An introduction to Point-Set-Topology (Part II)
Professor Anant R. Shastri
Department of Mathematics
Indian Institute of Technology, Bombay
Week 2 Lecture 28
Nets

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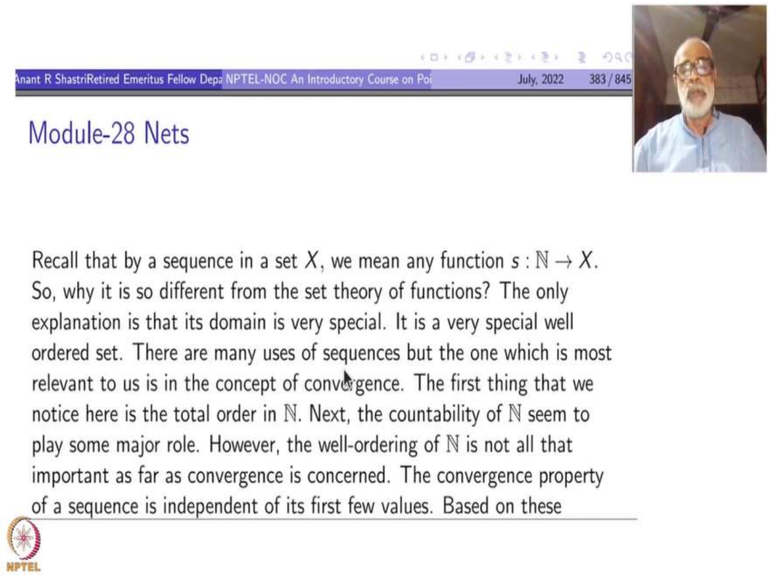

Hello welcome to NPTEL NOC an introductory course on point set topology Part 2, module 28. This chapter consists of Nets and Filters. So, module 28 will be on Nets.

Let me tell you a little bit background here. Recall that by a sequence in a set X, we mean any functions with the domain the set of natural numbers and the codomain X. So, why the study of sequences is not exactly a part of the study of set theoretic functions? Why it is so different?

A sequence is just a function the only specialty is that the natural numbers have a natural order and that too is actually a total order. More than that it is a well-order. The set of natural numbers is well order set. That is the explanation that a sequence is more special than just an ordinary function. Its domain being a well ordered set, allows you to make so many other mathematical statements about a sequence, beginning with the principle of mathematical induction.

So, there are many uses of sequences, but the one which is most relevant to us is the concept of concept of convergence. The first thing that we notice here is the total order of the natural numbers, we do not worry about the well-orderness so much. The second thing is countability of the natural numbers. That also seems to play a major role.

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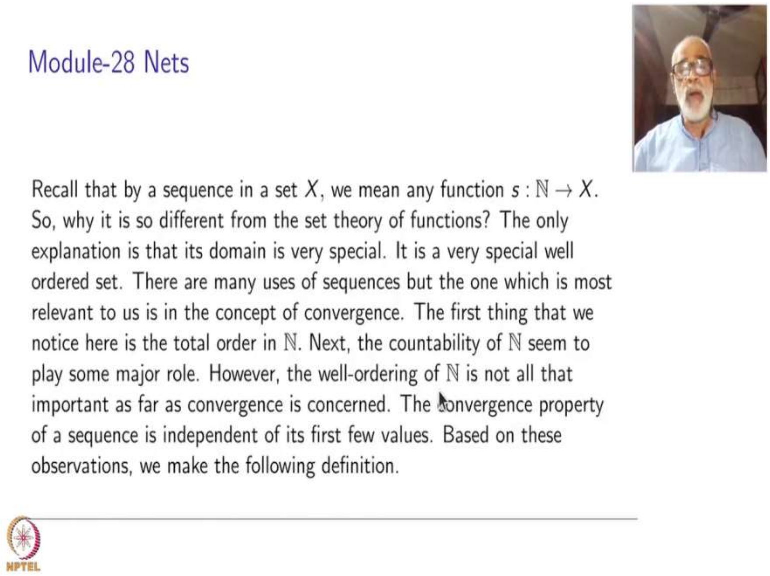

However, the well ordering of natural numbers, as I have told you, is not at all important for the convergence theory of sequences. Indeed, convergence property of a sequence is independent of the first few values of the sequence, to be very precise. That is why I think that the well-ordering is not all that important for discussion convergence.

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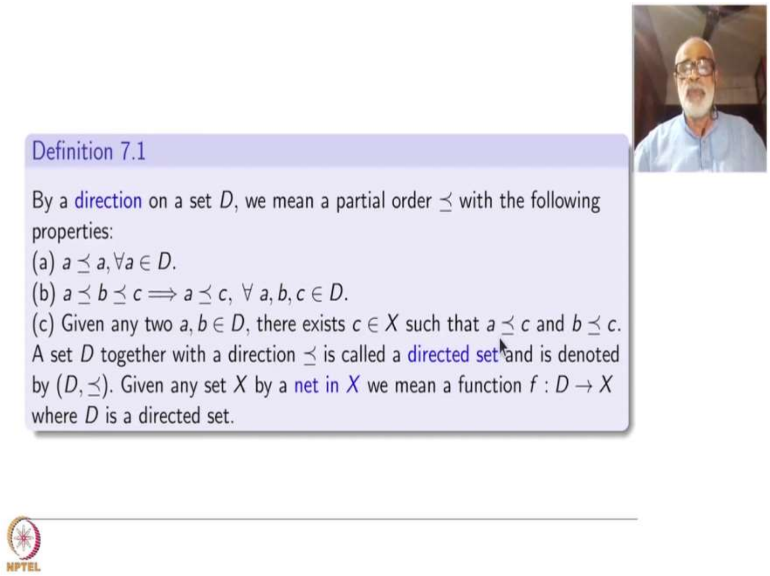

Based on these observation, we make the following definition wherein we want to enlarge the scope of this convergence theory. So, the enlargement or the generalization comes in the domain of the sequences. Naturally, when the domain is changed, you want to change the name also.

By direction on a set D, we mean a partial order, just to distinguish it from the standard partial order on the real numbers and natural numbers, we are going to use this latex notation. You can read it just as `$a$ followed by $b$', rather that saying `a is less than or equal to b' which could be somewhat misleading terminology. We are not comparing any quantitative thing here. But partial order is a partial order anyway and we shall use the terminolgy `a less than or equal to b' as well. So, this is a partial order with the following properties: (When somebody says partial order people already mean something, it does not matter. Actually, I will take a binary operation on D which just means it is a subset of $D \times D$ satisfying the following properties:)

(a) a is always followed by a. That is reflexivity. (Once you use the word `partial order' you I do not have to say reflexivity separately, that is included in the definition of partial order but I just wanted to be very clear about what we define here.)

(b) The second property is that a less than equal to b less than or equal to c implies a is less than or equal to c. That is transitivity.

Then there is a third one which makes a partial order into a `direction'. So, pay attention to it.

(c) Given any two elements a, b in D, there is a third one c in D which sits over both a and b, or you may say, which follows both of them or you may say which is greater than both of them.

If a binary relation on D satisfies the above three conditions then I will call it a direction.

A set together with a direction is called a directed set and is denoted by usually by $(D, \preceq)$. Quite often, as usual, we will not mention the direction separately and say that D is a directed set, especially, when the direction is understood.

Given any set X, (now, I am going to come to the sequence part now, replacing it by the word `net',) by a net in X, we mean a function from a directed set D to X. That is all.

Once again, a word of caution about my terminology here.

You see in general, a partial order to be anti symmetric also. In our case, if we assume anti symmetry, no harm is done, but the general definition of directed set does not make this assumption namely, there is no anti-symmetry assumption. It may happen that a may be less than or equal to b and b is also less than or equal to a, yet a may not be equal to b; that is allowed.

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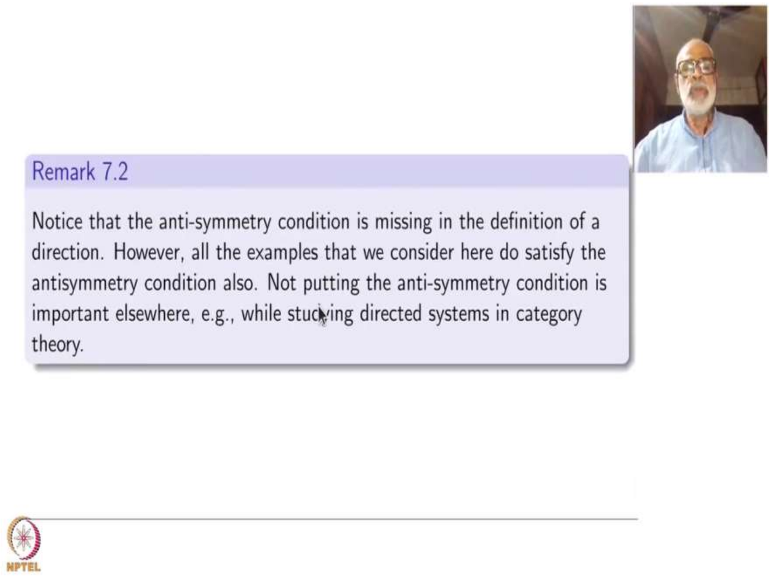

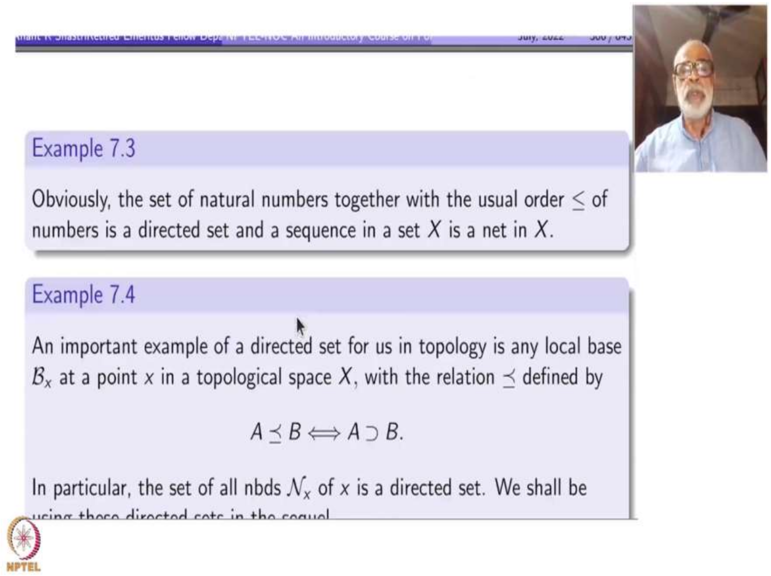

Such a generalization is not of much importance for us. So, if you do not want to bother about such details, you can assume anti-symmetry also, in the definition of a direction, no problem because all our examples are anti-symmetry. The general definition is necessary in what are called directed systems in the category theory.

Now, let us have some examples.

The set of natural numbers which was a motivating example is a directed set with the usual order. It has many more other properties those things we have sidelined for a moment now. So, a sequence is a net.

But now, there will be many more nets than sequences. So, we shall study them. First let us just concentrate on directed sets. Another important example of a directed set for us in topology is: any local base $\mathcal{B}_x$ at a point $x \in X$, where X is a topological space, with the usual inclusion of sets as the relation, only the thing is that I take it in the reverse order, so, I keep calling it reversed inclusion: A is less than or equal to B now implies for us and implied by B is a subset of A.

So, people do use the notation instead of this $\prec$ they will use the reversed $\prec$. There is absolutely no loss of generality at all. On the other hand, in general for subsets of a given set, following the common usage, it would have been appropriate to say A greater than equal to B if and only if A contains B. That is alright if you are studying only the partial ordering. Here we want to concentrate on directions.

I should say that by spending this much of time on our convention, I hope I have removed any confusion here.

In particular, the set of all neighbourhoods of point x in a topological space X, will be denoted by $\mathcal{N}_x$, and it will form a directed set under the reversed inclusion. We shall be using these directed sets in the sequel. These are important for us.

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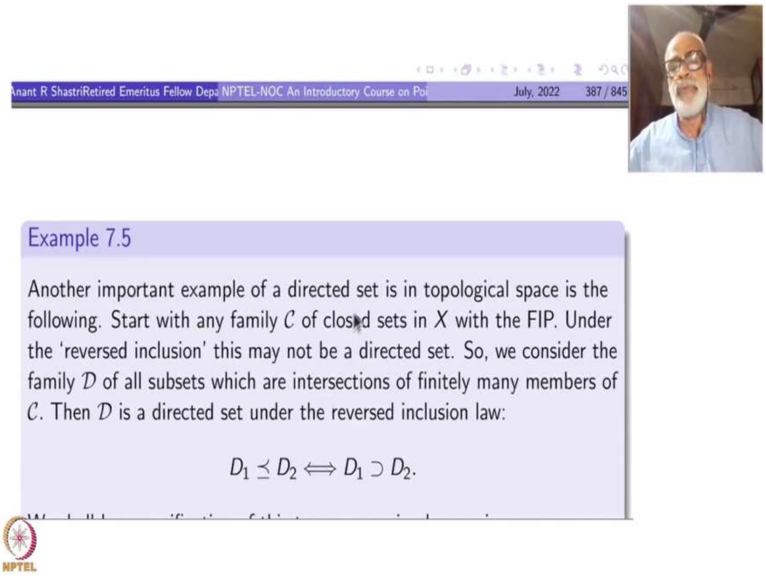

Some more examples: Another important example of a directed set in a topological space is the following: Start with any family $\mathcal{C}$ of closed subsets of X, with the finite intersection property. What is this finite intersection property? Intersection of any finitely many members of $\mathcal{C}$ must be non-empty. In particular, all members of $\mathcal{C}$ must be non-empty. Under the reversed inclusion this may not be a directed set.

So, we have an opportunity to study this kind of families. So, we will not leave it like this, but we will make it into a directed set by considering a larger family $\mathcal{D}$ of all subsets of X, which are intersections of finitely many members of $\mathcal{C}$. Put all of them inside $\mathcal{D}$. So, you have enlarged the family $\mathcal{C}$ to this family $\mathcal{D}$, for all members of $\mathcal{C}$ are in $\mathcal{D}$ as well because I can take just each one of them as being intersection with itself. All those are there, two of them are you take intersection may not be there put that one also like that finitely many intersections should be put inside $\mathcal{D}$. Then $\mathcal{D}$ becomes a directed set under the reversed inclusion. When you have two elements D_1 and D_2 in $\mathcal{D}$, which may not be comparable, you take $D_1 \cap D_2$ that will be a member of $\mathcal{D}$ and smaller than both of D_1 and D_2. 1 So, the condition(c) for direction is satisfied.

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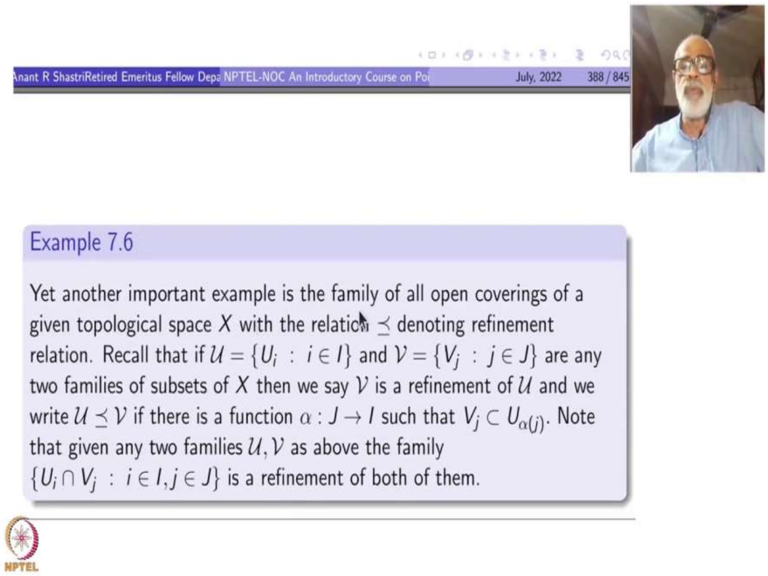

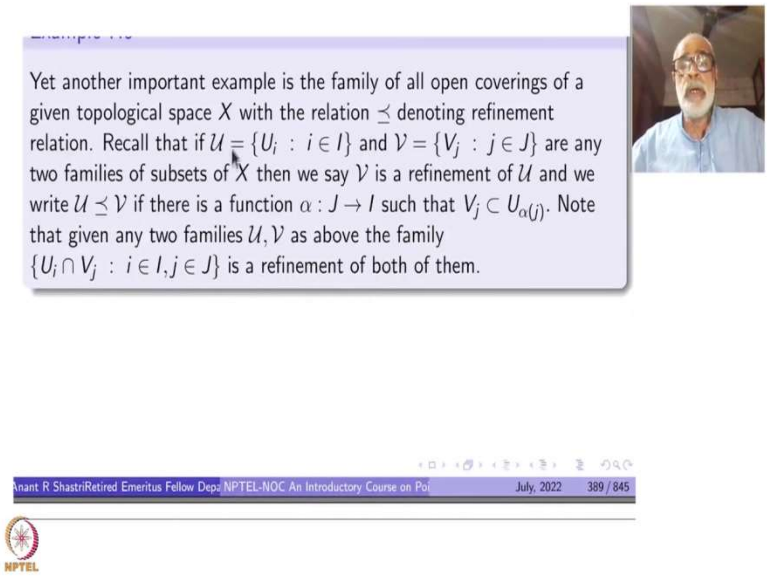

Yet another example is the family of all open coverings of a given topological space X with the relation, I am going to define this refinement relation now, what are the meaning of refinement relation? Suppose $\mathcal{U}$ and $\mathcal{V}$ are open coverings of X. Say $\mathcal{U}$ is less than or equal to ($\preceq$) $\mathcal{V}$ if $\mathcal{V}$ is a refinement of $\mathcal{U}$, viz., each member of $\mathcal{V}$ is contained in a member of $\mathcal{U}$; i.e., there is a refinement function α from the indexing set J of $\mathcal{V}$ to the indexing set I of $\mathcal{U}$ such that V_j is contained inside $U_{\alpha(j)}$. So, this is the refinement relation. Once you have 2 families like this, you can take $U_i \cap V_j$ where i runs over I and j runs over J, this will be a common refinement. So, that is why this is directed set. So, notice that here neither U is contained inside V nor V is contained inside U, members of V are contained inside some members each member is going to some member. So, that is the relation here.

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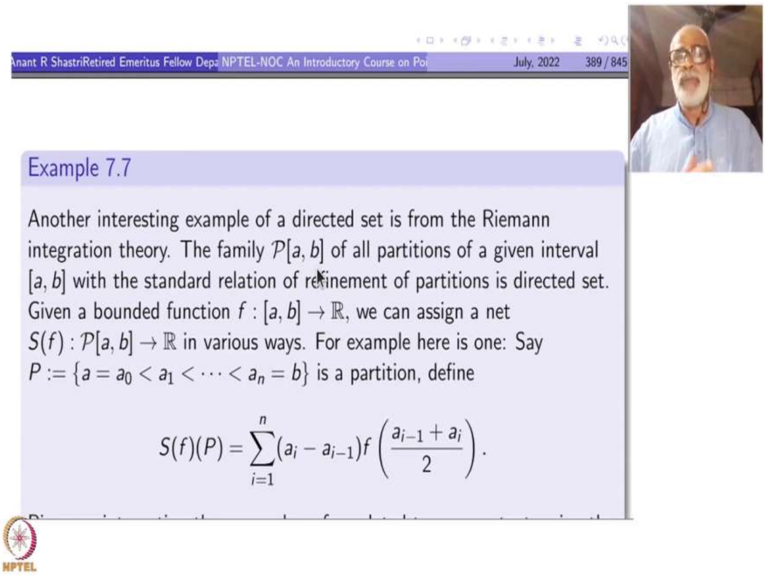

Another interesting example of a directed set which is more or less the mother of all these theories comes in Riemann integration theory. How? You start with a bounded function on a closed interval and then you start cutting down the interval into partitions. Then you are not satisfied with that you take any two partitions you want a refinement of both of them and that is precisely leads to the notion of directed system, family of all partitions with the relation of refinement of partitions.

You know what is a refinement of partition. You put some extra points in between to get another partition, that is a refinement. So, if we have two arbitrary partitions, you can always get a common refinement by interlacing the points of both the partitions.

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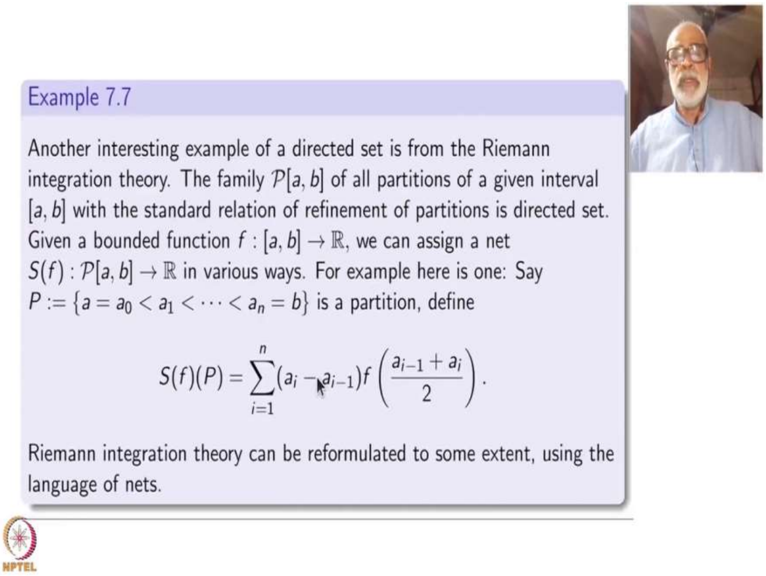

And what I want to just tell you I cannot go on doing Riemann theory here is the Riemann integration theory can be formulated beneficially if you use the terminology of directed sets, and directed systems.

So, like this you can mention other examples also from analysis. I will tell you what, directed systems are used very much in advanced topology as well as in complex analysis. In complex analysis, you are dividing rectangles or domains inside the complex plane into rectangles, smaller and smaller rectangles and so on. So, there is this famous method known as Runge's trick, you can use this to prove many things in complex analysis.

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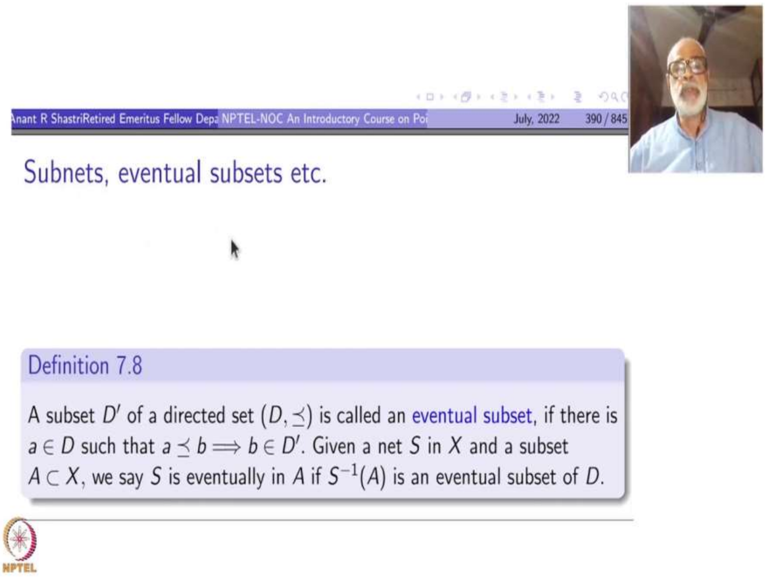

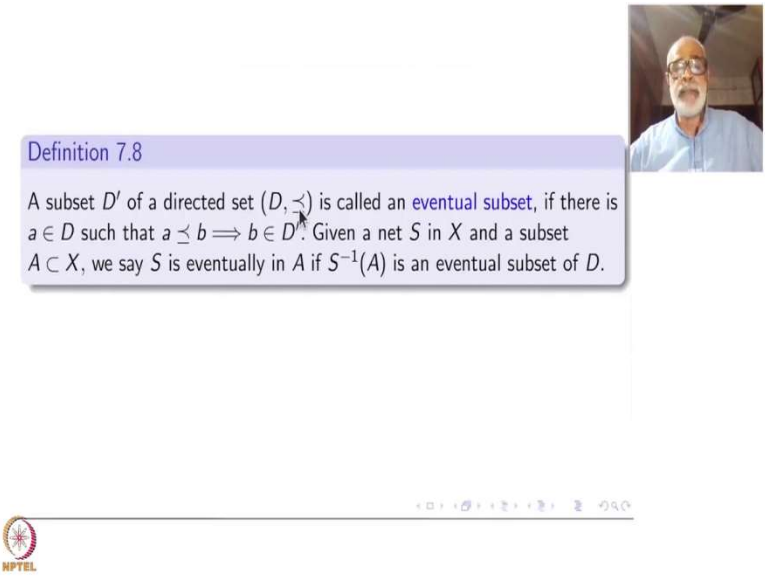

So, let us proceed with certain notions of sequences, which we use in the convergence theory and try to modify them or adopt them for the case of nets. So, here I am going to introduce subnets eventual subnets and so on.

Start with any directed set D, take a subset D' of D and then you want to tell what kind of subset you are taking.

It will be called an eventual subset if there is a inside D such that a is less than equal to b would imply b is inside D'. There is one point in D, everything following that point, all those points must be inside D'. So, such a thing is called an eventual subset.

We can apply this idea to nets also, borrowing it from sequences, what determine the limit property of the sequences, after all. This what I told you, first few values of a sequence do not matter as far as the limiting behavior of the sequence is concerned. While dealing with a net we cannot talk about first few and so on here.

So, you begin with some a and you are not worried about what happens before a. Everything after a must be inside D' such thing is called eventual subset.

Given a net S in X and a subset A of X, S is a net means what? S is a function from D to X now, we say S is eventually in A, if $S^{-1}(A)$, set of all points of D which come inside A under S, this must be an eventual subset of the domain D of the function S.

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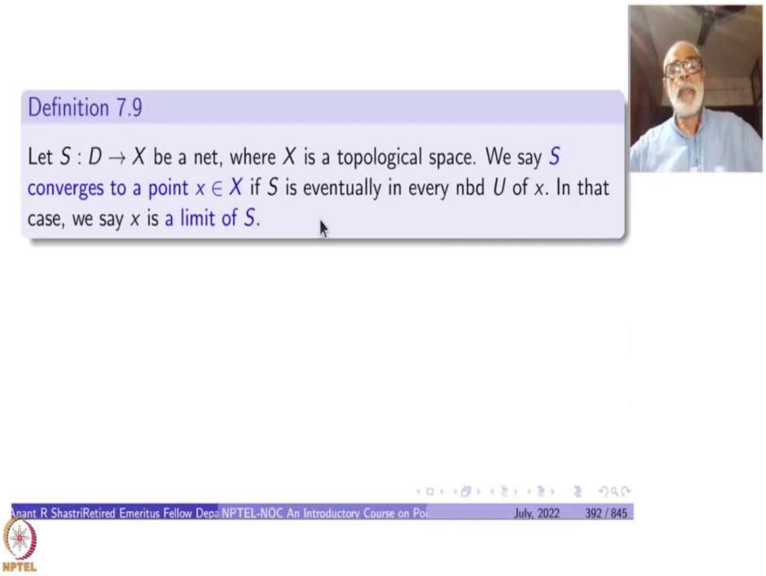

Next we consider a net S taking values in a topological space. See eventuality of net has nothing to do with the topology on X; it is a part of the definition of net. Now, we are coming to the convergence theory here and so X must be a topological space.

We say S converges to a point x inside X, if S is eventually in every neighborhood of x. (That is the meaning of convergence.) In that case, we say x is a limit point of S.

So, I will explain this definition. It just means that given a neighborhood U of x in X, $S^{-1}(U)$ must be an eventual subset of D, which is the same thing saying that given any neighborhood U of x, there exist a inside D such that a follows b implies $S(b)$ is inside U.

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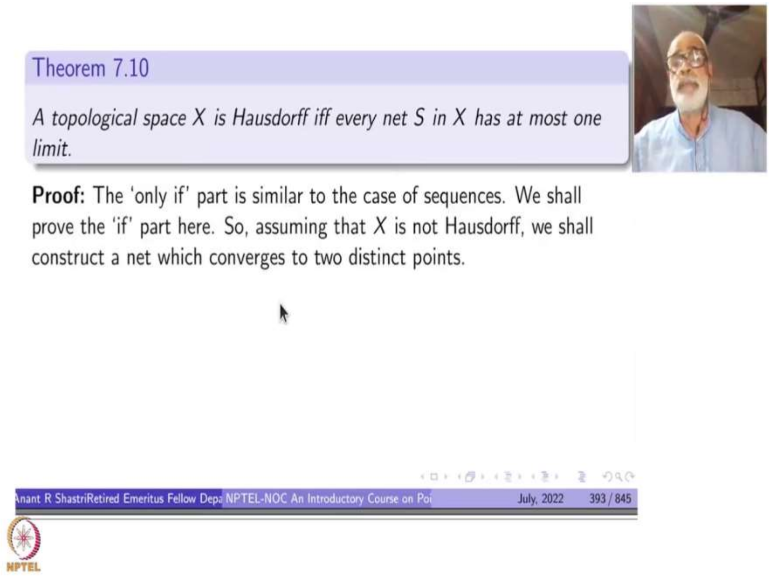

Now, we immediately come to a theorem here which is a one point higher score than the sequences. For a sequence, you do not have such a theorem. So, what is it?

Start with any topological space X. It is Housdorff if and only if every net S in X has at most one limit.

We know that in Hausdorff space every sequence has this property, but if every sequence has this property, we also know that we cannot say that the space is Hausdorff. However, we are saying that if this happens for every net inside X then X must be Hausdorff. The proof of `only if' part is similar to the case of sequences. So, I will leave it to you to figure it out as an exercise.

Now, I have come to the `if' part. So, assume that X is not Hausdorff. Then I will prove that this property does not hold, i.e., we have to produce some net S in X which converges to at least two points.

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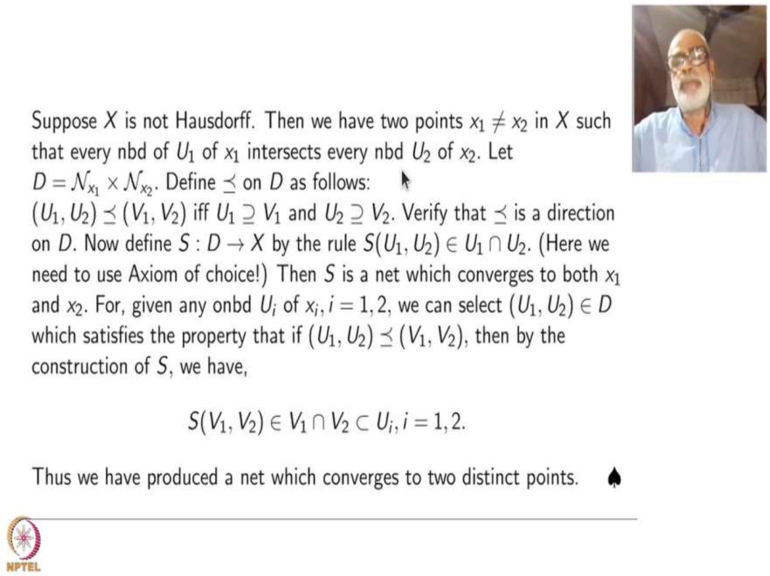

So, suppose X is not Hausdorff then there are two point $x_1 \neq x_2$ in X such that every neighborhood U_1 of x_1, intersects every neighborhood U_2 of x_2. This is the negation of Hausdorffness. Now you look at the product set $D := \mathcal{N}_{x_1} \times \mathcal{N}_{x_2}$, $\mathcal{N}_x$ denoting the family of all neighborhoods of x, which are directed sets. But what is the direction on the product? I will define it presently. There are many ways of defining it.

So, define this direction on D as follows: (U_1, U_2) is less than or equal to (V_1, V_2) if and only if, (this is a strict order) U_1 is less than equal to V_1 and U_2 is less than or equal to V_2, which is the same as U_1 contains V_1, and U_2 contains V_2. Both them should hold. Verify that this is a direction on D. It easy you can do that.

Now, define S from D to X by the rule: $S(U_1, U_2)$ is an element of the intersection of U_1 with U_2. This intersection is non-empty is our assumption, viz., every neighbourhood of x_1 intersects every neoghbourhood of x_2. So, I can pick up a point in the intersection and call it $S(U_1, U_2)$. We are using axiom of choice here. We are using axiom of choice all the time, anyway.

So then, S is a net. Once D is a directed set the function S is a net. This net I want to claim converges to both x_1 and x_2. It is very easy to see. Take any neighborhood U_i of x_i, $i = 1$ or 2. You can select (U_1, U_2) belonging to D. That satisfies the property that if (V_1, V_2) follows (U_1, U_2) that means V_i is contained inside $U_i, i = 1, 2$, then by the construction S we have

$S(V_1, V_2)$ is an element of $V_1 \cap V_2$, which will be both inside U_i, $i = 1$ or 2. Since this is true for arbitrary U_i, this just means that S converges to both x_1 as well as x_2. In one go, we have approved both of them.

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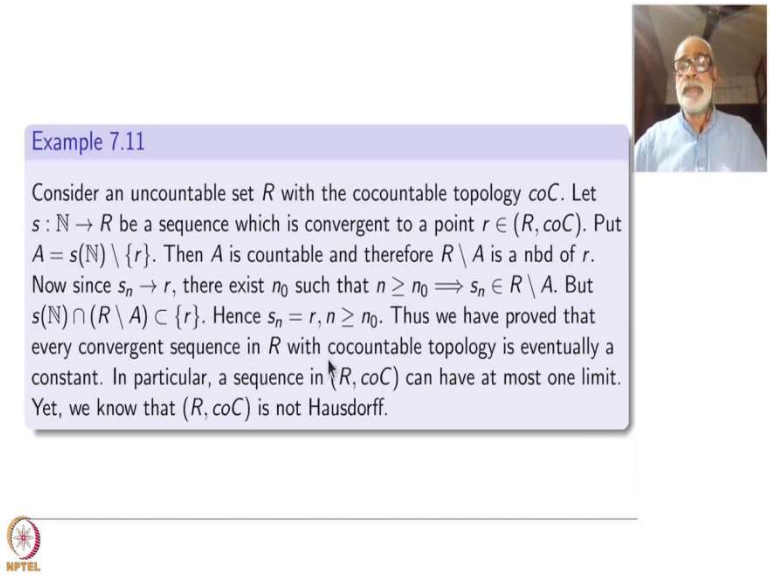

In any case, I want to recall this example which you must have seen in part I already. Take an uncountable set $\mathbb{R}$, like the real number itself with the cocountable topology. Remember cocountable topology means what? a set is open if and only if its complement is countable or it is the entire $\mathbb{R}$. Of course, the empty set is always allowed.

Now, take a sequence in $\mathbb{R}$, that is actually a net S from $\mathbb{N}$ to $\mathbb{R}$, which is convergent to a point r inside this space. Put $A = s(\mathbb{N}) \setminus \{r\}$. $S(\mathbb{N})$ is a countable set that is all, throw away r from it that is also countable set, hence A is a countable set. Therefore, $\mathbb{R} \setminus A$ is open, but little r belongs to this $\mathbb{R} \setminus A$, because A does not contain r.

So, $\mathbb{R} \setminus A$ is a neighborhood of r. Now $s(\mathbb{N})$ converges to r, we start with a sequence converging to r, that is what I said, (this is a short notation for convergence: S_n converges to r), there exists n_0 such that $n \geq n_0$ implies, $S(n)$ belongs to $\mathbb{R} \setminus A$, which is a neighborhood of r, but $S(\mathbb{N}) \cap (\mathbb{R} \setminus A)$ is contained in the singleton $\{r\}$, because everything else has been thrown away here, $\mathbb{R} \setminus A$, only r may survive. That means $S_n = r$ for all $n \geq n_0$.

What is it? What is this means? This means that the sequence is eventually a constant. Thus, we have proved that every convergent sequence in $\mathbb{R}$ with co-countable topology is eventually a constant. In particular, a sequence in $(\mathbb{R}, coc)$ can have at most one limit.

So, the property as in the above theorem is satisfied. However, we also know that the cocountable topology on an uncountable set is not Hausdorff. Any two non-empty open sets will intersect. So, this is one little small surprise for you or justification for doing something like a net, a general concept than sequences. So, we will have many such things.

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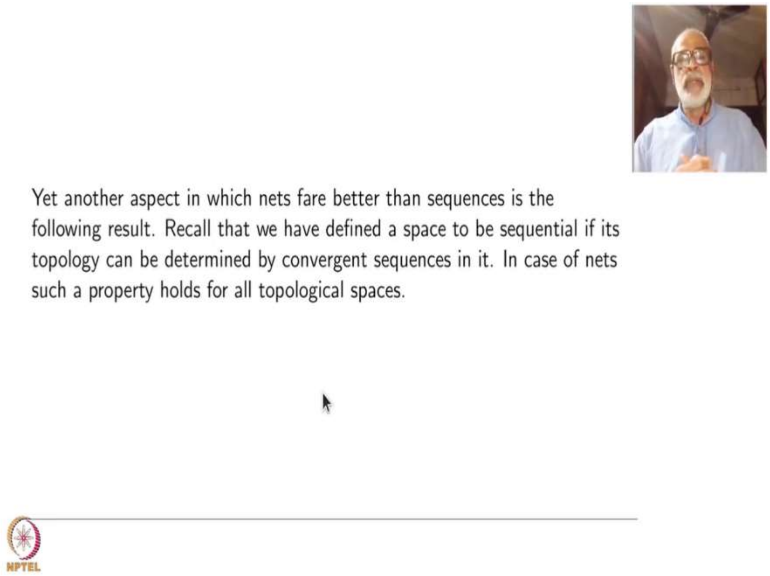

So, next one, yet another aspect in which nets fare better than sequences is the following result.

Recall that we have defined a space to be sequential if its topology can be determined by convergent sequences in it. So, sequences have this property while studying metric spaces and so on. In the case of nets if you try to have such a result, then you will get all the topological spaces. Just washes out the whole thing. So, that is what I want to say that means what? the nets will determine the topology completely.

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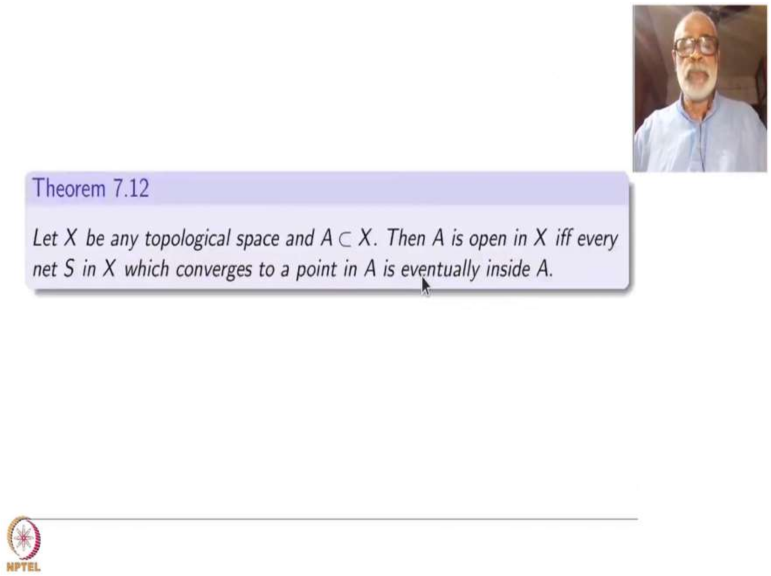

How so? That is what here, in this theorem:

Let X be any topological space and A be a subset of X. (You may prefer to take A to be non-empty subset, but that is not necessary, because if A is an empty set then the following statement is vacuously true.) statement). Then A is open if and only if every net S in X which converges to a point in A is eventually inside A.

So, that is a characterization of an open subset (other than non empty set other than empty set). Characterizing all open sets means its topology is defined completely, determined by the convergence behavior of the nets.

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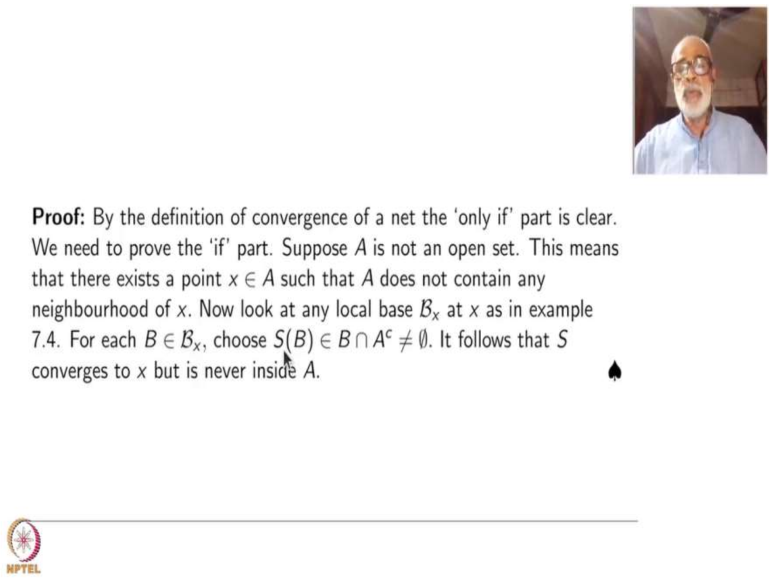

So, this is a theorem which is not at all difficult to prove. By the definition of convergence of a net, the `only if' part is clear. We need to prove the `if' part. Suppose A is not an open set. This means that there exists a point x inside A such that A does not contain any neighborhood of x.

Now, look at any local base $\mathcal{B}_x$. (You can take the whole of $\mathcal{N}_x$, if you like, but just any local base at x is enough. Sometimes you can verify this property only for a local base and that is why I am using local base at x, which we have studied earlier, local bases are directed sets. For each B inside $\mathcal{B}_x$, choose $S(B)$ to be a point inside $B \setminus A$, possible because this neighborhood B is not contained in A. (Once again this definition of S uses axiom of choice.) It follows, just as before that S converges to x, but it is never inside A, forget about eventualities.

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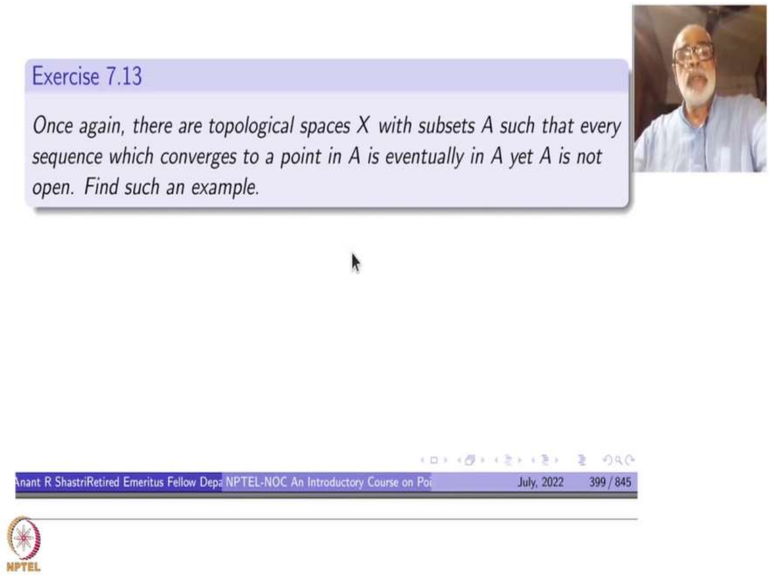

So, we have seen that convergence of nets determines the topology. So, it is so powerful. We will see more of its power. A few more properties of this convergence theory, we will study. Once again there are topological spaces with subsets A such that every sequence which converges to a point in A, is eventually in A. Yet A is not open. I will not give you a specific example here, but it is already there whatever you have seen today. So, just find it out.

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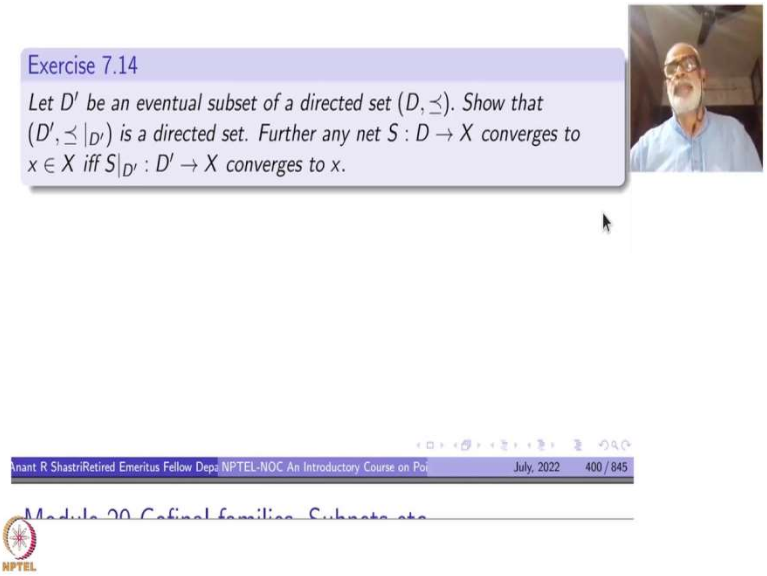

Here is an exercise: let D' be an eventual subset of a directed set D. Show that D' with the same partial ordering, same direction restricted to the subset D' is a direction on D'. So this is something non tirvial but not difficult.

Further take any net s D to a topological space X. Suppose s converges to x belong to X. Then the subnet s restricted to D' also converges to x and conversely. It is `if an only if'.

This is also not difficult to figure it out. Once you do that you will be familiar with the definition of convergence, definition of eventual set and so on. So, let us stop here. Next time we will study a little more properties of nets. Thank you.