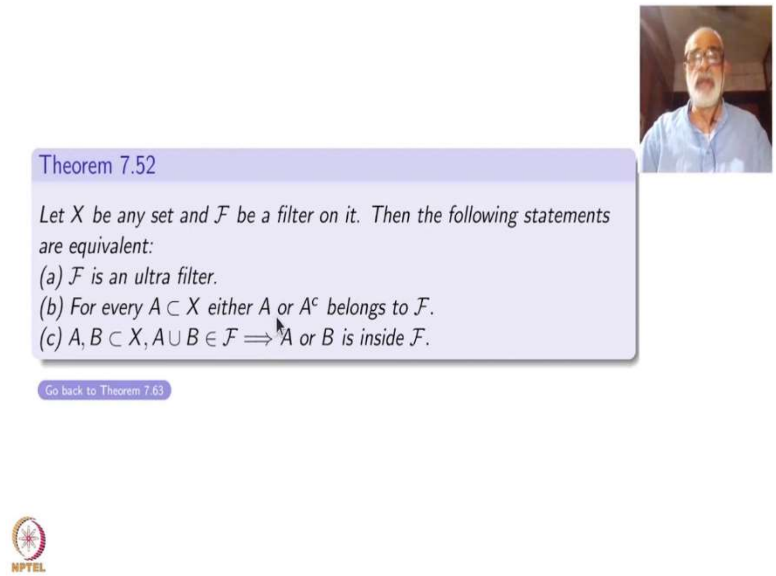

Let X be any set and $\mathcal{F}$ be a filter on it. Then the following statements (a), (b), (c) are all equivalent. So, we have two different criteria other than the definition (a) which are equivalent to the definition. So, in all, we will get three definitions of an ultra-filter.

(a) $\mathcal{F}$ is an ultra-filter.

(b) For every subset A of X either A or A^c belongs to $\mathcal{F}$. (Both of them cannot be there, that we know.) So, is quite a strong condition.

The third one is a weird one, but this is quite helpful also.

(c) A and B are subsets of X with the union belonging $\mathcal{F}$ implies A or B is inside $\mathcal{F}$.

(Remember by the very definition of filters, if A is there or B is the there, union would be there already, because any super set will be there. But now some super set is there which has been written as a union of two of its subsets. Then one of them must be inside $\mathcal{F}$. So, this is another condition for an ultra-filter.

Let us prove this one in a different order. For some pedagogical reason which you will come to know later, I have put these statements (a), (b), (c) in this order. Maybe I should have put (a) first, (c) second and then (b) third. But in my mind, (b) is easier to understand, after the definition (a) of ultra-filters. The definition of ultra-filter itself is easy to understand, namely, it is a maximal element. It is one of the maximal elements. So, that is easy to remember. And then this (b) is simpler than (c) and easy to remember also.

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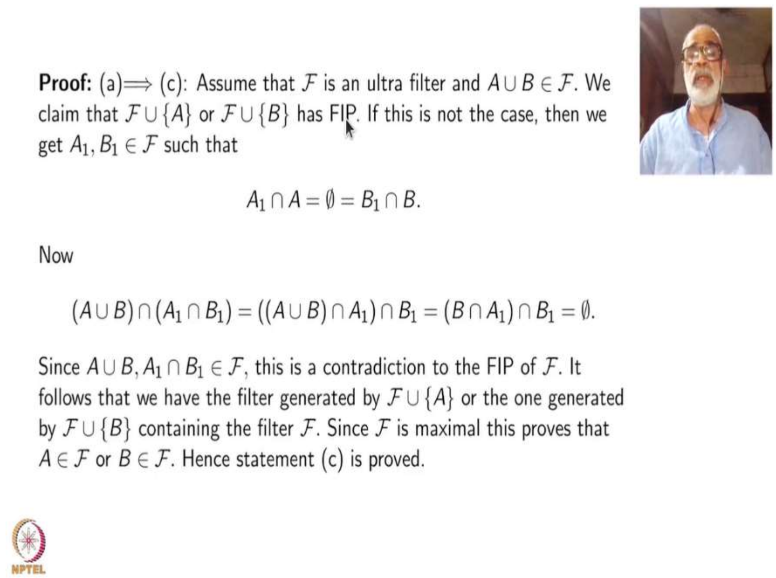

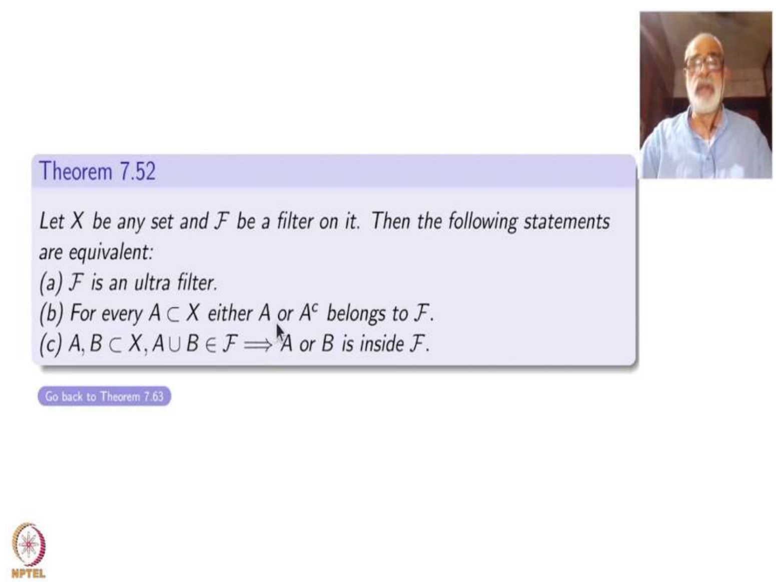

But, while proving their equivalence, I will find it economical to do (a) implies (c) implies (b) and then (b) implies (a).

So, first assume that $\mathcal{F}$ is an ultra-filter and union of two subsets is inside $\mathcal{F}$. First we claim that either $\mathcal{F} \cup \{A\}$ or $\mathcal{F} \cup \{B\}$ has finite intersection property. If this is not the case, then we get two subsets B_1 and B_2 inside $\mathcal{F}$, subsets of X, but they are members of $\mathcal{F}$, such that $A_1 \cap A$ is empty and $B_1 \cap B$ is empty, because each of these families violates the finite intersection property. The original family $\mathcal{F}$ has finite intersection property. Only because you have put the set A here and B there, it violates the finite intersection property. That just means that there are members A_1 and B_1 as described above.

But then look at $A \cup B$ which is already a member of $\mathcal{F}$. That is the hypothesis here. Intersect it with $A_1 \cap B_1$. See A_1 and B_1 are both elements of $\mathcal{F}$. Therefore, their intersection is also an element of $\mathcal{F}$. So, look at this intersection. This is one member. These two are two other members. Take their intersection, that is also a member of $\mathcal{F}$. So, this finite intersection this is nothing but $((A \cup B) \cap A_1) \cap B_1$. The first one is just $B \cap A_1$ because $A \cap A_1$ is empty. Now the whole thing is equal to $(B \cap A_1) \cap B_1$ which is empty because $B \cap B_1$ is empty.

That is a contradiction.

So, it follows that one of these two families has finite intersection property. Any family with finite intersection property will generate a filter.

So, one of them will generate a filter that will contain $\mathcal{F}$ and either A or B or both. Since $\mathcal{F}$ is maximal this proves that A must be inside $\mathcal{F}$ or B must be in $\mathcal{F}$. Here, it is not either A or B, both of them may also be in $\mathcal{F}$. That I do not care. So, statement only says that A or B is inside $\mathcal{F}$ not just only one of them.

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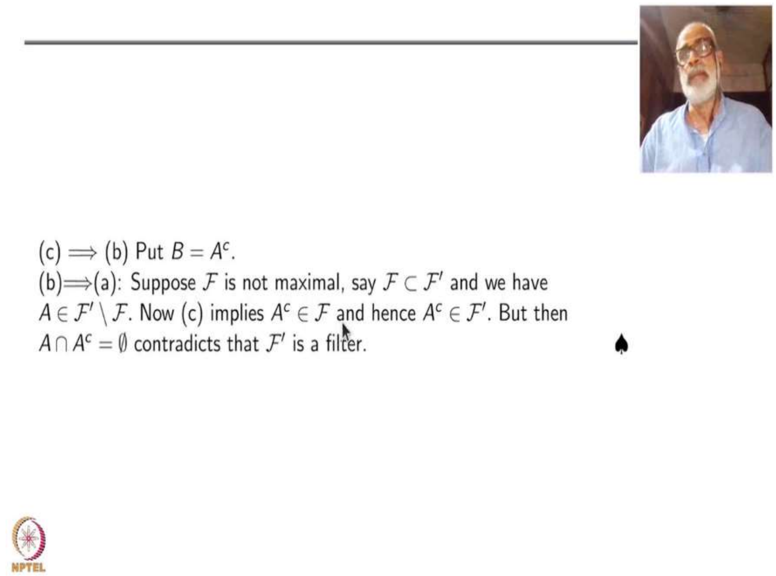

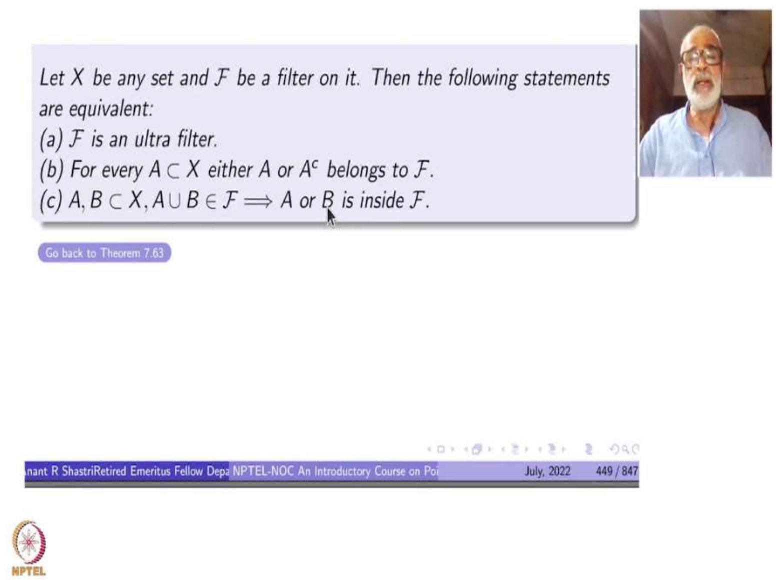

(c) implies (b) has one line proof. What I can do? I can take A^c to be B, then $A \cup A^c$ is the whole set X which is already in $\mathcal{F}$. Therefore, I can apply statement (c) to conclude that either A or A^c belongs to $\mathcal{F}$. Of course, this time we do not add the phrase `both of them'. So, (c) implies (b) is very cheap. That is why I have proved (a) implies (c) first. Now, let us prove (b) implies (a).

Suppose $\mathcal{F}$ is not maximal. That means what? there is a filter $\mathcal{F}'$ containing $\mathcal{F}$ properly. So, take a member A in $\mathcal{F}' \setminus \mathcal{F}$, a member which is in $\mathcal{F}'$ but not in $\mathcal{F}$. Now, (b) implies that A^c, the complement of A must be inside $\mathcal{F}$, because A is not inside $\mathcal{F}$.

Therefore, A^c is inside $\mathcal{F}'$ also, because $\mathcal{F}'$ is larger than $\mathcal{F}$. But then both A and A^c are inside $\mathcal{F}'$. That is a contradiction. So, $\mathcal{F}'$ must be equal to $\mathcal{F}$, there is no other choice.

Therefore, we have proved (a) implies (c) implies (b) implies (a). The proof of the theorem is over.

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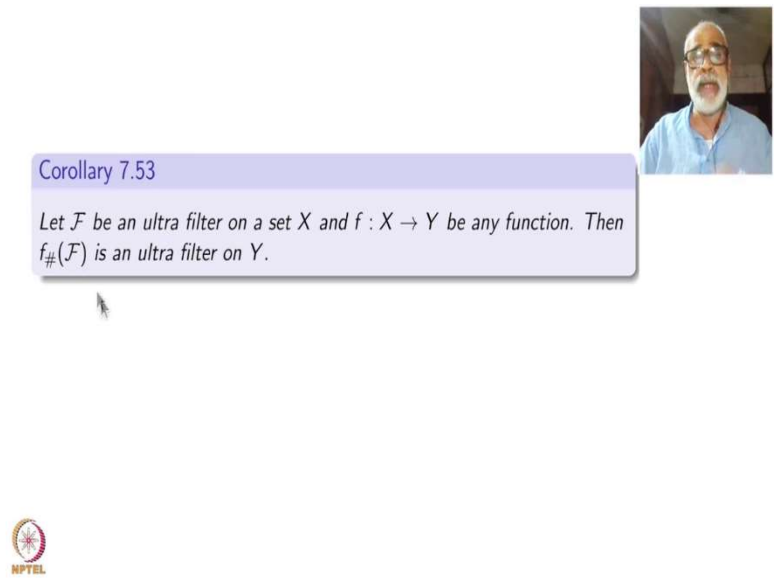

As a corollary, we prove something very, very important; very, very useful also.

Take any ultra-filter $\mathcal{F}$ on X and take a function f from X to Y. Then $f_{\#}(\mathcal{F})$ is an ultra-filter on Y.

So, ultra-filters behave very nicely. No condition on f. Remember, $f_{\#}(\mathcal{F})$ is a unique filter contains $f(A)$ for all $A \in \mathcal{F}$. It is generated by all those things.

In fact, all that you have to do is take all super sets of $f(A)$ where $A \in \mathcal{F}$. So, that is a definition of check. So, let us see how this comes.

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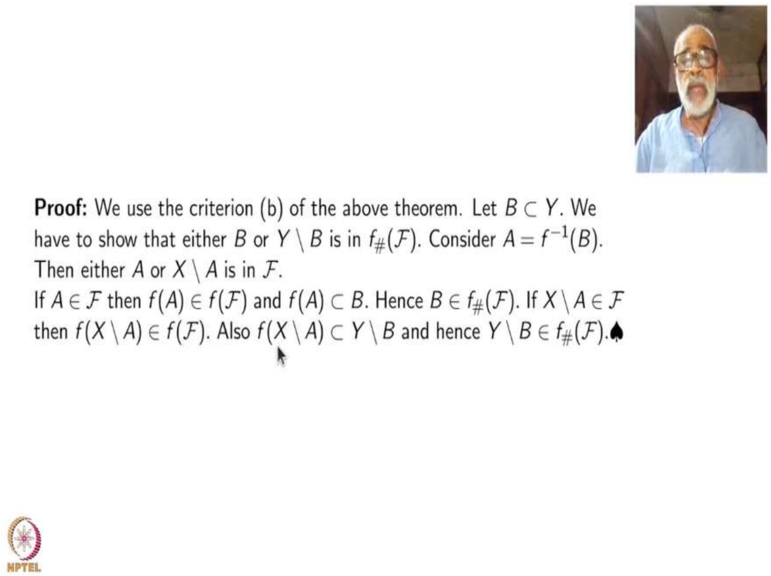

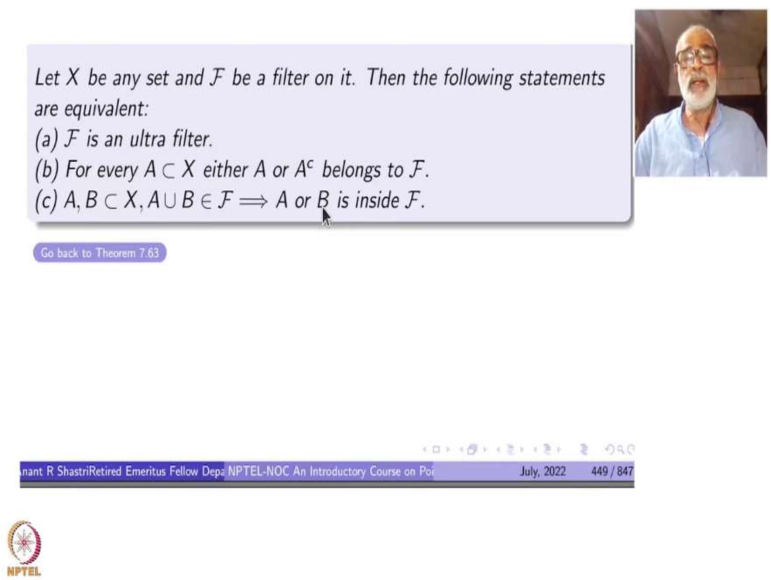

All that I have to do is to use the criterion (c). Actually I will use criterion (b) here. Let B be a subset of Y, we have to show that either B or $Y \setminus B$ is inside $f_{\#}(\mathcal{F})$.

Take $A = f^{-1}(B)$. That is subset of X. Since $\mathcal{F}$ is an ultra-filter, A or $X \setminus A$ is inside $\mathcal{F}$. If A is inside $\mathcal{F}$, $f(A)$ is inside $f(\mathcal{F})$. But we know that $f(A)$ is contained inside B, because A is $f^{-1}(B)$. Therefore, B is inside $f_{\#}(\mathcal{F})$. The other case is, suppose that $X \setminus A$ is inside $\mathcal{F}$. Then $f(X \setminus A)$ is inside $f(\mathcal{F})$, just by the set theoretic definition. Also, $f(X \setminus A)$ contained in $Y \setminus B$. Therefore, being a superset, $Y \setminus B$ will be inside $f_{\#}(\mathcal{F})$. So, we have shown that B or complement of B is inside $f_{\#}(\mathcal{F})$, for an arbitrary subset B of Y. That means $f_{\#}(\mathcal{F})$ is an ultra-filter.

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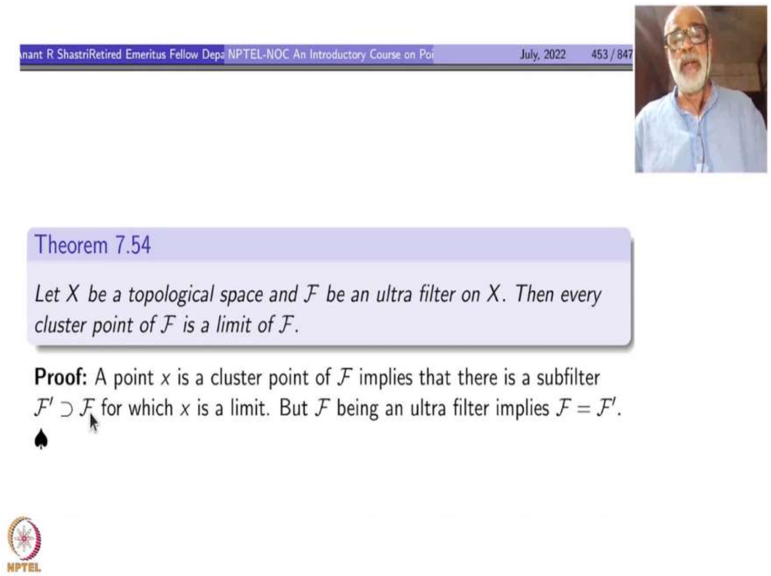

Now, X be a topological space, $\mathcal{F}$ be an ultra-filter on X. Then every cluster point of $\mathcal{F}$ is a limit point of $\mathcal{F}$.

Remember, for a filter every cluster point was a limit point of a sub-filter $\mathcal{F}'$. What is the meaning of a sub-filter? Sub-filter is a filter which contains the given filter. But $\mathcal{F}$ is an ultra-filter, so there is no larger filter than that one.

Therefore, $\mathcal{F}$ itself is its subfilter for which x is a limit point.

I repeat, if x is a cluster point of $\mathcal{F}$, earlier you have seen that there is a sub-filter $\mathcal{F}'$ for which x is a limit point. Sub-filter means what, containing $\mathcal{F}$. But $\mathcal{F}$ is ultra-filter, so there is equality here, over.

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Theorem 7.55

A topological space X is compact iff every ultra filter in X is convergent.

Proof: Combine the above theorem with Remark 7.40(1), theorem 7.50 and theorem 7.46.

Go to Remark7.40(1)

Let X be compact, and $\mathcal{F}$ be an ultra-filter on X. As a filter, $\mathcal{F}$ has a cluster point (theorem 7.46). $\mathcal{F}$ being an ultra-filter, from the above theorem, this cluster point will be a limit of $\mathcal{F}$.
Conversely, suppose every ultra-filter on X is convergent. Starting with any filter $\mathcal{F}$, we can put it inside an ultra-filter $\mathcal{G}$ (theorem 7.50). Now a limit of $\mathcal{G}$ is a cluster point for $\mathcal{F}$ (Remark 7.40(i)). Hence by theorem 7.46, X is compact. ♠

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Go to Theorem 7.50

Remark 7.40

1. If $\mathcal{F} \subset \mathcal{F}'$ then $\mathcal{F}$ converges to x implies $\mathcal{F}'$ also converges to x. Conversely, if $\mathcal{F}'$ converges to x then x is a cluster point of $\mathcal{F}$. For this reason, the bigger filter $\mathcal{F}'$ is called a subfilter of $\mathcal{F}$.
2. Every limit is a cluster point and as usual, the converse may not hold.
3. Let x be a limit of a filter $\mathcal{F}$. If $A \in \mathcal{F}$ then $x \in \bar{A}$. For otherwise, $\bar{A}^c \in \mathcal{N}_x \subset \mathcal{F}$, and this contradicts the FIP of $\mathcal{F}$.

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Theorem 7.50

Every filter is contained in an ultra filter.

Proof: We use Zorn's lemma. Consider the subfamily $\mathcal{A}$ of all filters each of which contains the given filter $\mathcal{F}$. If Λ is a chain in $\mathcal{A}$ then

$$\cup\{\mathcal{G} : \mathcal{G} \in \Lambda\}$$

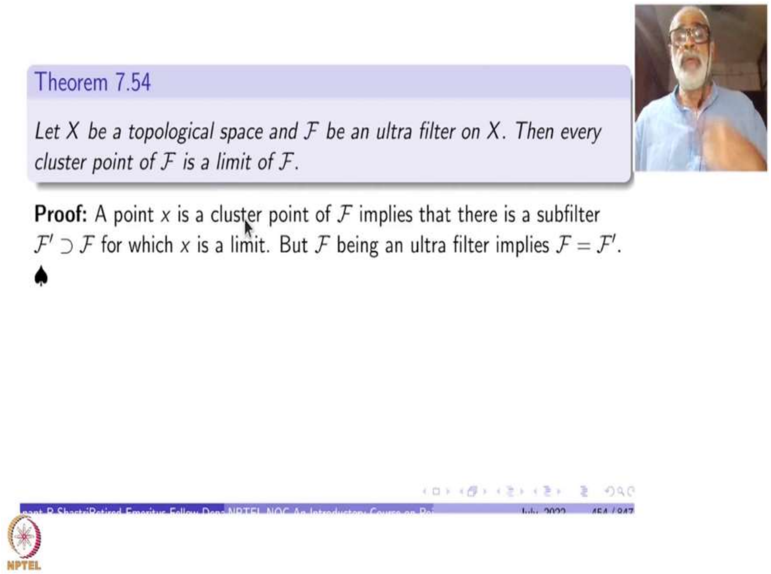
is easily seen to be an upper bound for Λ. Therefore, there is a maximal element in $\mathcal{A}$. ♠

Go to theorem7.46

Theorem 7.46

Let X be any topological space. Then the following conditions are equivalent:
(a) *X is compact.*
(b) *Every filter on X has a cluster point in X.*
(c) *Every filter has a convergent subfilter.*
(d) *Every net in X has a cluster point.*
(e) *Every net in X has a convergent subnet.*

Go to theorem7.55



A topological space X is compact if and only if every ultra-filter in X is convergent.

So, we have arrived at a very good theorem, by combining various earlier remarks. So, let me go through all these earlier remarks and theorems. 7.40 has actually three parts there. Theorem 7.50 and above theorem and theorem 7.46 which characterizes the compact spaces in terms of filters. If you do not remember maybe I should start just recalling this one.

So, first let us look at this remark, which says that if $\mathcal{F}$ is contained in $\mathcal{F}'$, then $\mathcal{F}$ converges to x implies $\mathcal{F}'$ also converges to x. Conversely, if $\mathcal{F}'$ converges to x, then x is a cluster point of $\mathcal{F}$. So, this is what just we have used. We will keep using this one again.

So, the next we have this one, every filter is contained in an ultra-filter. So, that also you will be using. Next, we have this criterion for compact spaces that every filter is a cluster point or every filter has a convergent sub-filter. And finally we have this, just now we have proved this theorem, namely, for an ultra-filter, a cluster point is a limit point.

So, if you combine these things, what you have is the following: Start with a compact space X and let $\mathcal{F}$ be an ultra-filter on X. As a filter, $\mathcal{F}$ has a cluster point. But just now we have proved that $\mathcal{F}$ being an ultra-filter, a cluster point must be a limit point. So, one way is over.

The converse is slightly more complicated. Suppose, every ultra-filter on X is convergent. Starting with my filter $\mathcal{F}$, we can put it inside an ultra-filter $\mathcal{G}$. Just now I quoted that theorem.

Being an ultra-filter, $\mathcal{G}$ has limit points. But limit points of $\mathcal{G}$ are cluster points of $\mathcal{F}$, by remark 7.40, which you have just seen. Therefore, again we apply this theorem 7.46, every filter has a cluster point. So that is, that condition is equivalent to say that X is compact.

So, what we have a neat theorem here now: A space is compact if and only if every ultra-filter is convergent.

Later on we will improve upon this one. One more improvement is there in a special case. I will come to that one soon. I am just preparing your mind for such an improvement.

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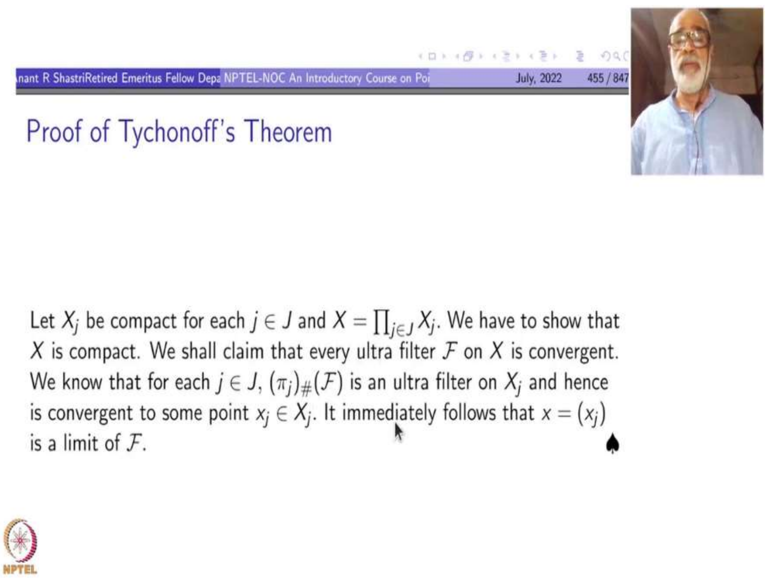

Now, we will prove Tychonoff's theorem in a very easy way, in a very canonical way. Recall, what is a Tychonoff's theorem? If you have a family of compact spaces then the product is compact and conversely. The converse part is easy and just put in the statement or completeness, that is all. That is not part of Tychonoff's theorem actually, because that can be proved much easily. Everybody knew that before Tychnoff. Namely, under a continuous function, image of a compact space is compact. The projection maps are continuous. So, each X_j will be compact if the product is compact. Our aim is to prove that if X_j's are compact then X is be compact. So, what we should do? Prove that every ultra filter $\mathcal{F}$ on this product is convergent.

For each j look at $\pi_{j\#}(\mathcal{F})$. Just now we prove that that $\pi_{j\#}(\mathcal{F})$ is an ultra-filter on X_j. But X_j is compact so it converges to some point x_j inside X_j. Now, it is a matter of easy verification, using the product topology here, that this point x whose j-th coordinate is x_j is the limit point of $\mathcal{F}$.

See, what you have to show? The neighborhood system N_x of x is contained in $\mathcal{F}$. That is the meaning of a filter converging to a the point x. So, maybe I will leave this as an exercise to you. I mean, there is no difficulty but you have to write down the details, that is all.

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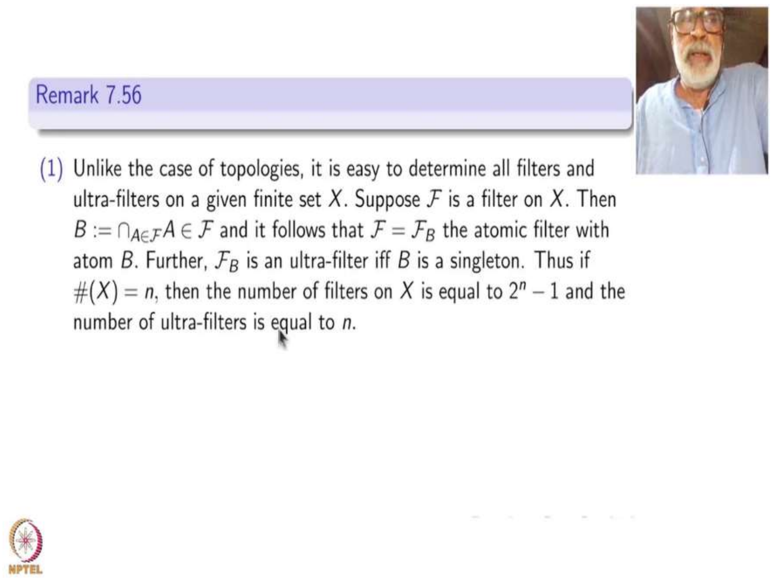

Unlike the case of topologies, it is easy to determine all filters and ultra-filters on a given finite set. You might have known that the study of finite topologies is indeed not a part of point set topology at all. As such, it is a topic in number theory, or combinatorics. It has quite a different flavour. And as far as topologists are considered, to some extent it provides some counter examples that is all. Beyond that topologists do not use it much. But the problems are there. And it is easy to find problems but difficult to find solutions here, of course.

So, here, similar questions if you ask for filters. And then you can combine them also. So, I am going to tell you one single example of such problems, namely, counting the number of filters and ultra-filters on a finite set. That is very, very easy. Suppose $\mathcal{F}$ is a filter on X, where X is a finite set. Then you can take intersection of all its members that must be a member of $\mathcal{F}$. Of course, this must be non-empty because it is a finite intersection. And it follows that that member

B is contained inside every other member which just means that $\mathcal{F}$ is $\mathcal{F}_B$, the atomic filter. Once B is there, all supersets are there, everything is there already, so $\mathcal{F}$ must be $\mathcal{F}_B$, the atomic filter. So, every filter is like that. Means what? There is a one to one correspondence between non empty subsets of X and the filters on X.

So, if you throw away the empty set, you exactly have 2^{n-1} number of filters on a set with n elements. Similarly, let us look at when is this filter an ultra-filter? That is also easy. The moment there are more than one point in B, $\mathcal{F}_B$ would not be an ultra-filter, because I can take a smaller one, say $x \in B$ and that $\mathcal{F}_x$ will contain this $\mathcal{F}_B$.

So, there are larger filters. Therefore, the only ultra-filters are $\mathcal{F}_B$, where B is a singleton. So, how many singletons are there? Precisely n of them. Over. So, that is just for getting a flavor, but I just want to tell you that I myself am not an expert in these kind of things. I want to ensure that there you can find many interesting questions here, but answers may not be easy.

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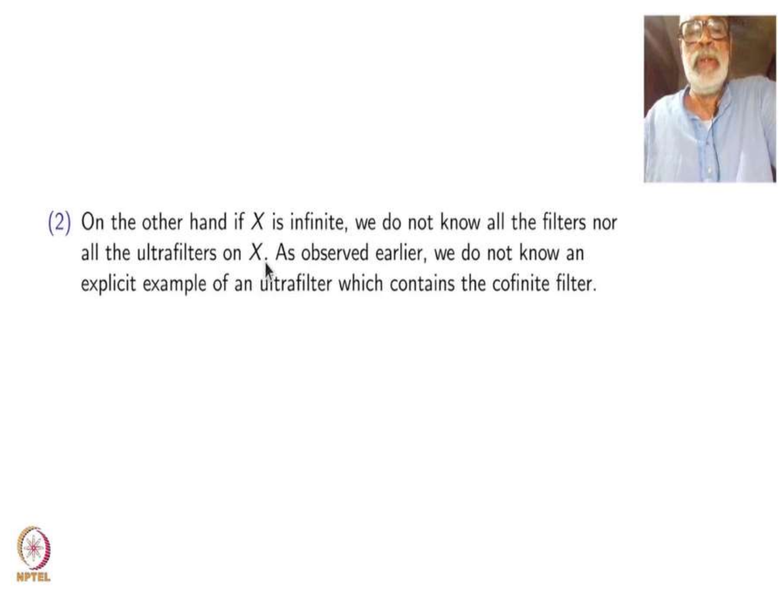

On the other hand, if X is an infinite set then the answer is much more difficult anyway. We do not know all the filters nor all ultra-filters on a given set. In particular, take a countably infinite set, say, natural numbers. Even there, we do not know. As observed earlier, we do not know an explicit example of an ultra-filter which contains the cofinite filter on the set of natural numbers, or on any countable set.

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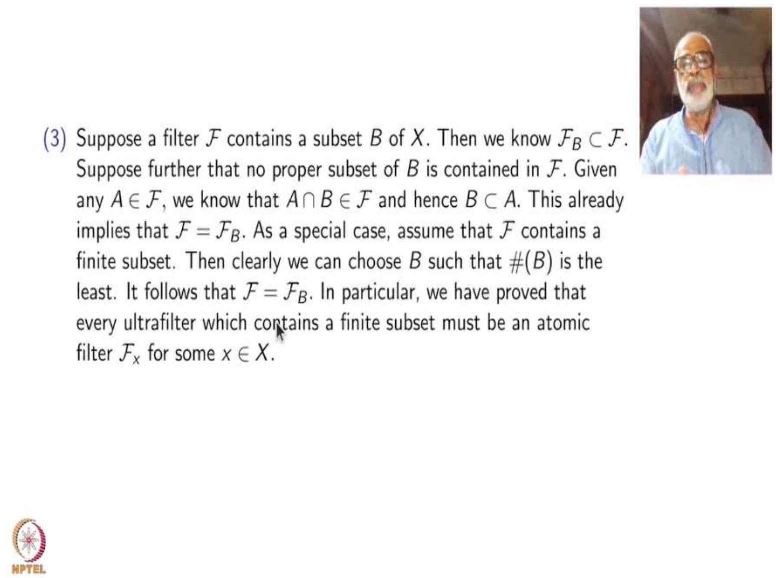

So, one more remark here. Suppose a filter $\mathcal{F}$ contains a subset B. Then obviously, the atomic filter $\mathcal{F}_B$ is contained inside $\mathcal{F}$, because $\mathcal{F}$ contains all supersets of B. Suppose further that no proper subset of B is contained inside $\mathcal{F}$. (See $\mathcal{F}$ may have even smaller subsets after all.) Then what happens, given any A inside $\mathcal{F}$, B is there, A is there, $A \cap B$ is there, but $A \cap B$ is not a proper subset of B just means that B is already contained inside A. Therefore, $\mathcal{F}$ must be $\mathcal{F}_B$.

As a special case, assume that $\mathcal{F}$ contains a finite subset. Then clearly we can choose B to be such that cardinality of B is the smallest amongst all finite subsets belonging to $\mathcal{F}$. It follows that once you choose B of that smallest cardinality, just cardinality smallest, it follows that $\mathcal{F}$ must be $\mathcal{F}_B$. So, you cannot have several subsets B such that all of them have the same finite cardinality. That is also clear anyway.

So, in particular, we have proved that every ultra-filter which contains a finite subset must be an atomic filter. So, this is something I would like to say, some attempt trying to understand what are ultra-filters are on an arbitrary set. So, the darkness is still there. Once you do not have this hypothesis, namely a filter which may not contain any finite subset then you would not know how to characterize them. So, let us stop here. Next time, we will study ultra-closed filters.