An introduction to Point-Set-Topology Part-II
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Lecture 31
Convergence Properties of Filters

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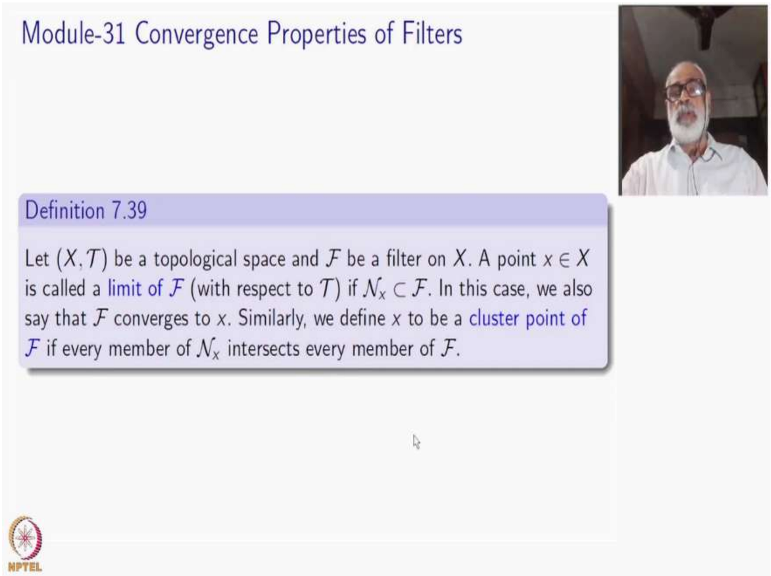

Hello, welcome to module 31 of, NPTEL point-set-topology part-II course. So, having studied some general properties of filters, today, we shall study convergence properties of filters. For that, we start with a topological space, not just a set. Remember that a filter can be talked about on any set, whereas now we have a topological space $(X, \mathcal{T})$.

Let $\mathcal{F}$ be a filter on X. A point x of X is called a limit of $\mathcal{F}$ or you can say it is a limit point, no problem, with respect to the topology tau (limit points will not be spoken about unless you have a topology, that is important), if the neighbourhood system, the entire neighbourhood system $\mathcal{N}_x$ the neighbourhood set of all neighbourhoods at the x, that itself is contained in the filter $\mathcal{F}$. In this case, we also say that $\mathcal{F}$ converges to x. Similarly, we define x to be a cluster point of $\mathcal{F}$ if every member of $\mathcal{N}_x$ intersects every member of $\mathcal{F}$.

So, here, you can see that this is a weaker condition than that one, because if $\mathcal{N}_x$ is contained in $\mathcal{F}$, then this property is also true because $\mathcal{F}$ itself has finite intersection property. So, that is a weaker condition.

Indeed, both these definitions are copied from what happens in the case of the filter associated to a sequence and x is either a limit point or a cluster of that sequence.

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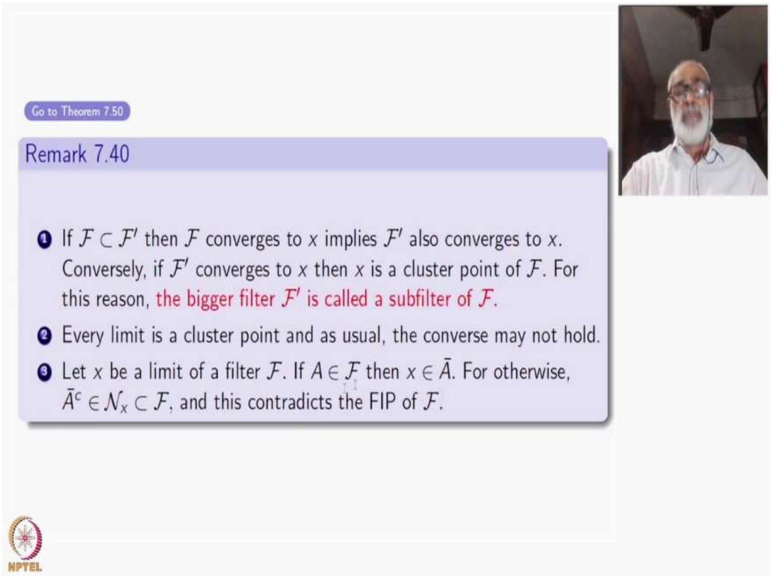

Some observations, which are, immediate from the definition are the following:

(1) If you have one filter $\mathcal{F}$ contained in another filter $\mathcal{F}'$, then $\mathcal{F}$ converged to x implies $\mathcal{F}'$ also converges to x; the larger filter so-called larger filter also converges to x. Conversely, if $\mathcal{F}'$ converge to x, then you cannot say that $\mathcal{F}$ converges to x, but x will be a cluster point of $\mathcal{F}$. So, that is a weaker condition.

For this reason, the bigger filter $\mathcal{F}'$is called the sub-filter. So, this is a bit strange terminology, which you have to digest, which you have to live with, because this is borrowed from what happens to sequences and subsequences. If we have a sequence which converges to a point then every subsequent will also converge the same point. If a subsequence converges, then the point of convergence of subsequence will be only a cluster point of the original sequence.

(2) Secondly, every limit point is a cluster point that I already told you, because being a cluster point is a weaker condition. But the converse is not true, just like in the case of sequences. So, we shall not bother to discuss it any further. Limit point is a cluster point, cluster point need not be a limit point.

(3) The third observation is that suppose x is a limit of a filter. If A is any member of $\mathcal{F}$, then x will be in the closure of that member, x belongs to $\bar{A}$. This is very easy to see, but I have added, explanation here.

If not, what happens? if x is not $\bar{A}$, that will mean x is inside the complement of $\bar{A}$, which is open, which means $\bar{A}^c$ is in $\mathcal{N}_x$ which is contained in the filter. So the filter has two disjoint members, contradicting the finite intersection property of the filter. So, a limit point is the closure of every member.

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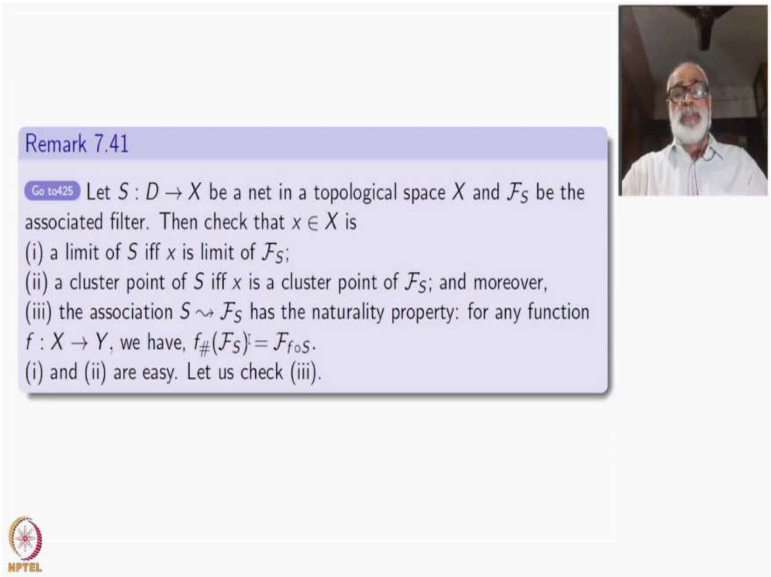

There is one more important remark. In fact, it has three parts here and we will use them. So, pay attention to this one.

(i) Now I have a net which is a generalization of a sequence. Start with a net S from D to X, where X is a topological space. Let us consider $\mathcal{F}_S$, the associated filter. Then the first thing you can check is that x belonging to X is a limit point of $\mathcal{F}_S$ if and only if x is a limit point of S considered as a net.

(ii) The second thing is that x a cluster point of $\mathcal{F}_S$ if and only if, it is a cluster point of S as a net. So, this is precisely what I already told you that both the definition of limit point and cluster point have been copied from what happens to the associated filter from a net or a sequence.

(iii) The third one is very important here. The association S to $\mathcal{F}_S$ has the naturality property: for any function f from X to Y. We have $f_\#(\mathcal{F}_s)$ is equal to the filter associated to the net $f \circ S$.

You take $f \circ S$ that is another net on Y. The associated filter on Y is nothing but $f_\#(\mathcal{F}_s)$.

So, (i) and (ii) were easy to verify. The third one is actually a general property and has nothing to do with the convergence. So, I should have done it earlier. In any case, let us check it. So, I will take some time.

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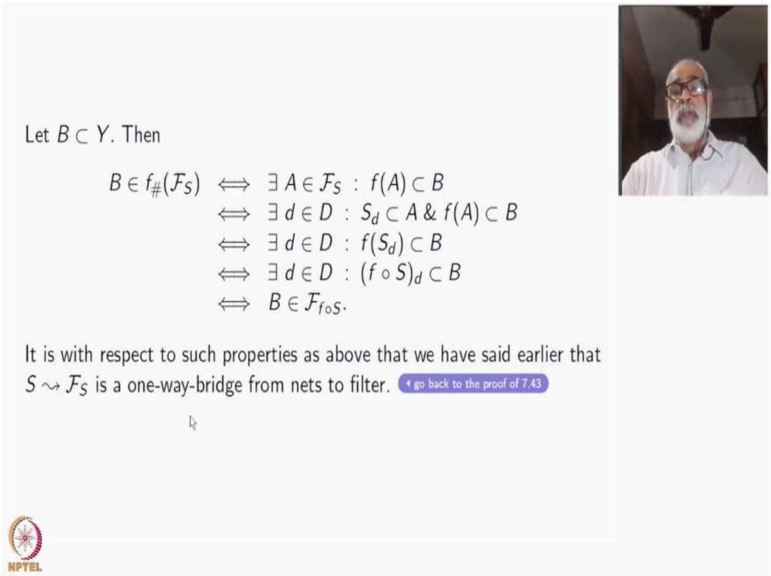

So, here is the proof: Start with any subset B of Y. The function f is from X to Y. So, things are happening inside Y. So, take a subset B of Y it is in $f_\#(\mathcal{F}_S)$ if and only if (by this by definition of $f_\#$) if and only if there exists a subset A belonging to $\mathcal{F}_S$ such that $f(A)$ is contained in B. (So, that is the definition, how to generate $f_\#(\mathcal{F}_S)$. It contains all the supersets of all the images of members of $\mathcal{F}_S$ under the map f, that is the definition.

Now this is same thing as saying that there exists d belonging to D such that S_d (what you call the right ray of S) is contained inside A, that is the this first part, viz., A belonging to $\mathcal{F}_S$ and the second part, $f(A)$ is contained inside B as it is.

But this is same thing as saying that there is d inside D such that $f(S_d)$ is contained inside B because $f(S_d)$ is inside $f(A)$, which is contained in B.

So, that like saying that the right ray of $f \circ S$, viz., $(f \circ S)_d$ is containing inside B. So, that is same thing as saying that B is inside the associated filter of $f \circ S$. That is all, a purely set theoretic result. There is no topology in this one, but this will be useful. So, this is important and this association S to $\mathcal{F}_S$ is going to be very important. It is a one-way bridge from nets to filters. The filters have much more generalities. You cannot always come back to the nets from filters.

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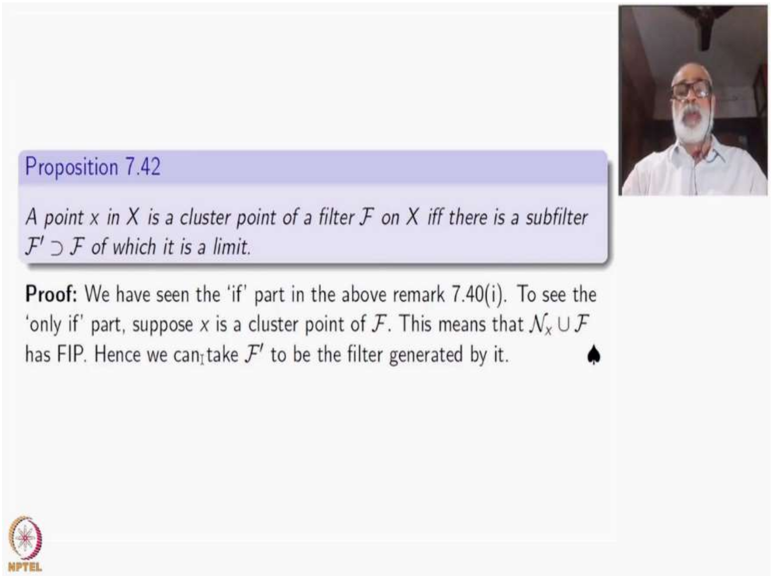

So, now let us come back to the study of limits and cluster points.

A point in X is a cluster point of a filter $\mathcal{F}$ if and only if there's a subfilter $\mathcal{F}'$ of $\mathcal{F}$ i.e., $\mathcal{F}'$ containing $\mathcal{F}$ I will read this as a subfilter, such that x is a limit point of $\mathcal{F}'$.

Here, I am actually giving you the converse part as well, since `if' part, we have already seen, in the earlier remark.

To see the `only if ' part, suppose x a cluster point of $\mathcal{F}$, this just means that the neighbourhood of system $\mathcal{N}_x \cup \mathcal{F}$ has finite intersection property. See within, $\mathcal{N}_x$ or within $\mathcal{F}$, we have finite intersection property. So any member here will intersect any member there, is the same as the union hving FIP. We can then take $\mathcal{F}'$ to be the filter generated by this family. Any family with finite intersection property will generate a filter. Namely, you now take all super sets of each member of that family. That will be our $\mathcal{F}'$. I have to produce one, there exists some filter. there may be many more. No problem.

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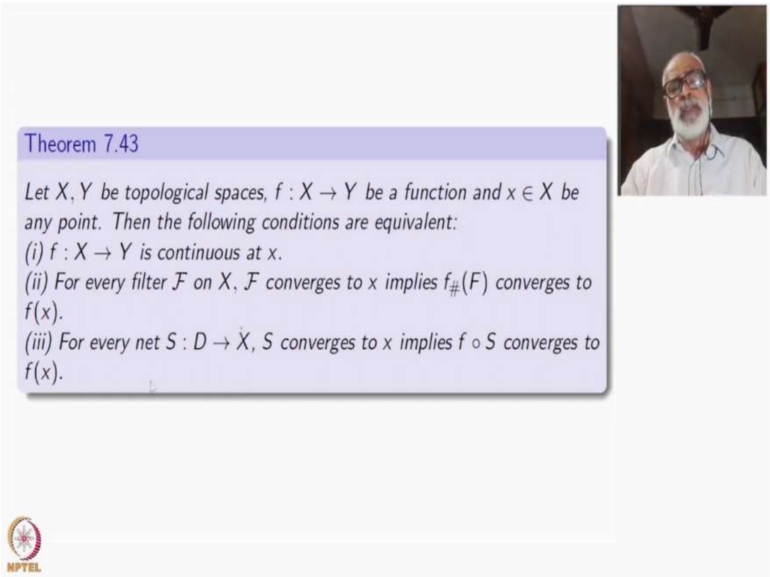

The next theorem is about functions. Take two topological spaces and a map from X to Y. Let x belong to X be any point. Then the following conditions are equivalent: As I told you this is about the continuity of functions, giving a characterization in terms of filters. So, start with any function. Take a point x in the domain.

(i) f is continuous at x. (This is the first statement. Second one is)

(ii) for every filter $\mathcal{F}$ on X which converges to x, we have $f_{\#}(\mathcal{F})$ converges to $f(x)$. (This is the second statement. The third statement is)

(iii) for every net S from D to X converging to x, we have $f \circ S$ converges to $f(x)$.

So, these are three statements. The theorem is that these three are equivalent. Part of it I could have proved while doing nets, but I deliberately postponed it because it can be done much easier now, once you go along with the filter as well. So, the three conditions are equivalent is the statement of the theorem. So, as usual we will prove (i) implies (ii) implies (iii) implies (i). That is what our plan.

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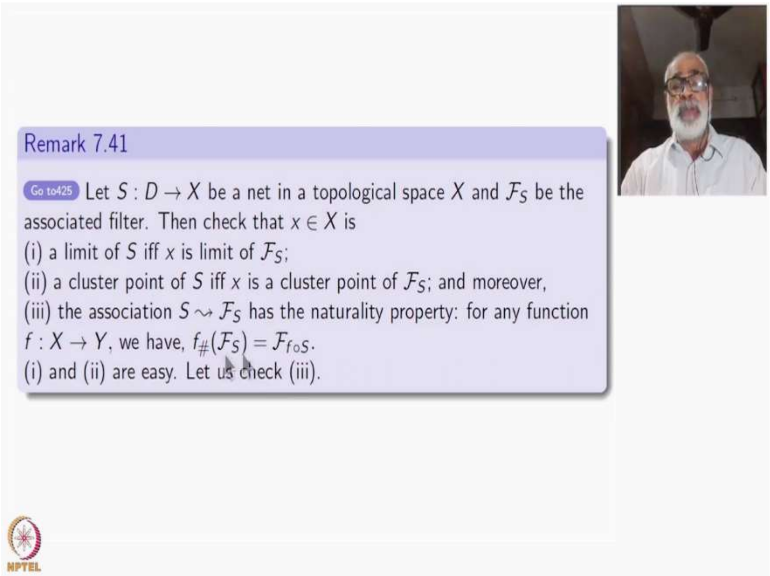

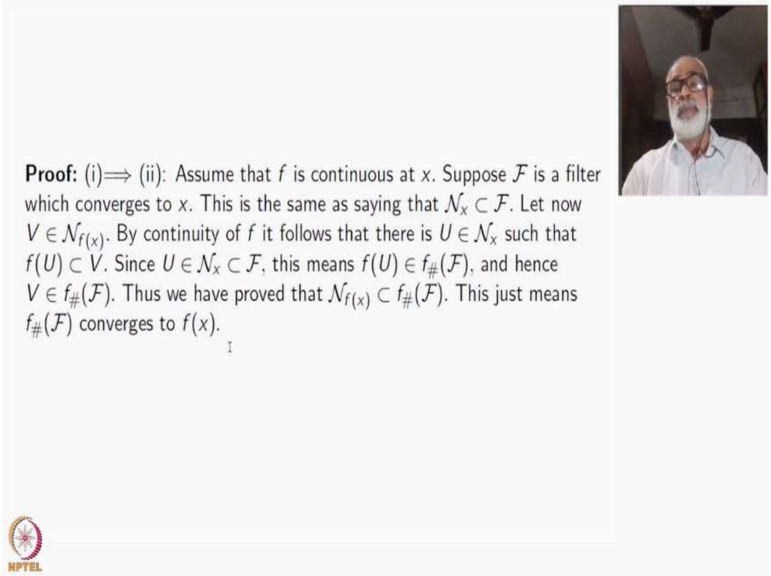

Assume that f is continuous at x. Now take a filter $\mathcal{F}$ which converges to x. This is the same thing as saying that $\mathcal{N}_x$ is contained in $\mathcal{F}$. Let now V belong to $\mathcal{N}_{f(x)}$. I have to show that V belongs to $f_\#(\mathcal{F})$. That is what I have to show. Start with a neighbourhood V of $f(x)$. By continuity of f it follows that there is a neighbourhood U of x, such that $f(U)$ is contained inside V. That is continuity.

But now U is in $\mathcal{N}_x$, and $\mathcal{N}_x$ is contained in $\mathcal{F}$. Therefore $f(U)$ belongs to $f_\#(\mathcal{F})$. Over. So, that just means that $f_\#(\mathcal{F})$ converges to $f(x)$. So, (i) implies (ii) is over. Let us do (ii) implies (iii).

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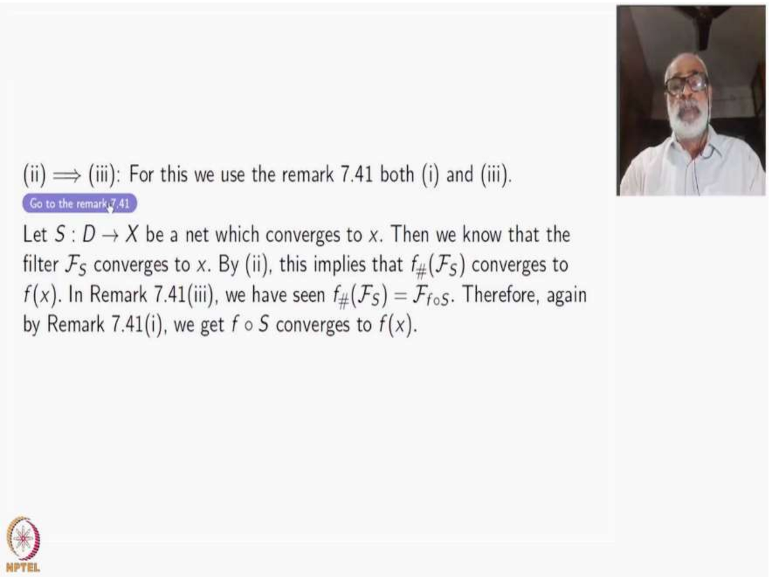

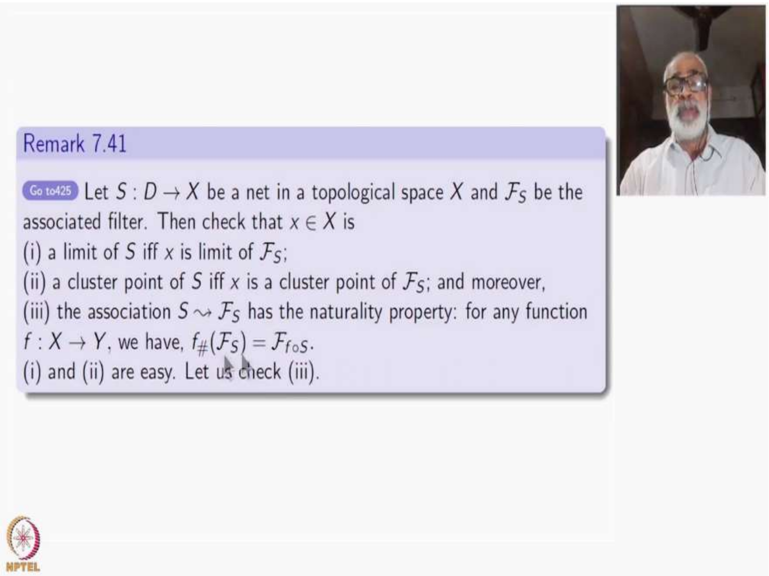

For this we use remark 7.41 both (i) and (iii). So, let me just show you 7.41. First one says that something is a limit point of S iff it's a limit point of $\mathcal{F}_S$. So, this is the, statement from a net to the associate filter and back. The third one says that the associated filter of $f \circ S$ is the same as $f_{\#}(\mathcal{F}_S)$.

So, both of them will be used now. So, let me go back to the proof of this (ii) implies (iii).

Start with a net S from D to X, converging to a point $x \in X$. Then we know that the filter associated to S also converges to x. Statement (ii) now implies that $f_{\#}(\mathcal{F}_S)$ converges to $f(x)$. But this filter is the same as the filter associated with $f \circ S$. Therefore, by (ii) of the remark 7.41, this implies the net $f \circ S$ converges to $f(x)$. So, we have proved (ii) implied (iii).

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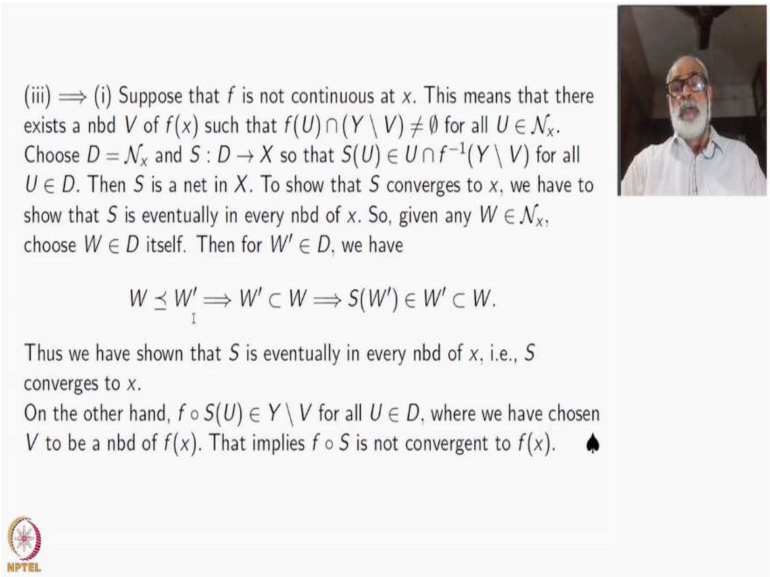

Now let us go to the proof of (iii) implies (i). So, here we prove it by contradiction. Namely, suppose f is not continuous at x. Then we will construct a net S in X which converges to x but $f \circ S$ does not converge to $f(x)$. So, that is the way we are going to prove (iii) implies (i).

Start with the assumption that f is not continuous at x. This just means that there is one neighbourhood V of $f(x)$ such that for no neighbourhood of U of x, $f(U)$ will be contained inside V, which is samething as saying that for all $U \in \mathcal{N}_x$, intersection of $f(U)$ with $Y \setminus V$ is non empty. Now I choose my directed set D to be and $\mathcal{N}_x$ itself with its usual reversed inclusion as the direction. Remember that. Now I am going to define the net S from D to X such that each $S(U)$ is chosen from this non empty set, $U \cap f^{-1}(Y \setminus V)$. $f(U) \cap (Y \setminus V)$ is non non empty. It just means that there is some point in $U \cap f^{-1}(Y \setminus V)$. So, we have got such a function S. So, that function is going to be our net. So, what is the property of S? For each U inside D, (D is nothing but $\mathcal{N}_x$), $S(U)$ is inside U as well as inside $Y \setminus V$ also. Since U ranges over $\mathcal{N}_x$. From this we claim that S converges to x.

For that we have to show that S is eventually in every neighbourhood of x. So, let W be any neighbourhood of x. Choose W itself as a member of D. Now if W' follows W. i.e., if W' is contained inside W, then $S(W')$ is inside W' and hence inside W. So, this means S inside W. Thus, we have shown that S is converging to x.

On the other hand, it is very clear that $(f \circ S)(U)$ belongs to $Y \setminus V$, for all U. V is fixed neighbourhood of $f(x)$, $f \circ S$ should have been inside eventually inside V but not possible because all the time, for all U, f($f \circ S)(U)$ is inside $Y \setminus V$, outside V. So, this means that $f \circ S$ is not converging $f(x)$.

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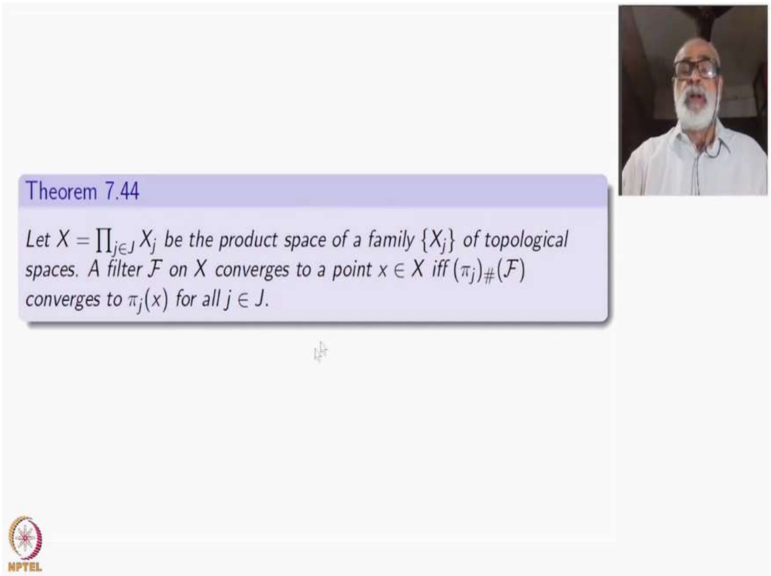

So, these were some fundamental properties. Now we shall prove some useful corollaries to these things, about convergence properties in products.

Take a family $\{X_j\}$ of the topological spaces, and take X to be the product of all X_j's. Let $\mathcal{F}$ be a filter on the product. Take a point x inside the product. The filter $\mathcal{F}$ converges to this point x, if and only if, look at all the projection maps, π_j's, look at the image filters, $\pi_{j\#}(\mathcal{F})$, look at the point $\pi_j(x)$ in X_j, what we want is that for every j, $\pi_j(x)$ is a limit of $\pi_{j\#}(\mathcal{F})$.

So, this is `if and only if'. One way, we have already seen, namely, because π_j are continuous functions. The general statement we have proved is that if $\mathcal{F}$ converges to x then $f_{\#}(\mathcal{F})$ converges to $f(x)$, take $f = \pi_j$.

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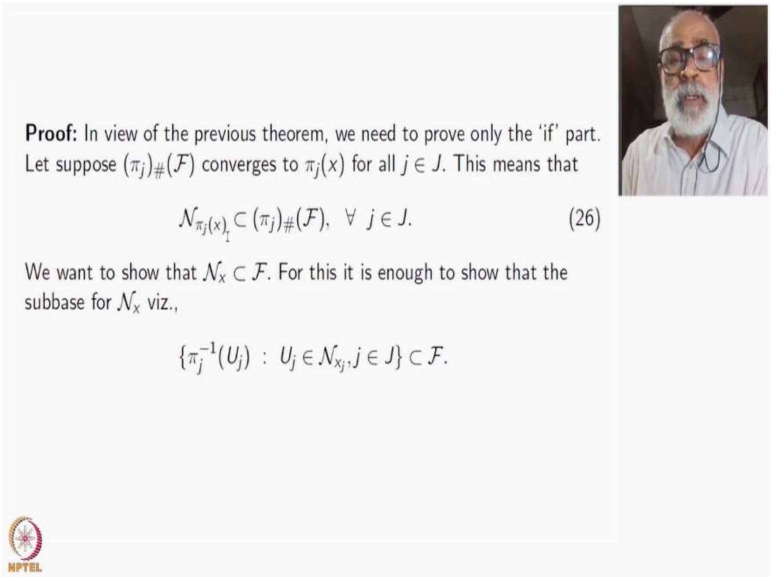

So, `only if' part we have seen. Now we come to the `if' part.

Let us now suppose that $\pi_{j\#}(\mathcal{F})$ converge to $\pi_j(x)$, for all j. This means that the neighbourhood system at $\pi_j(x)$ in the topological space X_j is contained in the filter $\pi_{j\#}(\mathcal{F})$. This is happening for all j. But what we want to show is that the neighbourhood system at x in the product space, should be contained in $\mathcal{F}$. For this it is enough to show that some subbase for $\mathcal{N}_x$ is contained in $\mathcal{F}$.

Now you have to remember what is the product topology. In the product topology a neighburhood system at x is generated by a subbase viz., $\{\pi_j^{-1}(U_j) : j \in J\}$, where U_j range over all the neighbourhoods of $x_j \in X_j$. Finite intersections of such members forming a local base at x etc.

Once a subbase is inside $\mathcal{F}$, finite intersections will be there and anything bigger than that will be also there. So, whole, $\mathcal{N}_x$ will be there. So, that is why it is enough to show that some subbase for $\mathcal{N}_x$ is contained in $\mathcal{N}_x$.

So, I have to show that for each J, $\pi_j^{-1}(U_j)$ belongs to $\mathcal{F}$ for all U_j in $\mathcal{N}_{x_j}$ and for all j.

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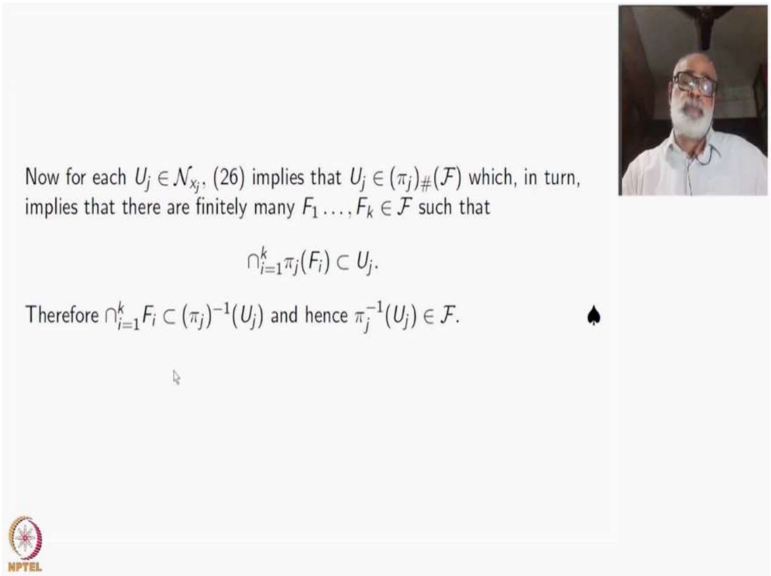

So, that is very easy. Look at (26), whatever we have, that is enough. This one is contained inside this one. So, what is definition of the $\pi_{j\#}(\mathcal{F})$? That is what I have to interpret. That implies that there are finitely many members of $\mathcal{F}$ say, $F_1, \ldots, F_k$ such that such the intersection $\pi_j(F_i)$'s is contained in U_j.

Because these images under π_j form a subbase for $\pi_{j\#}(\mathcal{F})$; take finite intersections and then all the supersets. So, intersection F_i's is contained in the $\pi_j^{-1}(U_j)$. I am applying π_j^{-1} on both sides here. I can just say contained in, not equality which is not needed either. This is contained inside this one. Therefore $\pi_j^{-1}(U_j)$ belongs to $\mathcal{F}$.

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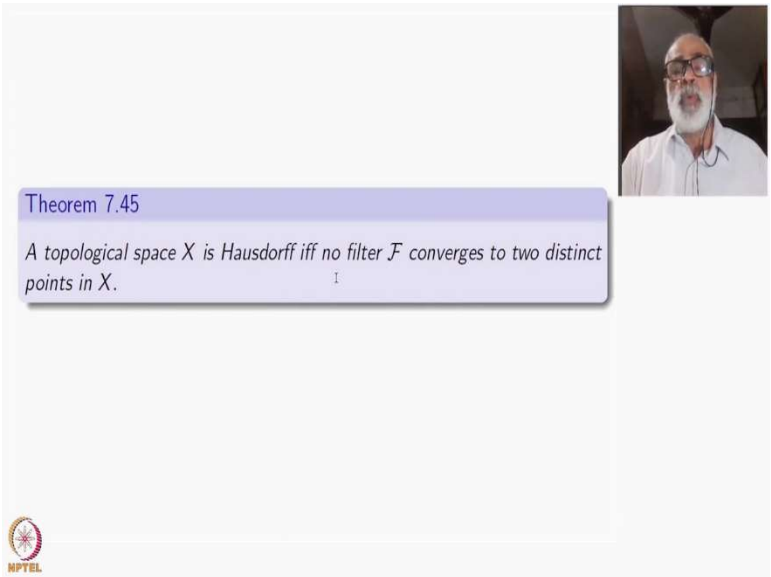

So, next, we have a theorem:

A topological space X is Hausdorff, if and only if, no filter $\mathcal{F}$ on X converges to two distinct points of X.

Remember when we defined a limit of a filter, we did not say that the limit is unique. There may be many limit points. In general. But you take a Hausdorff space, then a filter can have at most one limit point. That is not just the point. The point is this statement is `if and only if'. We get a characterization of Hausdorff spaces. Maybe this is the fifth or sixth characterization of a Hausdorff sapce. Remember that we had several characterizations of Hausdorffness in Part I. This Hausdorffness being very important, it will keep coming again. So, a topological space X is Hausdorff if and only if, no filter $\mathcal{F}$ on it converges is to more that one point.

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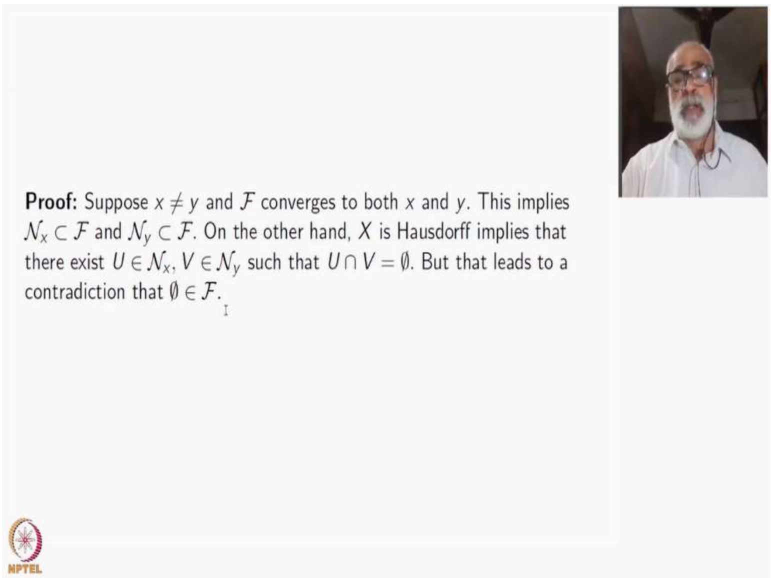

So, let us prove this one. Suppose x is not equal to y in X, and $\mathcal{F}$ converges to x and y. Then what happens? Both $\mathcal{N}_x$ and $\mathcal{N}_y$ are inside $\mathcal{F}$. On the other hand, if X is Hausdorff then there exists U in $\mathcal{N}_x$ and V in $\mathcal{N}_y$ such that $U \cap V$ is empty. So both U and V are inside $\mathcal{F}$ That cannot happen because then empty set will be inside $\mathcal{F}$. That means that $x = y$.

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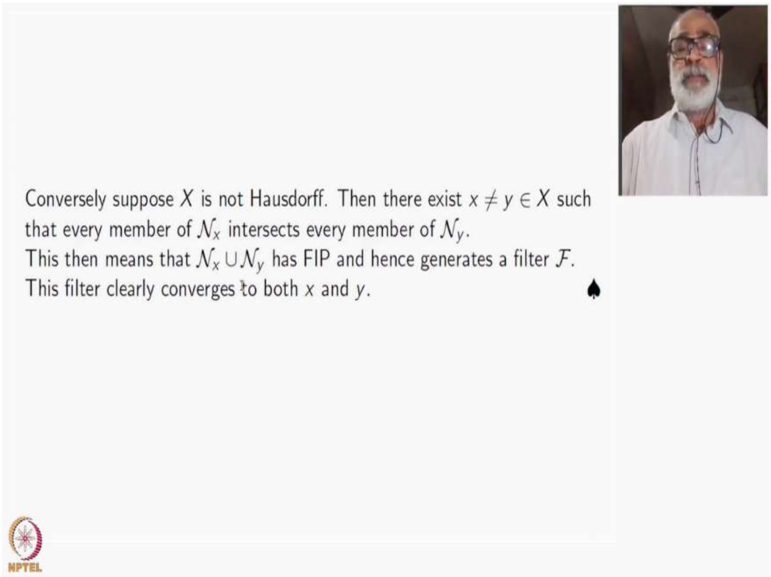

For the converse, suppose X is not Hausdorff. Then I will produce filter, which will have two distinct limit points. So, not Hausdorff just means that there are points x and y not equal to each other, such that every member of $\mathcal{N}_x$ intersects every member of $\mathcal{N}_y$.

This just means that $\mathcal{N}_x$ union $\mathcal{N}_y$ has finite intersection property. As soon as family has finite intersection property, we know that it is contained in a filter. Namely, the one generated by this this family. So, that filter will have both x and y as limits because both $\mathcal{N}_x$ and $\mathcal{N}_y$ are there. That is all.

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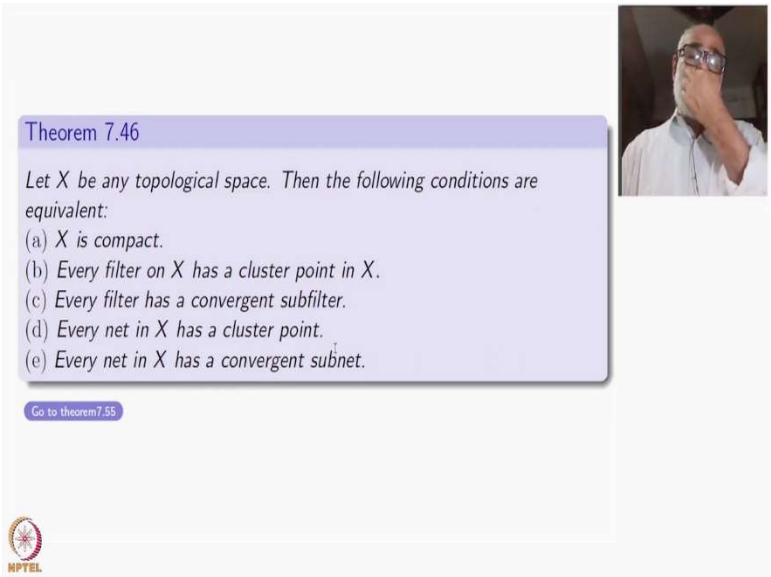

Finally, we have this big theorem here.

Let X be any topological space. Then the following conditions are equivalent. (Like Hausdorffness has the many characterizations we want to characterize compactness also).

(a) X is compact. (That is the first statement.)

(b) Everybody filter on X has a cluster point.

(c) Every filter has a convergent subfilter. (These are about filters. And I have promised you to do the similar things for nets also. So, two different characterizations of compactness here, similar ones with nets now.)

(d) Every net in X has a cluster point.

(e) Every net in X has a subnet which is convergent.

So, statements (a) (b) (c) (d) (e) are equivalent. These give you four more characterizations of compactness.

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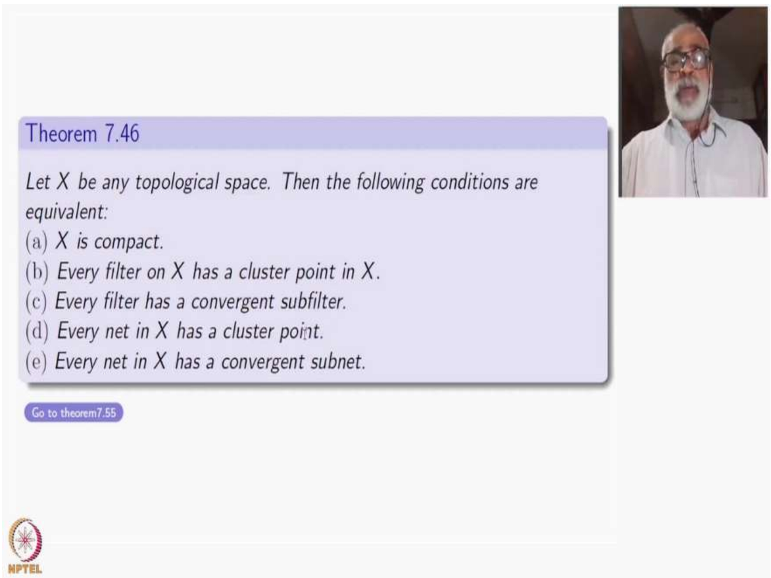

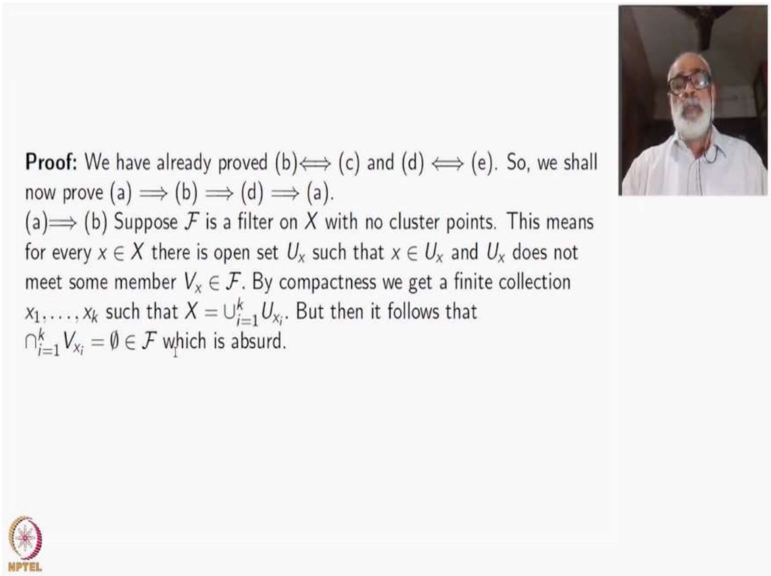

What we have already proved is (b) implies and implied by (c). Look at this. Every filter has a convergent subfilter. Limit of a subfilter is a cluster point. And converse this part also we have seen. Similarly, for net also, (c) implies and implied by (d), this much we have seen. So, we shall now prove (a) implies (b) implies (d) implies (a). Automatically (c) and (d) will be taken care. So, let us prove (a) implies (b).

Suppose $\mathcal{F}$ is a filter on X with no cluster points, then I have to show that X is non compact. (So, that is the contrapositive of (a) implies (b).) This means that for every x inside X, there is an open neighbourhood U_x of x which does not meet some member V_x of $\mathcal{F}$. So, there is one member here, and one member here which do not meet.

That is the negation of saying that $\mathcal{F}$ has cluster points.

By compactness of X, we get a finite collection $x_1, \ldots, x_k$ such that $\{U_{x_i}\}$ cover the whole of X, because, union of all U_x's forms an open cover for X. So, take a finite sub. But then you look at the intersection of all V_{x_i}. Corresponding to each x_i, you have members V_{x_i} of $\mathcal{F}$, finitely many of them. Their intersection must be empty. Because this intersection has empty intersection with each U_{x_i} and hence with their union also, which is the whole of X. So, the intersection of V_{x_i}'s is empty, which is absurd because these are finitely many members of a filter. So, that is not possible.

(It is not very hard to prove it directly also, you can try to prove it instead of proof by contradiction.)

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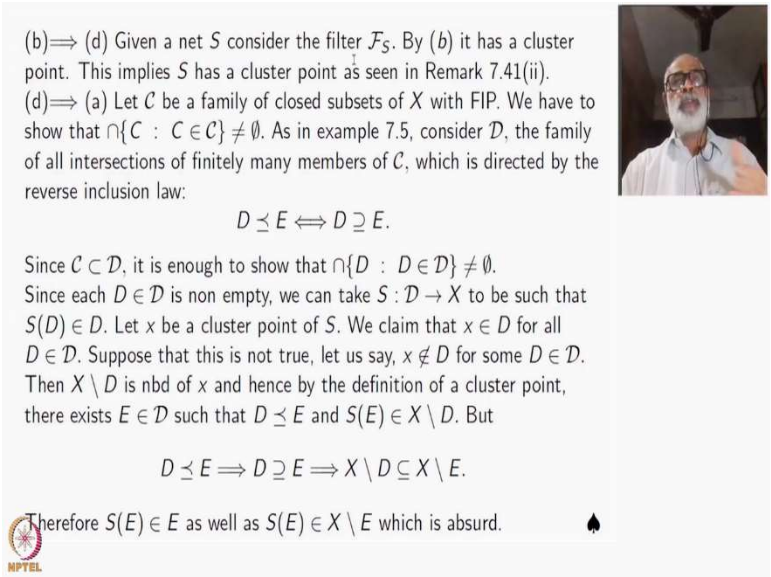

So proof of (b) implies (d) now. So, I have to go to nets now. Starting with a net S, consider the filter $\mathcal{F}_S$. By (b) this filter has a cluster point. This is the same thing as saying that S as a cluster point S as seen in 7.1.

Finally, I have to show that (d) implies (a). So, for this we have to work harder, so, please pay attention.

(So, this way, I am, I am saving time for instead of proving (b) implies (a) and then (d) implies (a) etc. This is easily done.)

Start with a family $\mathcal{C}$ of closed subset of X with finite intersection property. We have to show that the entire intersection of members of $\mathcal{C}$, is non empty. That will show X is compact. So, the argument is similar to one of the previous examples.)

Consider the family $\mathcal{D}$ of intersections of finitely many members of $\mathcal{C}$, which is directed by the reverse inclusion.

In particular, $\mathcal{D}$ contains $\mathcal{C}$. All the members of $\mathcal{D}$ are closed subsets. By the way, you take the reversed inclusion as direction on $\mathcal{D}$, so, that is the direct set for us.

Since $\mathcal{C}$ is contained inside $\mathcal{D}$ and I want show the intersection of members of $\mathcal{C}$ non-empty, I am actually going to show that intersection of all the members of $\mathcal{D}$ is non-empty though this is the intersection on a larger family. The intersection will be actually is smaller. If this itself is non empty, then that will be also non empty.

Now, look at members D of $\mathcal{D}$, they are non-empty by definition because I have taken finite intersection of members of $\mathcal{C}$, which has finite introduction property. So, we can take the net S from $\mathcal{D}$ to X to be such that $S(D)$ is inside D for each D. (So, once again, here we are using axiom of choice.)

Suppose x a cluster point of S. We claim that x is inside D for all D inside $\mathcal{D}$.

(This, we have actually seen before. I will repeat it because you might have forgotten how it came.)

Suppose this is not true. Suppose x is not inside some member D belonging to $\mathcal{D}$. Then $X \setminus D$ is a neighbourhood of x, because each member in $\mathcal{D}$ is a closed subset. That is the point. It is a neighbourhood of x, hence by the definition of a cluster point, there exist an E belonging to $\mathcal{D}$ such that for all members D' which follow E, $S(D')$ will belong to the neighbourhood $X \setminus D$. Now choose D' in $\mathcal{D}$ such that it follows both E and D. That means D' is contained in D and hence $X \setminus D$ is contained in $X \setminus D'$. Therefore $S(D')$ is in $X \setminus D'$ as well as in D' which is absurd.

(Reviewer's note: Please note that here are a few typos in the slide).

So, that completes today's, plan, model 31. Thank you. We will meet again, next time.