An introduction to Point-Set-Topology (Part-II)
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Week 2
Lecture 29
Cofinal families subnets

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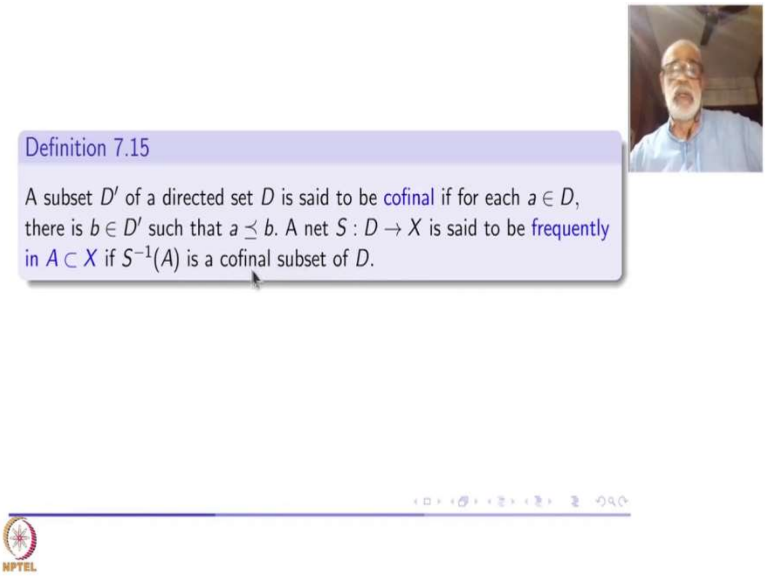

Hello, welcome to NPTEL NOC an introductory course on point set topology part 2. So today we will continue our study of nets: module 29.

Cofinal families subnets etcetera.

So, let us begin with a definition.

A subset D' of a directed set D is set to be cofinal if for each a inside D you must have some b inside D' such that this b follows a. If you have a net S from D to X that will be called frequently inside a subset A of X, if $S^{-1}(A)$ is cofinal subset of D.

This is the same as saying that for each $a \in D$ there is b in D' such that b follows a and $S(b)$ is in A. $S^{-1}(A)$ is cofinal the same thing as now S is frequently inside A.

Which is somewhat weaker than eventually inside A. Eventually constant function is much more stronger than having a just a subsequence, which is a constant. So, this is somewhat like this. Do we also have a concept similar to a subsequence. Yes, there is a such a notion: subnet. But wait for a while.

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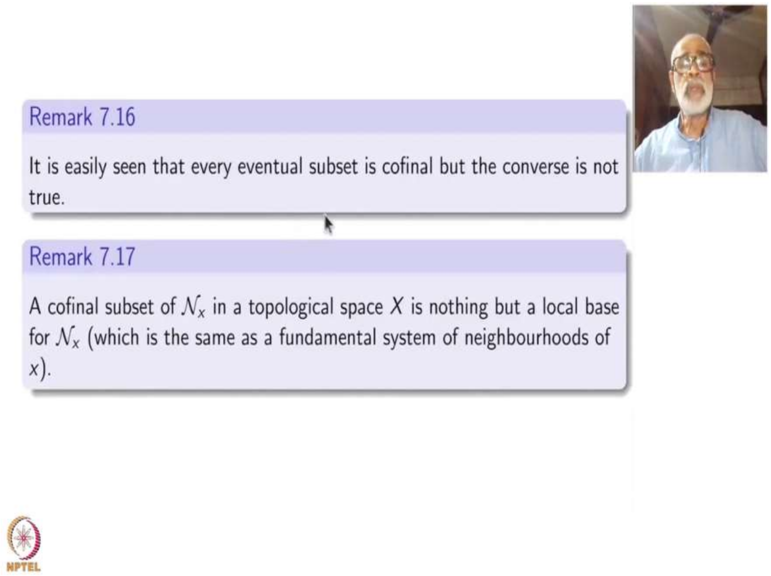

Anyway, it is easily seen that eventual subset is cofinal, but converse is not true. That I have indicated. You can have examples of sequences themselves where this is not true.

So, another remark is that cofinal subset of this $\mathcal{N}_x$ (remember, $\mathcal{N}_x$ is directed set), in a topological space X is nothing but a local base. Remember the definition of a local base? given any open neighborhood of x, there is a member of this family which is contained in the given neighbourhood. So, that will definition of local base. So, every local base is nothing but a cofinal subset of this $\mathcal{N}_x$. Therefore, they will do the job of this $\mathcal{N}_x$ quite often.

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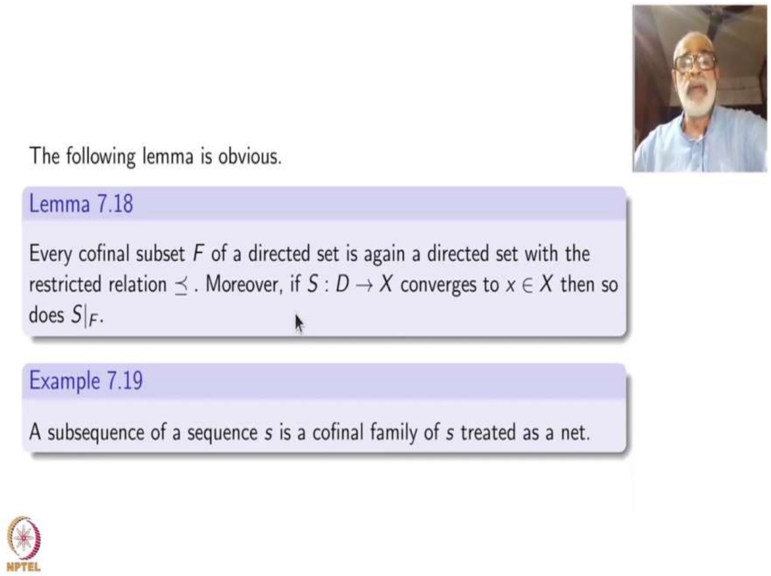

The following lemma is obvious. So, that is precisely what I mean that the cofinal families take care of the convergence properties of the original thing.

Every cofinal subset F of a directed set is again a directed set with the same direction restricted to the subset. Moreover, if S from D to X converges to $x \in X$, then so does S restricted to F.

So, that is why they do the job is what I said, but they may not do everything that the original thing can do.

Of course, if you take all of them together, then they will do the job. That is the whole idea.

Note that a subsequence of a sequence is a cofinal family of S treated as a net. A subsequence can be thought of as a net where sequence itself is thought of as a net. So, both of them you can treat as nets then a subsequence will be a cofinal family. Cofinal word is used even within sequences also anyway.

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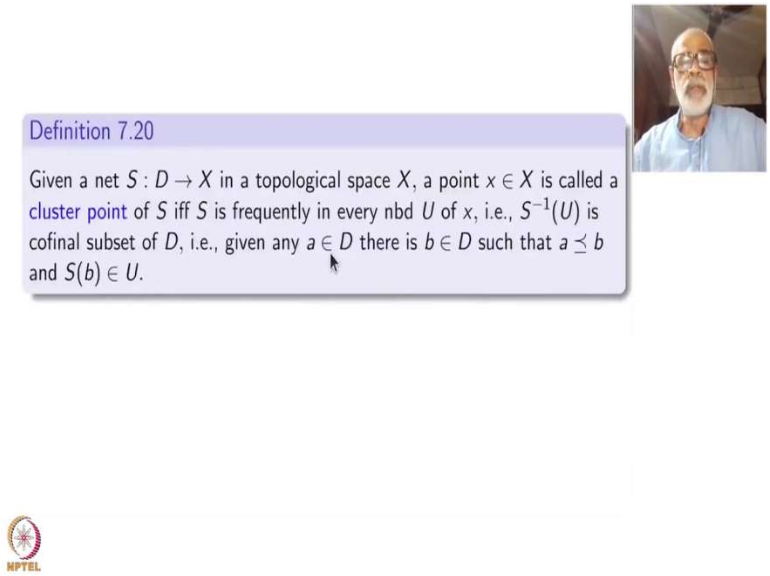

One more definition. Now we are coming closer and closer to the convergence property. Take any net S in a topological space. A point is called a cluster point of S, (earlier we defined a limit of a net) if and only if S is frequently in every neighborhood of x which is the same thing as saying that $S^{-1}(U)$ is a cofinal subset of D, for every U, where U is a neighborhood of x. It is just same thing as saying that given $a \in D$ there must be $a, b \in D$ such that a is less than or equal to b implies $S(b)$ is inside U. So, this is the notion of cluster point just like subsequence converging. So, if a cofinal net is converging to a point, we will call that point a cluster point.

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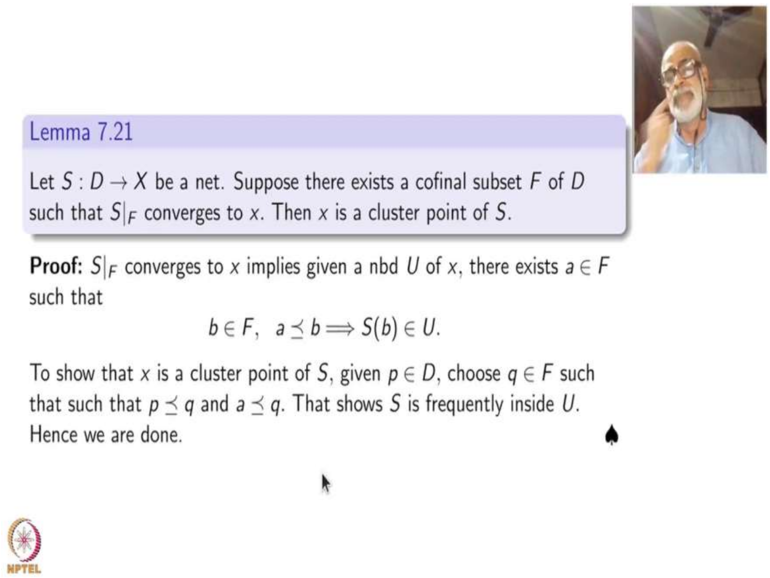

Now, here is the lemma which relates the property of the cluster points: Take any net S. Suppose there exist a cofinal subset F of D such that S restricted to F as a net converges to x. Then x is a cluster point.

Proof: As $S|_F$ converges to x implies that given a neighborhood U of x, there exists a belongs to F, such that for all $b \in F$ which follow $a, S(b)$ is inside U. So, this is the meaning of S restrict to S converges to F.

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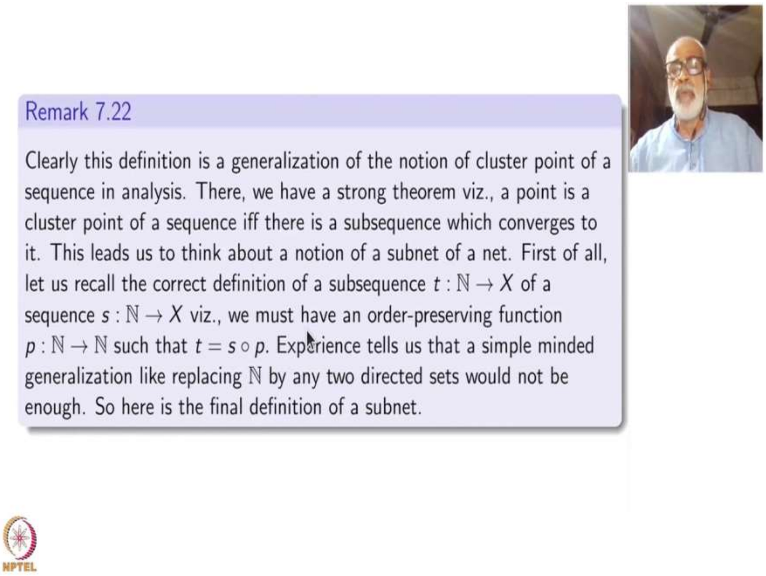

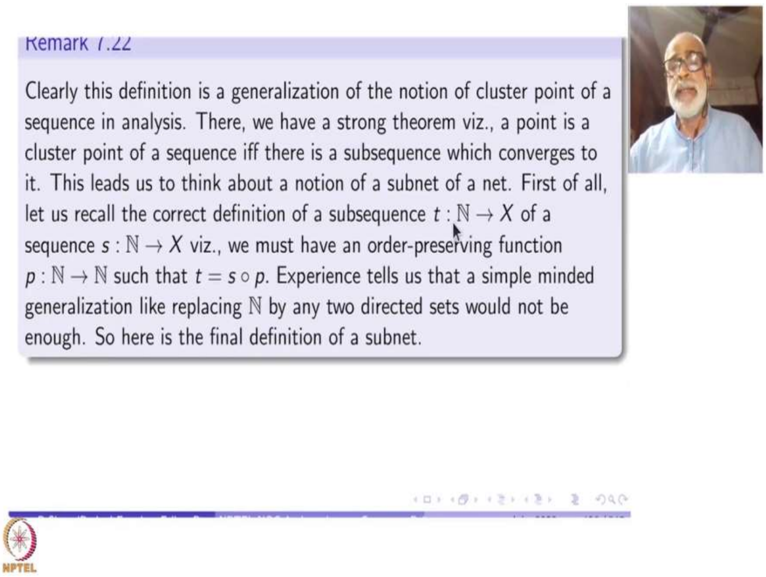

Now to show that x is a cluster point of S, given any p inside D, first choose $q \in F$ such that p less than or equal to q and a is less than or equal to q. You have two points a, and p in D, so you can take q to be bigger than both of them, i.e., which follows both of them. That shows that S is frequently inside U. Hence we are done. See, since a is less than or equal to q, $S(q)$ is inside U.

So, what is the conclusion here? The cluster point of a net is a generalization of a cluster point of a sequence in analysis. There, we have a strong theorem namely a point is a cluster point of a sequence if and only if there exists a subsequence which converges to it. This leads us to think about a notion of a subnet of a net. Like a sub sequence, first of all, let us recall the correct definition of a subsequence, sometimes some books do not give you this.

So, I am going to give you that one, t from $\mathbb{N}$ to X, is a subsequence of a sequence s from $\mathbb{N}$ to X, namely, we must have an order-preserving function p from $\mathbb{N}$ to $\mathbb{N}$, such that this t is s composite p. It is like re-parameterization of the domain, So, order preserving map one way, then this t is a subsequence of s.

Experience tells us that a simple-minded generalization like this replacing $\mathbb{N}$ by any directed subsets would not be enough. See all sequences have their domain the set of natural numbers $\mathbb{N}$. But when you take all nets, the domains keep changing here. So, if you just say there is an order preserving map from one to the other, that may not be enough. So you have to be cautious here. So, we need sufficient experience, in order to come out with this definition. There is scope for improving it or making it more complicated or whatever. I would like to say all these things are not hard and fast rules. So, you are free to think of doing something different also. So, here is final definition as far as the existing theory is concerned.

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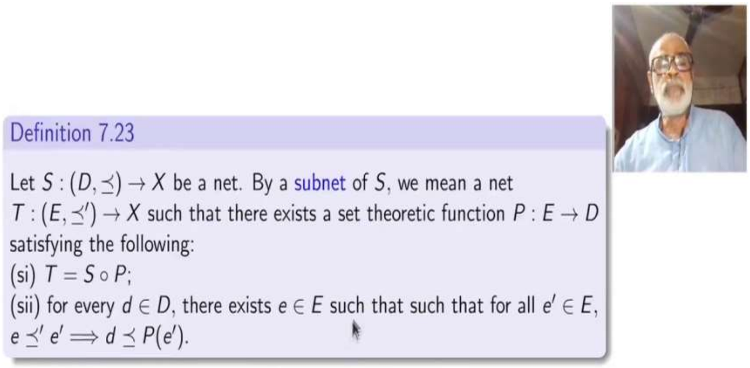

Start with a directed set and a net inside X. By a subnet T of S, we mean another net from another, directed set $(E, \preceq')$ such that there exist a set theoretic function P from E to D, such that

(s1) our T is nothing but $S \circ P$ (this is the first thing).

Now, in the definition of a subsequence subsequence we have order-preserving relation. Here it is replaced by a weaker condition

(s2) says the following: for every $d \in D$, you must have an $e \in E$, such that for all e' which follows e, (e less than or equal to e'), d must be followed by $P(e')$.

So, $P(e')$ should come after d. So, that is a condition on P.

So, (s2) is reflecting cofinality here. Of course, we have to bring P here, if P is the inclusion map, (s2) will be just like a corfinal condition. If E' is a sub-direction of of D, then it would have been the same thing. But we want to cover more general set-up. So, the two nets are related by a function P which has this property.

So, this becomes a far superior definition indeed. Let us see whether it works or not.

If you make too weak a definition then it may not be good enough. That is right.

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Remark 7.24

The first condition (si) is, of course, as in the case of sequences but we have given up on the order preserving requirement from P. Instead (sii) ensures that

'as $e' \to \infty$ we have , $P(e') \to \infty$.'

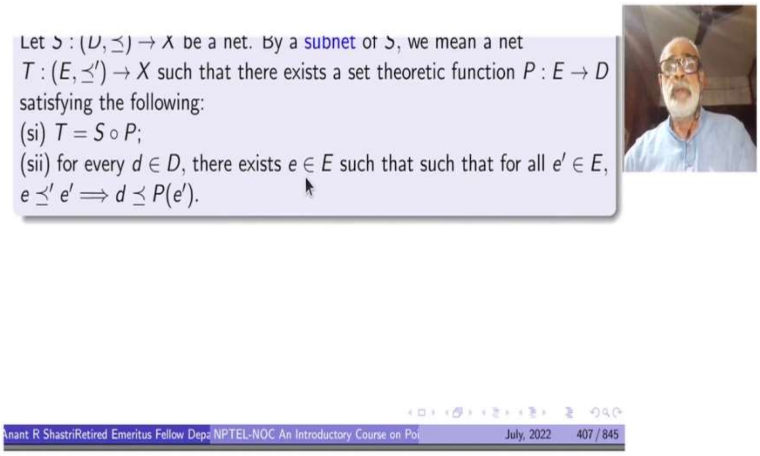
Notice that every subsequence of a sequence s is a subnet of the net s. Also, we can easily get examples of subnets of a sequence which are not subsequences. Thus this definition of subnet is a liberal generalization of a subsequence.

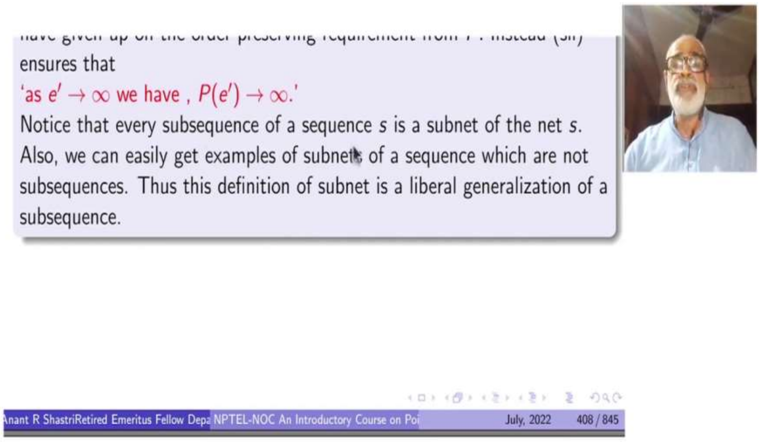
Definition 7.23

Let $S : (D, \preceq) \to X$ be a net. By a subnet of S, we mean a net $T : (E, \preceq') \to X$ such that there exists a set theoretic function $P : E \to D$ satisfying the following:

(si) $T = S \circ P$;

(sii) for every $d \in D$, there exists $e \in E$ such that such that for all $e' \in E$, $e \preceq' e' \implies d \preceq P(e')$.





The first condition says as in the case of a subsequence. But we have given up on the condition of order-preserving requirement of P. Instead of that the second condition is something like if e' goes to infinity, $P(e')$ also goes to infinity. You see e''s is large should imply $P(e')$ lshould be large. That is like $\epsilon - \delta$ definition of function going to infinity if you should do correctly. So, it is similar to that.

So, analysis is the guide for all new definitions. After all these are derived from experience in analysis that is all. Notice that every subsequence of sequence s, is a subnet when you treat s as a net. Also, we can easily get examples of subnets of a sequence which are not subsequences. Very easy to get just obliterate 1, 1 element. So, that is not order preserving. It has no effect on the rest of them, it will be a subnet, but it will not be a subsequence. Thus, this definition of subnet is a liberal generalization of a sub sequence.

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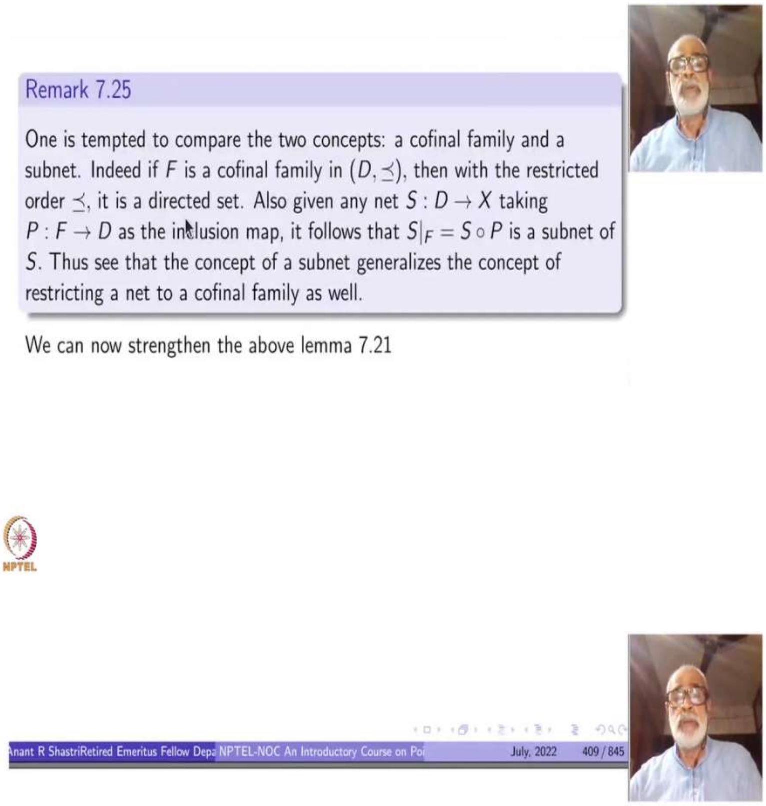

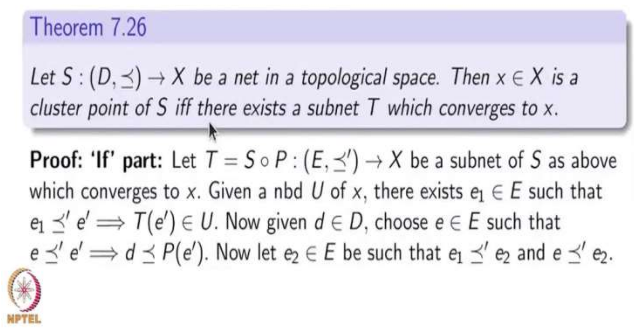

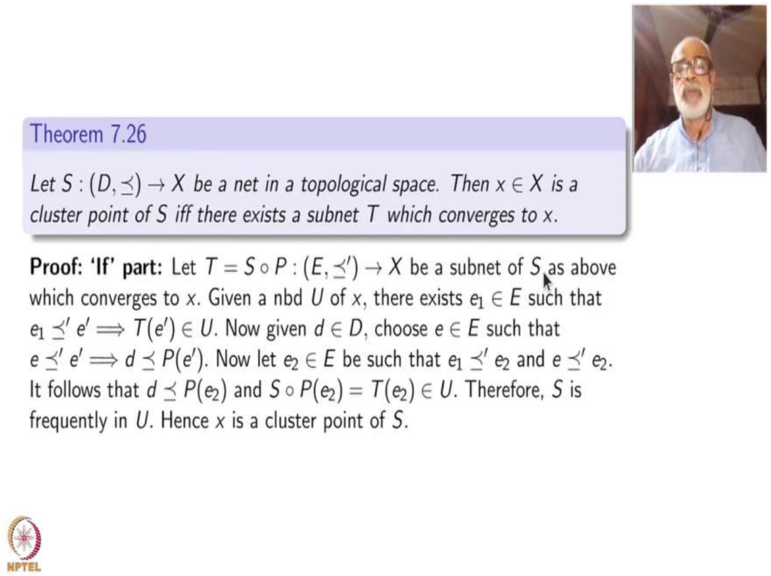

So, one more remark here. One is tempted to compare the two concepts a cofinal family and a subnet. Indeed, if f is a cofinal family in D, (whatever partial order direction) then with the restricted order it is a directed set. Also, given any net S from D to X taking, P as the inclusion map, it follows that S restricted to F, is $S \circ P$, is a subnet. So, that is what I have told you that cofinal families do give you subnets. Thus, we see that the concept of a subnet generalizes the concept of restricting a net to a cofinal family as well.

So, we can now strengthen the earlier lemma 7.21, that we had proved, namely, bring in `if and only if' etc.

Take any net s in a topological space X. Then a point $x \in X$ is a cluster point of this net S if and only if there exists a subnet T of S converging to x.

We had seen a weaker version of `if' part, i.e., there is a cofinal family $F$ such that $S$ restricted to $F$ converges to $x$. Now, in terms of subnets you will get `if and only if'.

Let us look at the `if' part. If you have subnet converging to x, then we have to show that x must be a cluster point of S. So, let $T = S \circ P$, from E to X etc. That T be a subnet of S and let us assume that T converges to x. Take a neighborhood U of x. Then there exists some e_1 inside E such that e_1 is less than or equal to e' implies $T(e')$ is inside U. This is from the convergence of T.

So, d is less than or equal to $P(e_2)$ and $(S \circ P)(e_2)$ which is $T(e_2)$, that will be inside U, because as soon as something is bigger than e_1, T of that is inside U. Therefore, S is frequently inside U. Hence x is a cluster point of S. So, one way is done.

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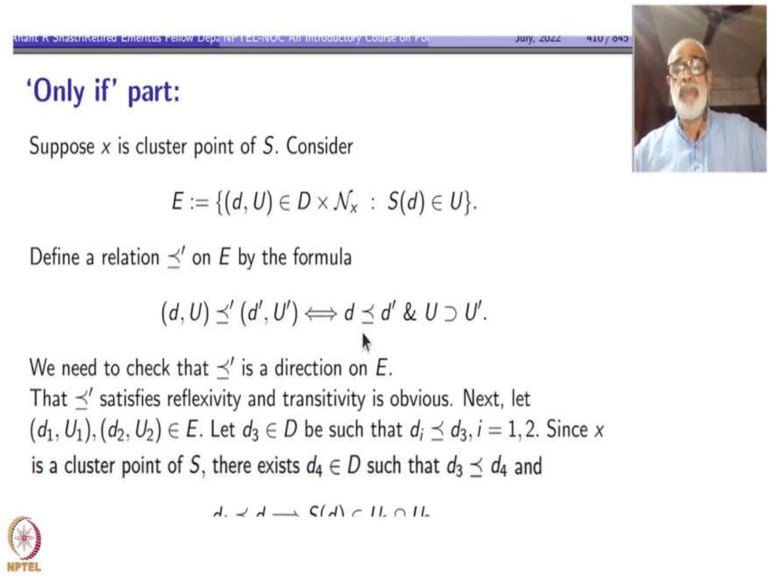

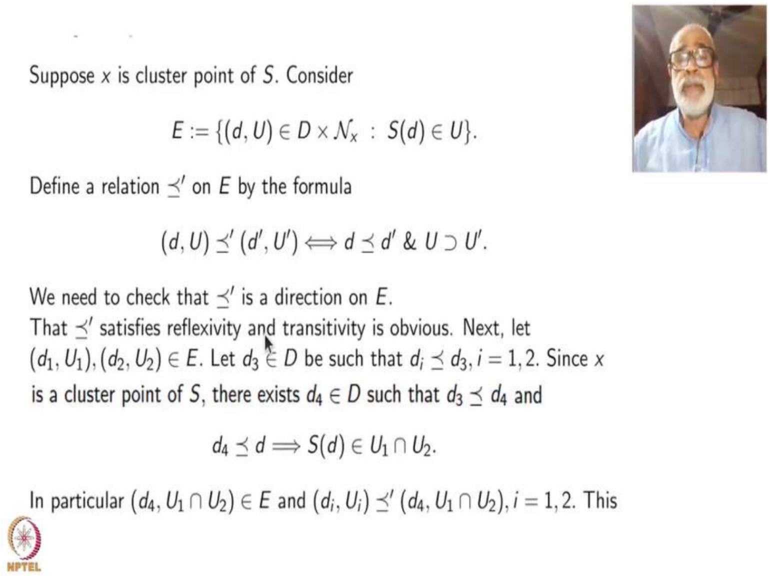

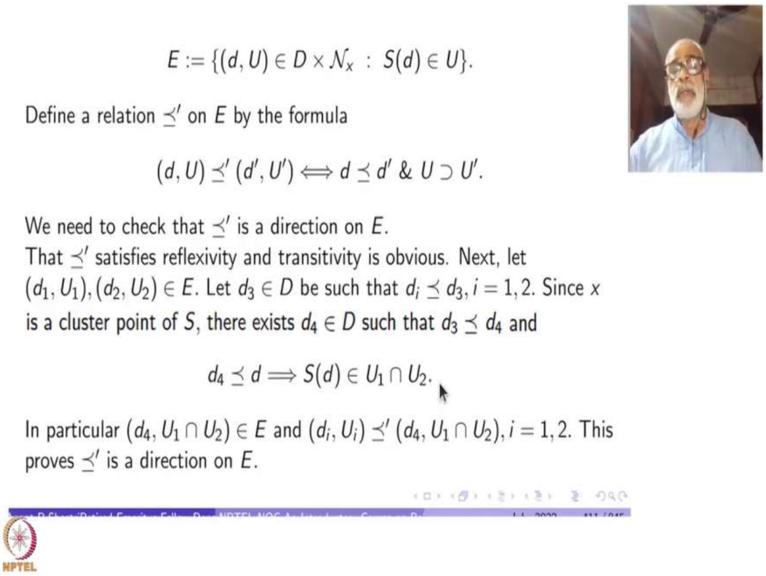

Now, the `only if' part. You have to work little harder.

Suppose x is a cluster point of S, I have to construct a subnet which converges to x. Such a thing is not possible in an arbitrary topological space with sequences. There might not be enough sequences at all. So, this is something in which the nets have an advantage over sequences. So, let us see how it is done.

We take ordered pairs (d, U) belonging to $D \times \mathcal{N}_x$, with the property that S of the first coordinate, viz., $S(d)$ belongs to U. What are U's? U ranges over neighborhoods of x, Obviously, every U, every element of $\mathcal{N}_x$ is non-empty.

So, that is my definition of the set E. Now, what is the relation in E? Again take the strict relation coming from $D \times \mathcal{N}_x$. $\mathcal{N}_x$ has a relation, D has a relation take the strict relation on the product, viz., (d, U) is less than or equal to (d', U') if and only if d is less than equal to d' with respect to the order in D and with respect to the direction in $\mathcal{N}_x$ viz., the reverse inclusion, U contains U'.

We need to check that this is a direction on E. This is not true in general. We have to use somehow the property of S, namely, x is a cluster point of S. Only because of that this is working.

In any case transitivity and reflexivity are obvious there is no problem, they not depend upon any special property of S; they are true inside $D \times \mathcal{N}_x$ and so, they will be true hear also.

The problem is about the third property namely direction. Suppose (d_1, U_1) and (d_2, U_2) belong to E. We must find a (d_3, U_3) which is bigger than both (or follows both the elements).

So, first of all find $d_3 \in D$, such that both d_1 and d_2 are less than or equal to d_3. So, this is possible because D is a directed set. Now, since x is a cluster point of S, you will get a d_4 inside D, such that d_4 less than or equal to d would imply $S(d)$ is inside $U_1 \cap U_2$. We have $U_1 \cap U_2$ as a neighborhood of x for which I must have a d_4 as above. That just means that $(d_4, U_1 \cap U_2)$ is an element of E because, $S(d_4)$ itself is inside $U_1 \cap U_2$. It will be bigger than both (d_1, U_1) and (d_2, U_2). Why? because d_1 for example is less than d_4, d_1 is less than equal to d_3 and d_3 is less than equal to d_4. This proves that this relation is a direction on E.

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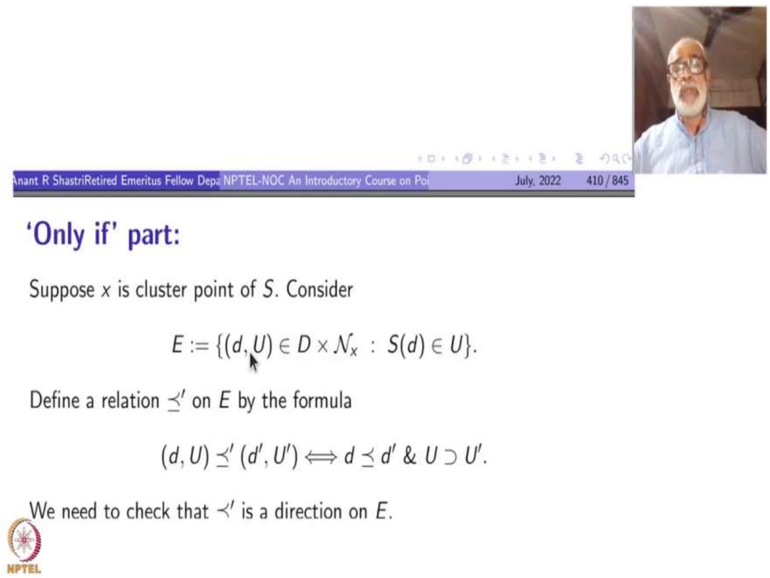

proves $\preceq'$ is a direction on E.

Take $P : E \to D$ to be the first projection, viz., $P(d, U) = d$. Let us verify that P satisfies (sii): Given $d \in D$ choose $(d, X) \in E$. Then $(d, X) \preceq' (d', U)$ implies $P(d', U) = d'$ and $d \prec d'$. This is (sii).
Now take $T = S \circ P$. So, we have constructed a subnet of S. It remains to see that T converges to x.

Suppose x is cluster point of S. Consider

$$E := \{(d, U) \in D \times \mathcal{N}_x \ : \ S(d) \in U\}.$$

Define a relation $\preceq'$ on E by the formula

$$(d, U) \preceq' (d', U') \iff d \preceq d' \ \& \ U \supset U'.$$

We need to check that $\preceq'$ is a direction on E.
That $\preceq'$ satisfies reflexivity and transitivity is obvious. Next, let $(d_1, U_1), (d_2, U_2) \in E$. Let $d_3 \in D$ be such that $d_i \preceq d_3, i = 1, 2$. Since x is a cluster point of S, there exists $d_4 \in D$ such that $d_3 \preceq d_4$ and

$$d_4 \preceq d \implies S(d) \in U_1 \cap U_2.$$

In particular $(d_4, U_1 \cap U_2) \in E$ and $(d_i, U_i) \preceq' (d_4, U_1 \cap U_2), i = 1, 2$. This

proves $\preceq'$ is a direction on E.

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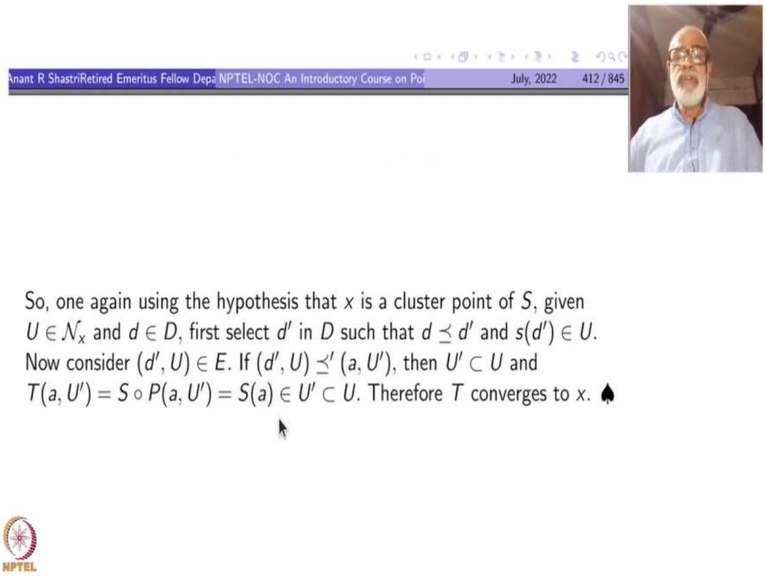
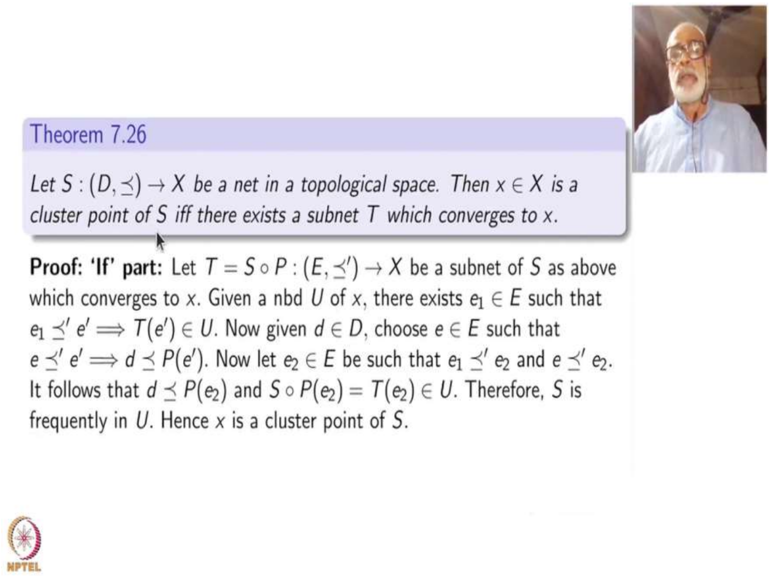
Take $P : E \to D$ to be the first projection, viz., $P(d, U) = d$. Let us verify that P satisfies (sii): Given $d \in D$ choose $(d, X) \in E$. Then $(d, X) \preceq' (d', U)$ implies $P(d', U) = d'$ and $d \prec d'$. This is (sii).
Now take $T = S \circ P$. So, we have constructed a subnet of S. It remains to see that T converges to x.





So, we have constructed a directed set. We have yet to construct a net that will be a subnet of S. For that I take P from E to D to be the first projection; $P(d, U) = d$. E is after all a subset of the product of two sets, take the first coordinate projection. Let us verify that P satisfies (sii), only then this P followed by S will be a subnet of S.

Given a d inside D, first I have to choose a neighbourhood of x, so, choose the whole of X, then d this will be inside X already. So, (x, X) is a member of E. Now (d, X) is less than or equal to (d', U') just means that d is less than or equal to d', which is the same as saying $P(d, X)$ is less than or equal to $P(d', U')$. Because d must be less than or equal to d' is built in here. So, that verifies (sii).

So, once you have that, I can take T is equal to $S \circ P$. So, we have constructed a subnet of S. All that is fine, but we have to verify that this T converges to x. Starting with a cluster point of S, we have got a subnet I will now show that, this converges to x. That is not difficult. Once again use the hypothesis that x is a cluster point of S. Given U inside $\mathcal{N}_x$, there is a $d \in D$, such that, d is less than or equal to d' implies $S(d')$ is inside U.

Now, (d, U) is inside E, because $S(d)$ itself is inside U. This is a member of E which will satisfy the convergence condition for T. For, suppose (d, U) is less than or equal to (d', U') in E. This implies d less than or equal to d' and hence by the choice of d we have $S(d')$ is inside U. But then $S(d') = S \circ P(d', U')$. Therefore, T converges to x.

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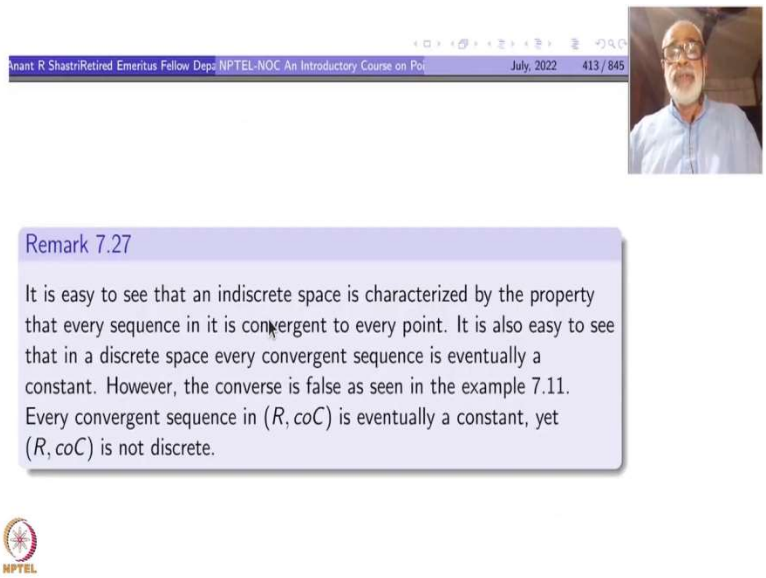

It is easy to see that, in an indiscrete space, we have this property, namely, every sequence in it is convergent to every point. And you can characterize indiscrete space by this property. Namely, if every sequence converges to every point, then it must be an indiscrete space. So, this is an easy thing, which you must have seen in part I.

It is also easy to see that in a discrete space, on the contrary, (see, I am taking these 2 extreme examples and what sequences can do to in them), if you take a discrete space, every convergent sequence is eventually a constant. However, the converse is false as seen in the earlier example, we have seen that one, namely, the uncountable set with the cocountable topology. Every convergent sequence in $(\mathbb{R}, coC)$ is eventually a constant yet, $(\mathbb{R}, coC)$ is not discrete space.

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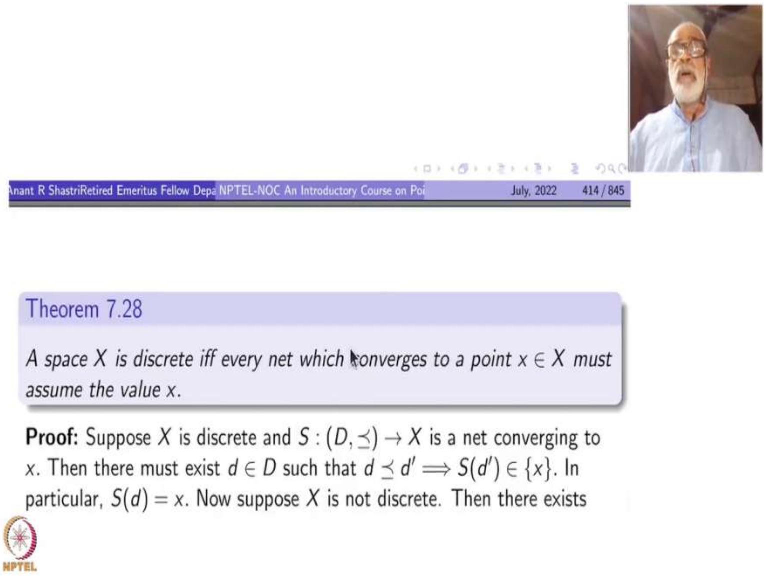

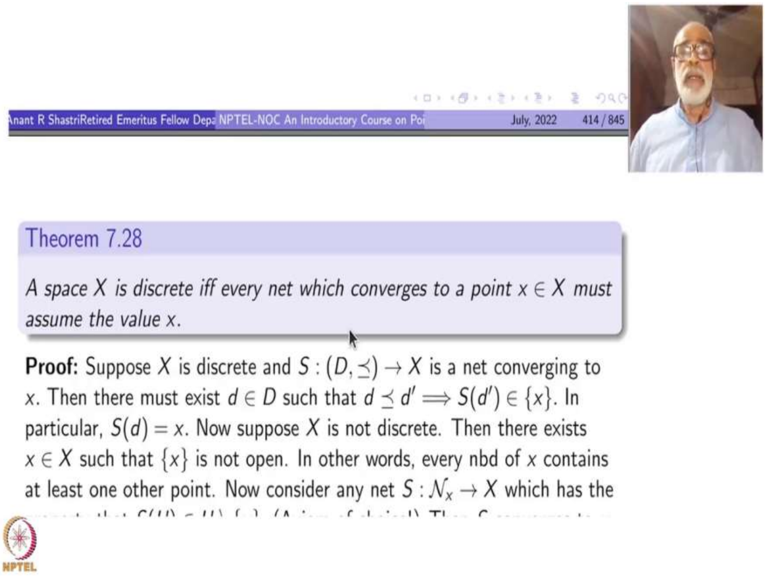

So, what is the role of a net here? The net, on the other hand has this property. A space X is discrete, if and only if, every net which converges to a point x must assume the value x, that is all. We do not say `eventually a constant x'; just that S assumes the value x. If this happens for every net, that space must be discrete.

So, this is a powerful characterization of discrete spaces in terms of convergence of nets. A space X is discrete if and only if every net which converges to a point must have the value x, that is all.

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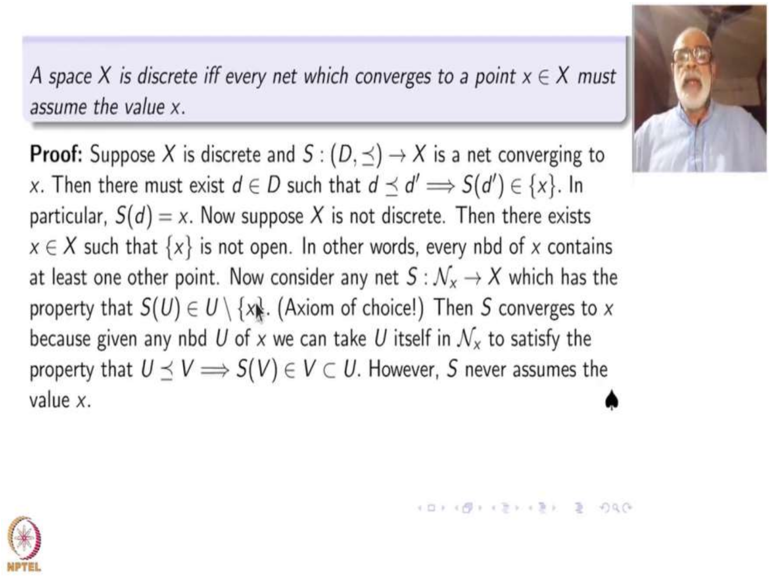

So, let us prove this one. Suppose X is discrete and S is a net converging to some point x. Then, there must exist d belonging to D, such that d is less than or equal to d' implies $S(d')$ is inside this neighborhood $\{x\}$, because every singleton is open. So, I am choosing this neighborhood and for that I must get a some d with with this property. In particular, putting d' equal to d here, $S(d)$ must be equal to x; there is no other element here.

So, the value is assumed over.

Now comes the converse. Suppose x is not a discrete space. Then we will produce a net which converges to some point in X and yet it does not assume that value at all. By the way, there are many sequences themselves, which are convergent and which do not assume the limit value. The sequence $\{1/n\}$, for example. So, it is not a surprise. You should realize that. But this ordinary property characterizes the discrete spaces is something new out of nets not out of sequences. So, you pay attention to that.

So, let us prove the converse here. Suppose X is not a discrete. Then there exists a point x such that singleton $\{x\}$ is not open in X, because if every singleton is open then X will be a discrete space.

In other words, every neighborhood of x contains at least one other point. We can now construct a net S from this neighborhood system $\mathcal{N}_x$ to X, which has the property that $S(U)$ is inside $U \setminus \{x\}$; $U \setminus \{x\}$ is non-empty, just by our choice of x inside a non-discrete space X (So, this function again exists because of axiom of choice.)

This S converges to x. That is easy to see. However, S never assumes the value x, because it is always inside $U \setminus \{x\}$, for all U, all the time.

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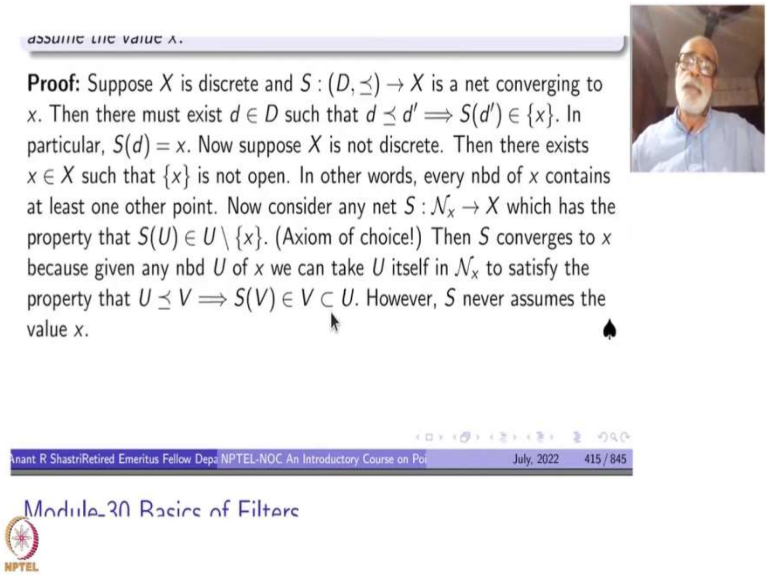

So, one can go on doing quite a bit of net theory, but I would like to stop here and then take up the study of filters and then as declared earlier, we will study a little bit of nets also in between, namely, whenever the properties of filters comes close to the properties of nets also. In particular, there will be many more results on filter than just what you have seen for nets. Thank you.