An introduction to Point-Set-Topology

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Module 27

Nagata-Smyrnov Metrization theorem

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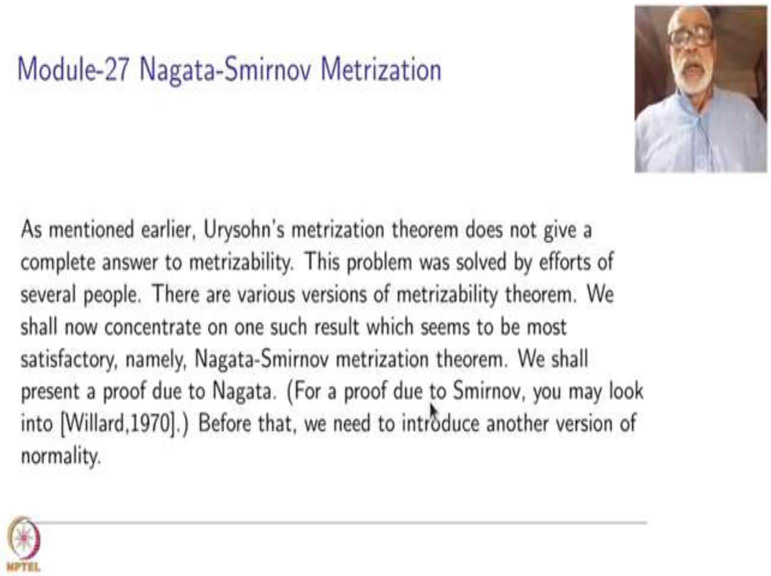

Hello, welcome to NPTEL NOC an introductory course on point set topology, part 2, module 27. Today, we shall study Nagata-Smirnov Metrization Theorem as mentioned earlier. Urysohn's metrization theorem does not give a complete answer to metrizability problem. It was solved by efforts of several people. There are various versions of metrizability theorem. We shall now concentrate on one such result which seems to be most satisfactory, namely, Nagata-Smirnov Metrization Theorem. We shall present the proof due to Nagata. A proof due to Smirnov can be found in Willard's book.

Before we begin with the theorem, we need to introduce another version of normality.

So, the Nagata's proof of metrization theorem is in spirit similar to Urysohn's, but since second countability is removed now, the embedding will be in a much larger product space and also, we need a little more strengthening than normality. So, that is what we want to introduce now.

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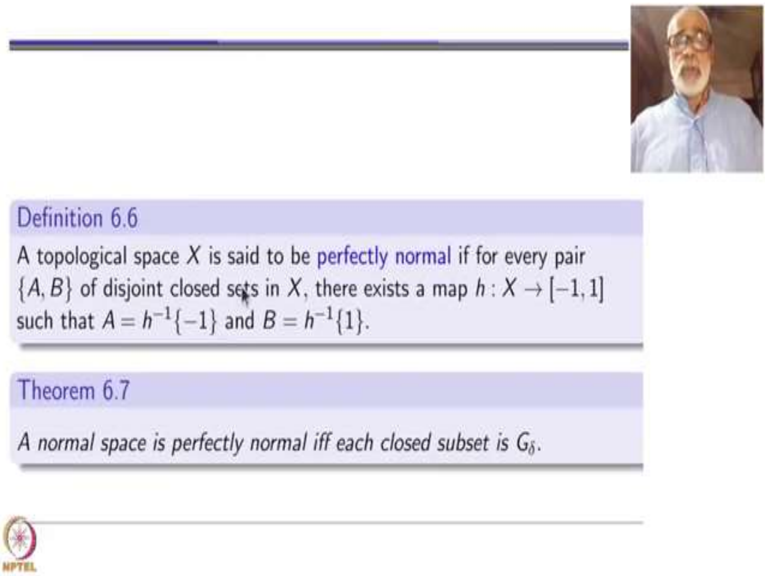

A topological space X is set to be perfectly normal if for every pair (A, B) of disjoint close sets inside X, there exists a map h from X to $[-1, 1]$ such that A is the precise inverse image of -1 and B is the precise inverse image of 1. So, I have said A, B are disjoint closed subsets. If one of them is empty, say, B is empty then the assertion is that h does not take the value 1. Similarly, if A were empty, then it means that h does not hit -1.

But generally, while talking about normality, we take non empty disjoint closed sets. Getting precise inverse images is stronger condition than normality. So, even in a normal space, you will have A going to -1 and B going to 1 that much is possible. But there may be more points that go to -1 and more points to 1. So that, that is where the perfect normality comes into business. A is the précise set of points wherein h takes value -1 similarly, B is the precise set of points where in h takes value 1.

A normal space is perfectly normal if only if each closed subset is G_δ. So, this is where the link between perfectly normal and normal is lies. G_δ is the key. Remember that under a continuous function into a metric space, inverse image of any single point is a G_δ.

Clearly perfectly normal is normal, there is no problem. So, every closed subset in X is G_δ is what you have to show under perfect normality. Given any closed subset A of X, you can choose B to be the empty set and get a continuous function h from X to $[0, 1]$ such that A is the precise inverse image of $\{0\}$ (and of course h does not take the value 1). It then follows that A is a G_δ.

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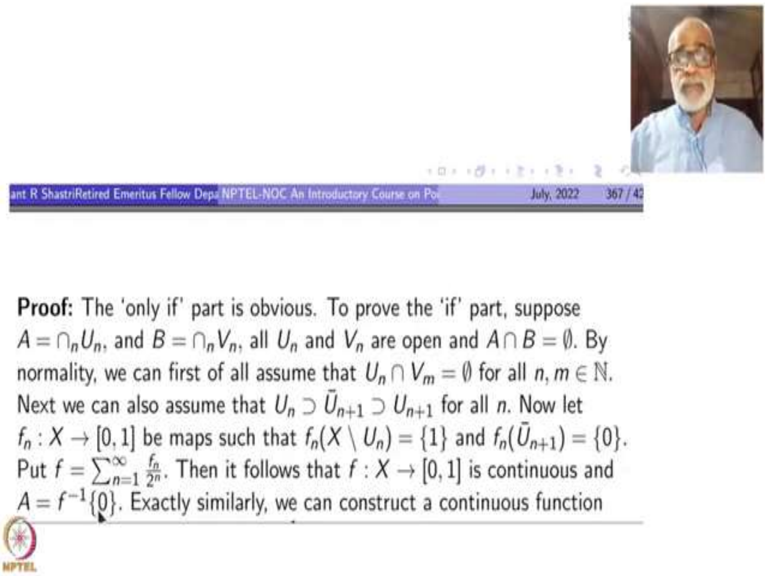

Now to prove the `if' part, start with A and B as disjoint closed subsets. And write A as intersection of a countable family of open sets U_n's because G_δ, similarly let B be the intersection of V_n's. Using normality, we can first show that owe may assume that U_n intersection V_n is empty for each n.

So, you have A is contained in U_n, B is contained inside of V_n, but they may not be disjoint. A and B are disjoint closed subsets and so inside these open subsets you can choose a smaller open subsets U'_n and V'_n respectively such that U'_n and V'_n are disjoint. But in contain A. So, by replacing U_n with U'_n etc, we may as well assume that $U_n \cap V_n$ is empty for all n.

Next we can also assume that U_{n+1} is contained in $\overline{U_{n+1}}$ contained inside U_n for all n, as follows: Once you have the family U_n as above, inductively choose an open subset W_{n+1} such that A is contained W_{n+1} contained in $\overline{W_{n+1}}$ contained in the intersection of $U_1, U_2, \ldots, U_{n+1}$. This is possible by normality of X. Now ignore the old U_{n+1} and rename W_{n+1} as U_{n+1}.

Now let f_n from X to the closed interval $[0,1]$ be a continuous function such that $f_n(X \setminus U_n)$ is singleton 1 and $f_n(\overline{U_{n+1}})$ is 0. So, this is possible by normality of X again. Here I may not get these sets as precise inverse images. That is OK.

So, you have got a sequence of functions with a certain property. Take f to be some of these f_n's only thing is before taking the sum, you should divide by some constant factors to ensure convergence. So, I am dividing $f_n/2^n$, because I know that these f_n's are bounded by 1, so this sum will be less than the summation $1/2^n$. So, this will also convergent. So, this dividing by some number is a general principle which you have to learn in analysis.

It then follows that f from X to $\mathbb{R}$ is non negative continuous function because the sum is uniformly convergent being dominated by summation $1/2^n$ and all terms are non negative. Now what happens that A is precised inverse of 0 under f, Because if x belongs to A, it must be in every U_n so each term in the sum is zero so $f(x)$ is zero. But if x is not in A, then it cannot be in U_n for some n and hence it is not in $\overline{U_{n+1}}$ and hence $f_{n+1}(x)$ is 1. Since the sum is over nonnegative values, it follows that $f(x)$ is not zero.

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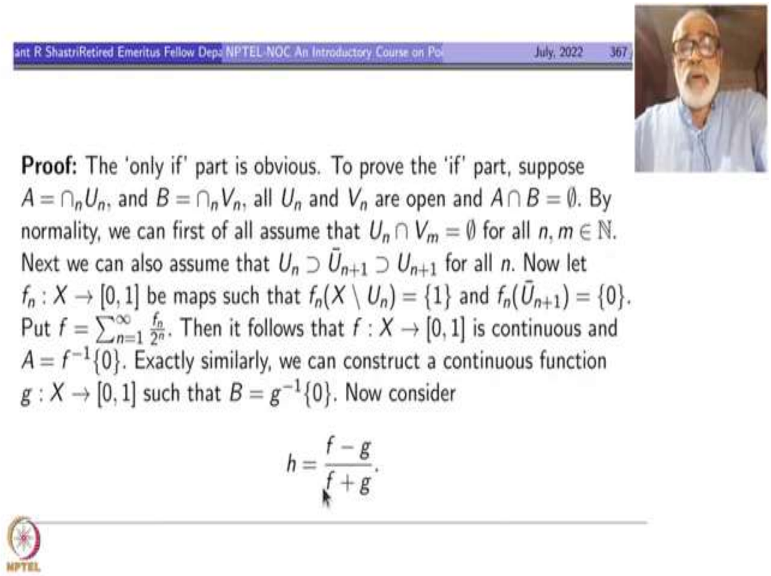

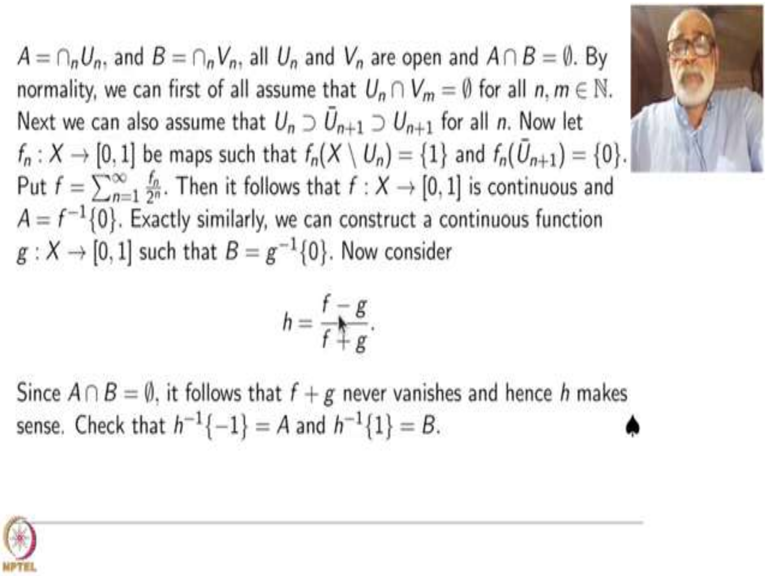

Similarly, we can construct a non negative continuous function g from X to $\mathbb{R}$ such that B is precisely the inverse image of zero. All that you have to do is same construction as above with V_n's instead of U_n's. That is all.

So, you have two different functions. I want a single function to do that job. So, this function f is such that the inverse image of 0 is A and g is such an inverse image of 0 is B. Now I want one single function h such that under that the inverse image of 0 is A and inverse image of 1 is B, the other way round.

So, for that, I just take h equal to $(f-g)/(f+g)$. Note that dividing by $f+g$ makes sense because this sum is never 0. See, first of all both f and g are non-negative. Next, if $f(x)$ is 0, that x must be inside A, which is disjoint from B. And outside B, g is not 0 so $g(x)$ is not zero. A and B are disjoint, so $f+g$ is never 0. So, I can divide by $f+g$. But $|f-g|$ is always smaller than modulus of $f+g$ and hence h takes values in the interval $[-1,1]$.

Finally, suppose this $h(x)$ is 1. What does that mean? This is so if and only if these two are equal: $f-g = f+g$. Therefore $g(x)$ must be 0. So, $h(x) = 1$ iff x is a point of B. Similarly, $h(x)$ is equal to -1 iff $g - f = f + g$ iff $f(x) = 0$ iff x is in A. So A is the inverse image of -1 and B is the inverse image of 1 under h.

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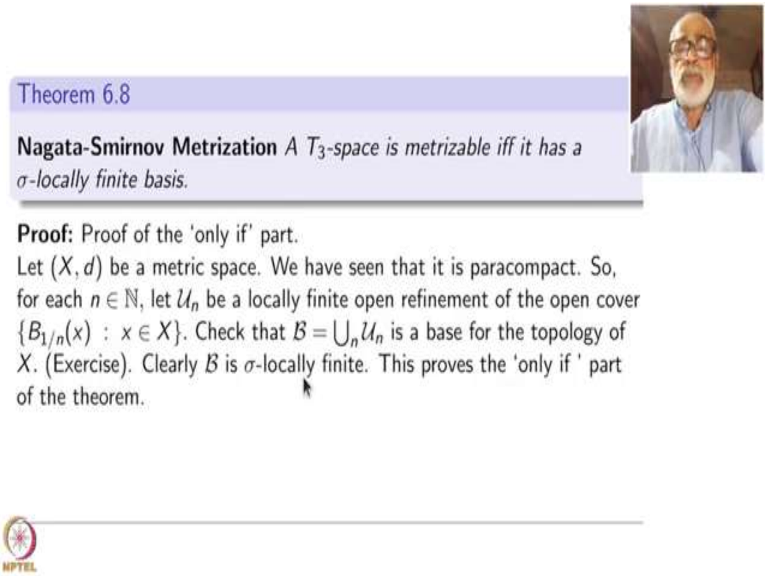

So, this is the perfect normality given by normality plus every close subset being G_δ.

Now, let us come to Nagata-Smirnov Metrization Theorem, proof which will be much simpler now, because we have perfect normality here.

A T_3 space is meitrizeable (see there is no second countability condition now) if and only if X has a σ-locally finite basis. The term σ-locally finite basis, should ring a bell! That is something to do with paracompactness.

So, proof of only if part: So, that all that you have to do is start with a metric space and show that it has a σ-locally finite basis. X is a metric space, we have seen that it is paracompact. So, for each $n \in \mathbb{N}$, let $\mathcal{U}_n$ be a locally finite open refinement of the open cover $\{B_{1/n}(x); x \in X\}$. It is an open cover of X, for each n.

So, that take $\mathcal{U}_n$ to be a locally finite open refinement of this one. So, for each n, I have a got a locally finite open refinement. All that I want to do is now, take the union of all these $\mathcal{U}_n$'s, call it $\mathcal{B}$. This is a base for the topology on X. This fact I am leaving it as an exercise, because you have done so much of this kind of things. So, for all n is in all that is all you have to use. Clearly this $\mathcal{B}$ is σ-locally finite, because by choice, each $\mathcal{U}_n$ is locally finite. So by definition $\mathcal{B}$ is σ-locally finite.

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Proof of the 'if' part:
So, let $(X, \mathcal{T})$ be a T_3-space and

$$\mathcal{B} = \bigcup \{\mathcal{B}_n \; : \; n \in \mathbb{N}\}$$

be a base for the $\mathcal{T}$, where each $\mathcal{B}_n = \{B_{n,\alpha} \; : \; \alpha \in \Lambda_n\}$ is a locally finite family. Put $\Lambda = \sqcup_n \Lambda_n$. The proof is going to be similar to the proof of Urysohn's metrization theorem, in the sense that we are going to produce an embedding $F : X \to \ell_2(\Lambda)$.

Let us recall the definition of $\ell_2(\Lambda)$, the generalized Hilbert space. For any function $f : \Lambda \to \mathbb{R}$, let us denote

$$s(f) := \{\lambda \in \Lambda \; : \; f(\lambda) \neq 0\}.$$

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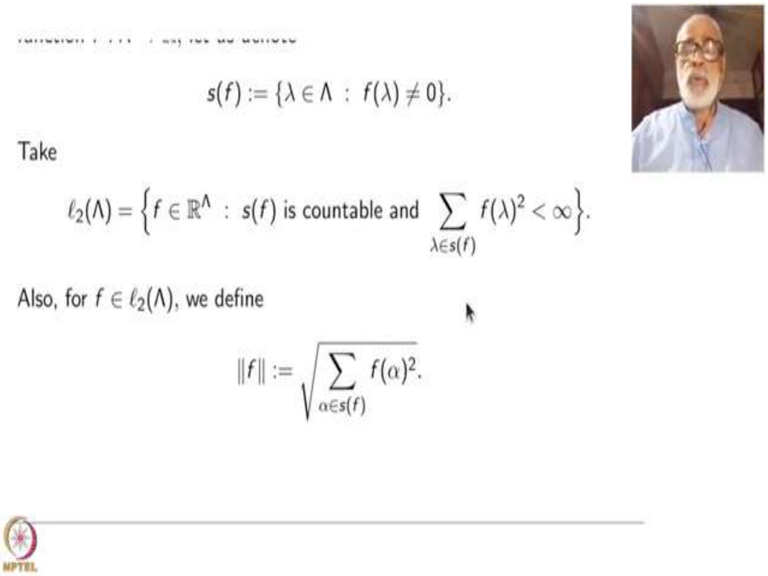
Take

$$\ell_2(\Lambda) = \left\{ f \in \mathbb{R}^{\Lambda} \; : \; s(f) \text{ is countable and } \sum_{\lambda \in s(f)} f(\lambda)^2 < \infty \right\}.$$

Also, for $f \in \ell_2(\Lambda)$, we define



Let us know prove the `if' part. Let $(X, \mathcal{T})$ be T_3 space and $\mathcal{B}$ equal to a countable union of $\mathcal{B}_n$'s, be a base for $\mathcal{T}$, with each $\mathcal{B}_n$ being locally finite. Remember each base is also an open cover for X. I am writing down each $\mathcal{B}_n$ as the collection $\{B_{n,\alpha}\}$ where α is in an indexing set Λ_n. So, $B_{n,\alpha}$'s are members of $\mathcal{B}_n$'s. And each $\mathcal{B}_n$ is a locally finite family. This is all the starting hypothesis now.

Now, take Λ to be the union of all these Λ_n's over n. So, this Λ is going to be our indexing set for the product. Last time you took $\mathbb{I}^{\mathbb{N}}$, where $\mathbb{N}$ is the set of natural numbers. Now, I would like to take $\mathbb{I}^{\mathbb{N}}$ but I will do a slightly better job here, than taking the product space. However, the proof here is now going to be similar to the proof of Urysohn's Metrization theorem, in the sense that we are going to produce an embedding of X inside the Hilbert space $\ell_2(\Lambda)$, with the ℓ_2-norm.

So, I will recall what is this $\ell_2(\Lambda)$. On any set A, $\ell_2(A)$ makes sense. That is what I am going to tell you. I am just recalling the generalized Hilbert space here. For any function f from Λ to $\mathbb{R}$, let us denote by $s(f)$ (you may read it as `support of f') the set of all those $\lambda \in \Lambda$ such that $f(\lambda)$ is not 0.

Now, take $\ell_2(\Lambda)$ to be the subset of the space of all functions from Λ to $\mathbb{R}$ such that $s(f)$ is countable, and when it is countable look at the sum of all the $f(\lambda)^2$, that sum is convergent, i.e., sum over λ of $t(\lambda)^2$ is finite.

In other words, it is the collection of all square summable functions, that is the terminology, square summable functions form a Hilbert space. I will just use the word Hilbert space because the inner product here is not being exactly used only the norm is used and what is the norm, norm is square root of summation of f_α^2, if you want to know what is the inner product, it is nothing but inner product of f and g is summation $\langle f_\alpha, g_\alpha \rangle$; if you have complex valued functions you take summation $f_\alpha \bar{g_\alpha}$, that is all. We are not interested in that part, we are just interested in the norm, under the induced metric, this space is complete.

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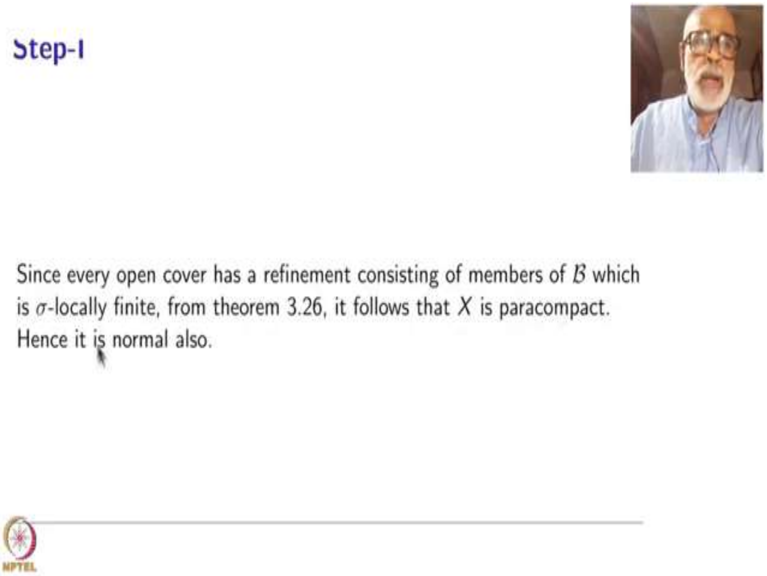

The first step is since every open cover has a refinement consisting of members of $\mathcal{B}$ because $\mathcal{B}$ is a base right? (For every member U of the open cover and for each point in U you can select a member of the $\mathcal{B}$ which contains the point and is contained inside U. That will verify this statement.) But $\mathcal{B}$ is σ-locally finite, therefore, from 3.26 it follows that X is paracompact. So, that is the link between the condition σ-locally finite base and this situation. In particular X is normal. What we want is that X is perfect normal. We will see why.

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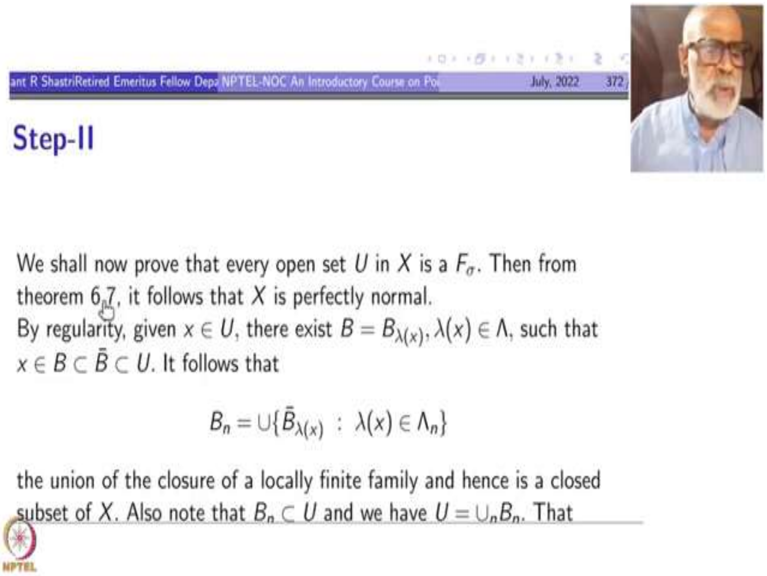

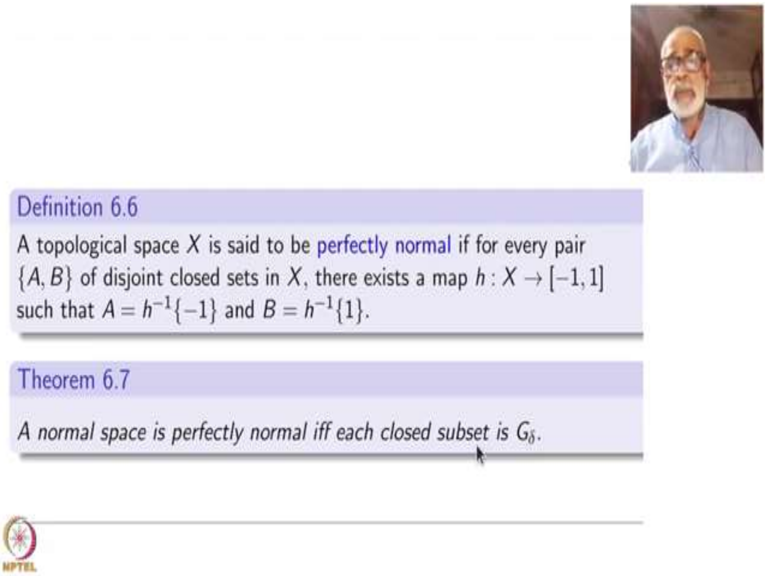

So, first we have seen that it is normal space. Secondly, we shall now prove that every open set U in X is F_σ.

So, now you see that the concepts of G_δ and F_σ are both coming together here. F_σ means what? union of countably many closed subset. Every open subset in X is F_σ. Then from theorem 6.7, it follows that X is perfectly normal. You want to recall theorem 6.7. A normals space is perfectly normal if and only if every close subset is G_δ.

Take the complements, to get every closed subset is G_δ.

Next by regularity, given x belonging to U open, there exist a basic open set $B = B_{\lambda(x)}, \lambda(x) \in \Lambda$, such that x is in B and $\bar{B}$ is contained inside U. It follows that if you put B_n equal to the union of all these $\overline{B_{\lambda(x)}}$, where $\lambda(x)$ ranges over Λ_n, is a closed set, because of local finiteness of $\mathcal{B}_n$.

Also note that B_n is contained inside U because each $\overline{B_{\lambda(x)}}$ is contained in U. Finally, it follows that U itself is union of all these B_n's. For each point $x \in U$ is inside some member of $\mathcal{B}_n$ for some n and then it is in B_n. So, what we have shown is that U is a countable union of closed sets. So, that completes the proof that every open subset is F_σ. Therefore X is perfectly normal.

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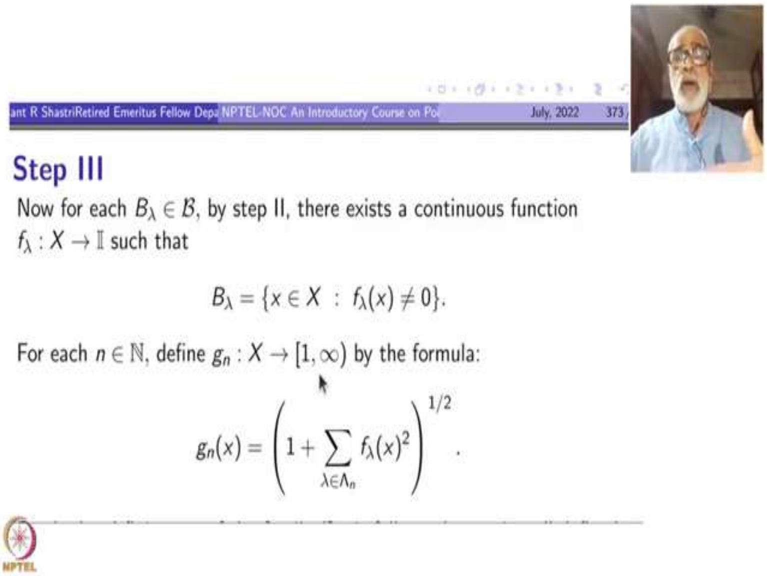

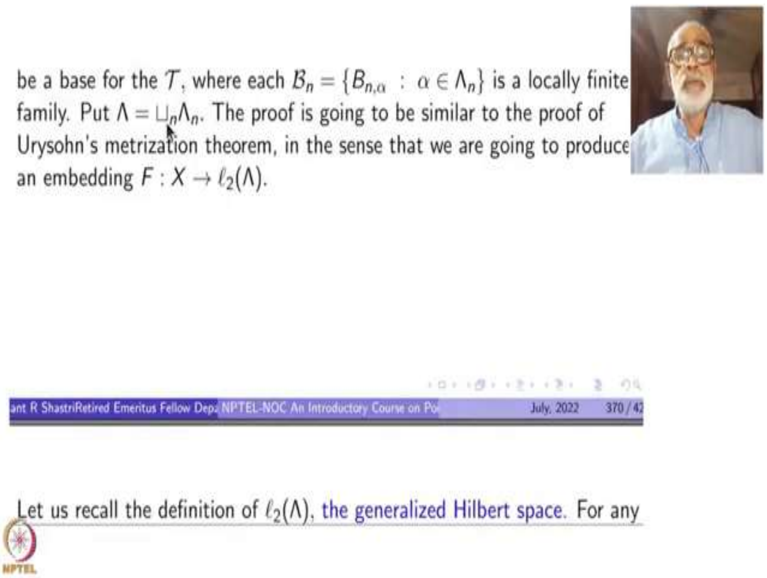

Now for each $B_\lambda \in \mathcal{B}$, by step II, there exists a continuous function $f_\lambda : X \to \mathbb{I}$ such that

$$B_\lambda = \{x \in X \ : \ f_\lambda(x) \neq 0\}.$$

For each $n \in \mathbb{N}$, define $g_n : X \to [1, \infty)$ by the formula:

$$g_n(x) = \left(1 + \sum_{\lambda \in \Lambda_n} f_\lambda(x)^2\right)^{1/2}.$$

By the local finiteness of the family $\mathcal{B}_n$, it follows that g_n is well-defined. For the same reason, it is also continuous. Note that each $\lambda \in \Lambda$, there is a unique $n(\lambda) \in \mathbb{N}$ such that $\lambda \in \Lambda_{n(\lambda)}$. It follows that the function

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$$h_\lambda(x) = \frac{f_\lambda(x)}{n(\lambda) g_{n(\lambda)}(x)}$$

are all continuous and take values inside $\mathbb{I}$.

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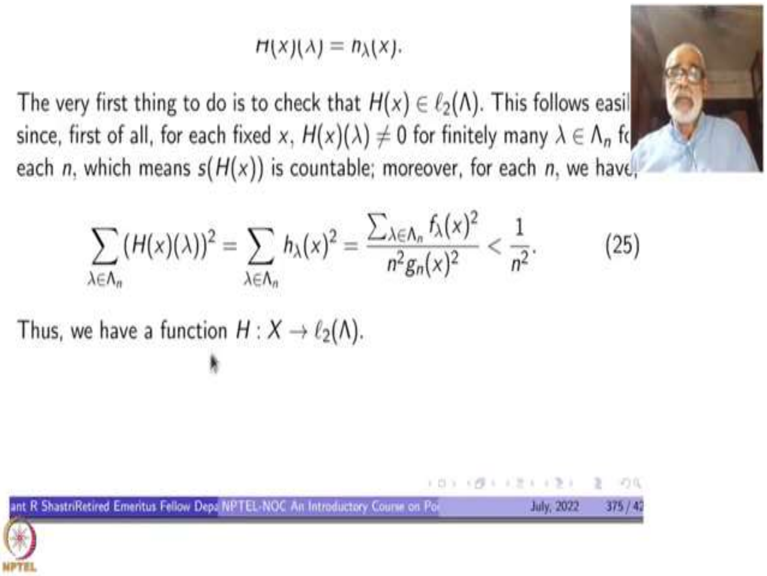
Now we define $H : X \to \mathbb{I}^\Lambda$ by the formula

$$H(x)(\lambda) = h_\lambda(x).$$

The very first thing to do is to check that $H(x) \in \ell_2(\Lambda)$. This follows easily since, first of all, for each fixed x, $H(x)(\lambda) \neq 0$ for finitely many $\lambda \in \Lambda_n$ for each n, which means $s(H(x))$ is countable; moreover, for each n, we have,

$$\sum_{\lambda \in \Lambda_n} (H(x)(\lambda))^2 = \sum_{\lambda \in \Lambda_n} h_\lambda(x)^2 = \frac{\sum_{\lambda \in \Lambda_n} f_\lambda(x)^2}{n^2 g_n(x)^2} < \frac{1}{n^2}. \qquad (25)$$



Third step is to take for each $\lambda \in \Lambda$, by perfect normality, a continuous function f_λ from X to $\mathbb{I}$ such that B_λ is precisely equal to the set of all points $x \in X$ such that $f_\lambda(x)$ is not equal to 0. B_λ's are open subsets, so their complements are closed subsets, they will be precise zero sets of continuous functions.

For each $n \in \mathbb{N}$, define g_n from X to $(0, \infty)$, (we are constructing these functions, so that finally we can take all of them and put them inside the generalized Hilbert space), by the following formula. I am taking a sum and then adding 1 just to be careful that it is positive, you could have added any positive epsilon, no problem.

So take 1 + summation of $h_\lambda(x)^2$, where λ ranges over Λ_n, this is a locally finite family and so. So, this sum is actually makes sense. So, 1 plus that makes sense, take the square root, that makes sense. So, that is $g_n(x)$. By the local finiteness of the family B_n's here, each term is actually finite now, finite, it follows that g_n is well defined. For the same reason it is continuous also because it is a locally finite sum of continuous functions. This 1 plus that is continuous and it is never 0. That is why I put 1 (or some positive epsilon here), I can take the square root the square root is also continuous.

Now, I can define H from X to $\mathbb{I}^\Lambda$ itself, in particular it will be inside $\mathbb{R}^\Lambda$ no problem, by the formula $H(x)(\lambda)$ equal to $h_\lambda(x)$. $H(x)$ is a function from Λ to $\mathbb{I}$.

Note that for each λ inside Λ, there is a unique $n \in \mathbb{N}$ such that λ is in Λ_n. Why? look at this definition of Λ. This is a disjoint union, each λ here belongs to exactly one of the Λ_n's. Taking this disjoint union is important, so that, I can now take h_λ to be f_λ divided by this number $n(\lambda)$ times the function $g_{\lambda(n)}$, which is a non zero function. It follows that h_λ are continuous, and taking values inside $\mathbb{I}$ now, because f_λ is only 1 of the summands involved in the definition of $g_{n(\lambda)}$; so, the numerator is always smaller than the denominator and they are all non negative. So, all h_λ take values inside $\mathbb{I}$.

So, the λ^{th} coordinate of $H(x)$ is $h_\lambda(x)$. You can think of H as taking values in the product space. First of all, you have defined $H(x)$ as an element of $\mathbb{R}^\Lambda$. But I want it to be inside $\ell_2(\Lambda)$; $\ell_2(\Lambda)$ is a subspace of $\mathbb{R}^\Lambda$.

So, this follows easily now. First of all for each fixed x, $H(x)(\lambda)$ is not equal to 0, for only finitely many $\lambda \in \Lambda_n$, for each n, therefore $s(H(x))$ countable Recall that $s(H(x))$ is the set of λ where in it is not 0, the support. That is countable.

Second thing is that for each fixed n, if you take the sum of all of them after squaring where λ runs over Λ_n, this is a finite sum and that is nothing but summation $h(\lambda)(x)^2$ where λ runs inside Λ_n. But by definition of $h(\lambda)(x)$, the numerator is just the sum of $f_\lambda(x)^2$ whereas in the denominator we have $n(\lambda)^2$ into $g_{n(\lambda)}(x)^2$, but λ is in Λ_n, so this $n(\lambda)$ is equal to n itself.

So, this is less than $1/n^2$. If, for each n this is less than $1/n^2$ when you take summation of all of them that will less that the summation of $1/n^2$ which is convergent. Therefore, $H(x)$ is an element of ℓ_2. So, we have got a function from X into ℓ_2.

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Next we check that H is injective. Given $x_1 \neq x_2$ in X, since X is T_2, there exist B_λ, such that $x_1 \in B_\lambda$ and $x_2 \notin B_\lambda$. This immediately implies that $f_\lambda(x_1) \neq 0$ and $f_\lambda(x_2) = 0$. But then $h_\lambda(x_1) \neq 0$, whereas $h_\lambda(x_2) = 0$. This just implies $H(x_1)(\lambda) \neq H(x_2)(\lambda)$ and hence $H(x_1) \neq H(x_2)$.

Now we define $H : X \to \mathbb{I}^\Lambda$ by the formula

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The very first thing to do is to check that $H(x) \in \ell_2(\Lambda)$. This follows easily since, first of all, for each fixed x, $H(x)(\lambda) \neq 0$ for finitely many $\lambda \in \Lambda_n$ for each n, which means $s(H(x))$ is countable; moreover, for each n, we have,

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For each $n \in \mathbb{N}$, define $g_n : X \to [1, \infty)$ by the formula:

$$g_n(x) = \left(1 + \sum_{\lambda \in \Lambda_n} f_\lambda(x)^2\right)^{1/2}.$$

By the local finiteness of the family $\mathcal{B}_n$, it follows that g_n is well-defined. For the same reason, it is also continuous. Note that each $\lambda \in \Lambda$, there is a unique $n(\lambda) \in \mathbb{N}$ such that $\lambda \in \Lambda_{n(\lambda)}$. It follows that the function

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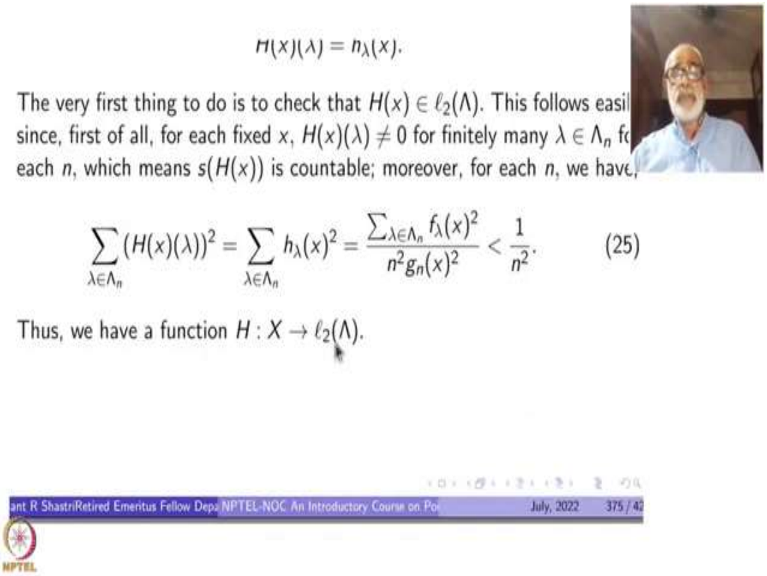

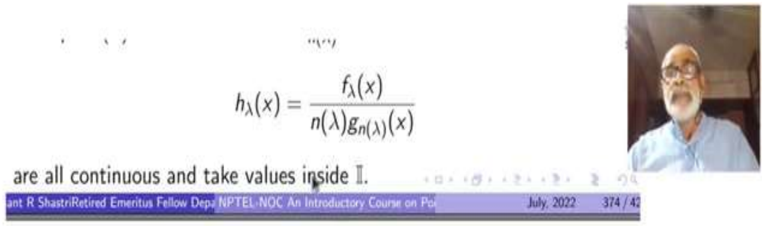

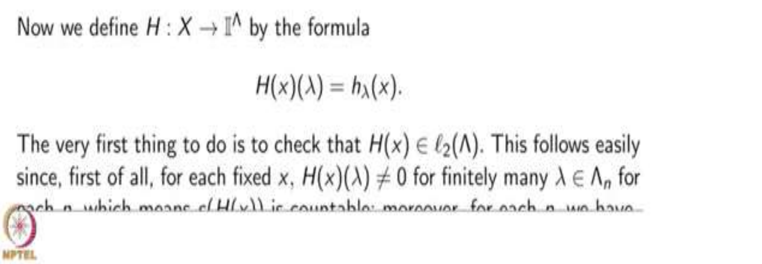

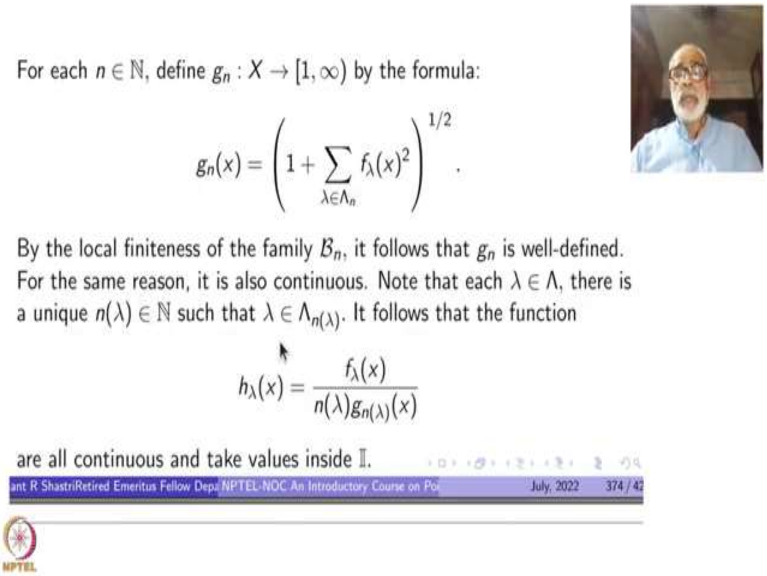

Student: Sir, may you please repeat the step? So, we had a, from starting with every element of X, we wanted to associate an element of $\ell_2(\Lambda)$, that was from step 2, we found that every

B_λ, what was concept so, we can associate a function so that it is exactly non zero. That was the first step, First part of step.

Professor: that will come now in the proof of injectivity and continuity of H. That this is an open subset and that is the precisely zero set etc is not used so far. Only local finiteness is used. so, that we have the supports of these elements is countable, $s(h)$. So that $H(x)$ is actually an element of $\ell_2(\Lambda)$.

Student: divided by $n, n(\lambda)$?

Professor: this $n(\lambda)$ term is brought for that, so that the sum is dominated by a convergent sum. Not only that, why in the definition of g_n, I have put 1 here? For two different purposes. First of all, I should be sure that when I take square roots continuity is assured. I should not bump into 0, if I want continuity is preserved after taking square root. Secondly, this term is bigger than just the sum. So, there are two purposes here. So, $g_n(x)$ is defined like this, I told you that adding any positive constant would have done the job here instead of adding 1. So, look at here now. These values are are inside $\mathbb{I}$. That point is not very crucial. But this whole thing is less than $n(\lambda)^2$ that is important, not just less than some constant.

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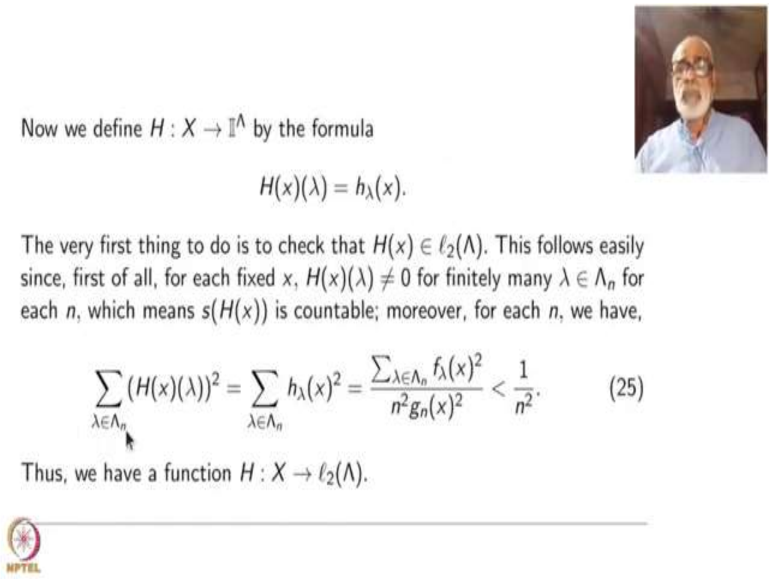

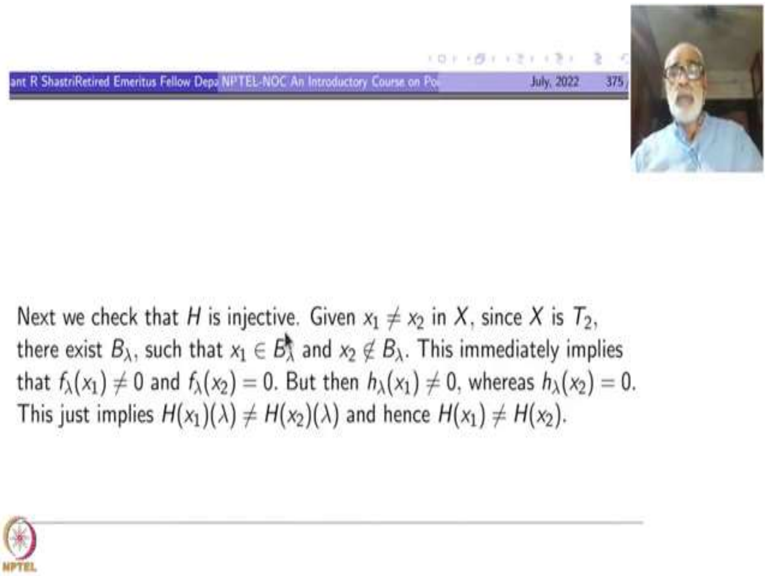

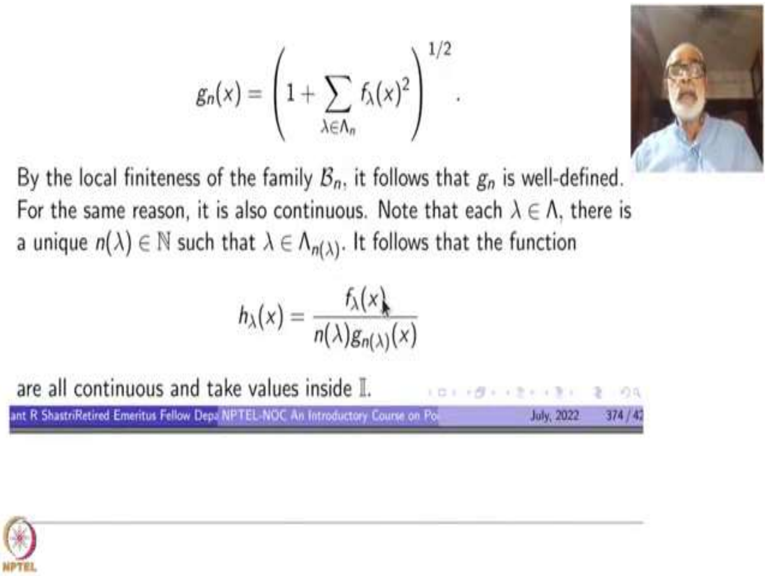

So, how to ensure that. For each fixed λ, λ in Λ_n, what is $n(\lambda)$? It is n. So, the same n is there. So, holding n fixed, first you take the sum that is a finite sum which is less than $1/n^2$ and then you take the sum over n so that is convergent. So, you are inside ℓ_2.

Now comes the role of all these open subsets etc., till now they are in the back ground.

Now, let us check that this H is injective: given $x_1 \neq x_2$ in X, since X is Hausdorff space (we have assumed it is T_3), you will have B_λ, a basic element so that x_1 is in B_λ and x_2 is not in B_λ.

So, this immediately implies that the corresponding $f_\lambda(x_1)$ is not 0, but $f_\lambda(x_2) = 0$. Therefore, $h_\lambda(x_1)$ is not 0, whereas, $h_\lambda(x_2) = 0$, because what is h_λ? In the numerator we have f_λ. That is what it is. So, h_λ separate points. Therefore, $H(x_1)(\lambda)$ will not be equal to $H(x_2)(\lambda)$. So, this lambda coordinates will be different as soon as $x_1 \neq x_2$. So, this proves H is injective.

Note that a countable family of such functions would not have been able to do this job. This is a local base we have used. Taking a countable base was possible if X were second countable. However, some countability was necessary. What ensures that? σ-locally finiteness. So, that plays the role here. Exploit it, to get into ℓ_2. Otherwise you would not be able to get inside ℓ_2.

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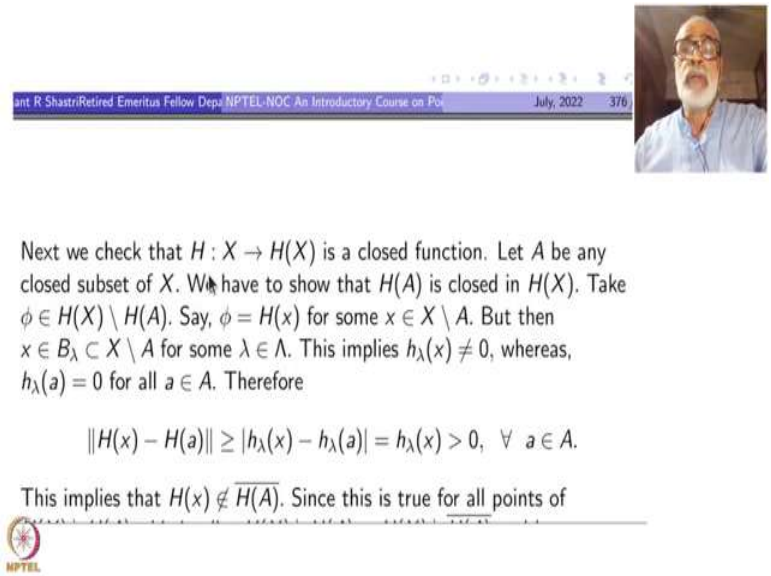

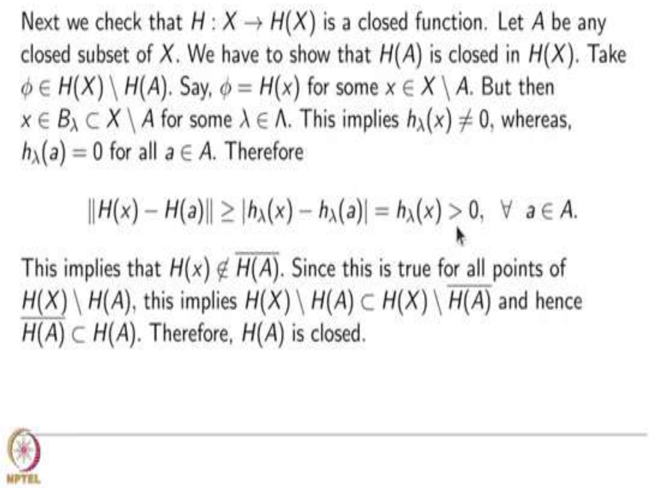

Next, we check that H from X to $H(X)$ is a closed function. Injective, closed, continuity these 3 things we have to show one by one, continuity is taken at the last.

If A is any closed subset of X, you have to show that $H(A)$ is closed in $H(X)$. Take a point phi outside of $H(A)$ but inside $H(X)$. We are working inside $H(X)$ and the entire ℓ_2. To show that something is an embedding you have to take the only image and work inside of the image. So, $\phi = H(x)$, for $x \in X \setminus A$, because it is not in $H(A)$. But then x belongs to B_λ contained inside $X \setminus A$. A is a closed subset to start with. So, you will have some λ such that x belongs to B_λ which is contained in $X \setminus A$. This automatically implies that $h_\lambda(x)$ is not 0,

Whereas, $h_\lambda(a) = 0$ for all $a \in A$. B_λ is the precisely equal to the set where f_λ is non zero. Therefore, the square of norm of $H(x) \setminus H(a)$, this is a summation of non negative terms and one of the term is square of the $h_\lambda(x) \setminus h_\lambda(a)$ which is equal to $h_\lambda(x)^2$, which is positive. There will be many other terms they are all non-negative terms. Therefore it follows that the distance between $h(x)$ and $H(A)$ is bigger or equal to $h_\lambda(x) > 0$.

Therefore, $H(x)$ cannot be in the closure of $H(A)$.

Since this is true for all points of $H(X) \setminus H(A)$, this implies that $H(X) \setminus H(A)$ itself is contained $H(X) \setminus \overline{H(A)}$. By DeMorgan law this just means that $\overline{H(A)}$ is contained inside $H(A)$. The bar of a set is contained inside the set itself, that means that set is closed. (Maybe you can directly prove that that $H(A)$ is closed or prove that $H(X) \setminus H(A)$ is open etc.)

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Finally, we shall check that H is continuous. Given $x \in X$ and $\epsilon > 0$, we must produce an open set U around x such that

$$x' \in U \Longrightarrow \|H(x') - H(x)\| < \epsilon.$$

First choose $N \in \mathbb{N}$ such that $\sum_{N+1}^{\infty} \frac{1}{n^2} < \epsilon/4$. By local finiteness of $\mathcal{B}_n$'s, we get an open set V around x such that V meets only finitely many members of $\mathcal{B}_n$ for all $n \le N$. Let us denote these members by $B_{\lambda_i}, i = 1, 2, \ldots, k$. Now choose a nbd U of x such that $U \subset V$ and

$$|h_{\lambda_i}(x') - h_{\lambda_i}(x)| < \frac{\sqrt{\epsilon}}{\sqrt{2k}}, \;\; 1 \le i \le k, \;\; \forall\, x' \in U.$$

$$x' \in U \Longrightarrow \|H(x') - H(x)\| < \epsilon.$$

First choose $N \in \mathbb{N}$ such that $\sum_{N+1}^{\infty} \frac{1}{n^2} < \epsilon/4$. By local finiteness of $\mathcal{B}_n$'s we get an open set V around x such that V meets only finitely many members of $\mathcal{B}_n$ for all $n \le N$. Let us denote these members by $B_{\lambda_i}, i = 1, 2, \ldots, k$. Now choose a nbd U of x such that $U \subset V$ and

$$|h_{\lambda_i}(x') - h_{\lambda_i}(x)| < \frac{\sqrt{\epsilon}}{\sqrt{2k}}, \;\; 1 \le i \le k, \;\; \forall\, x' \in U.$$

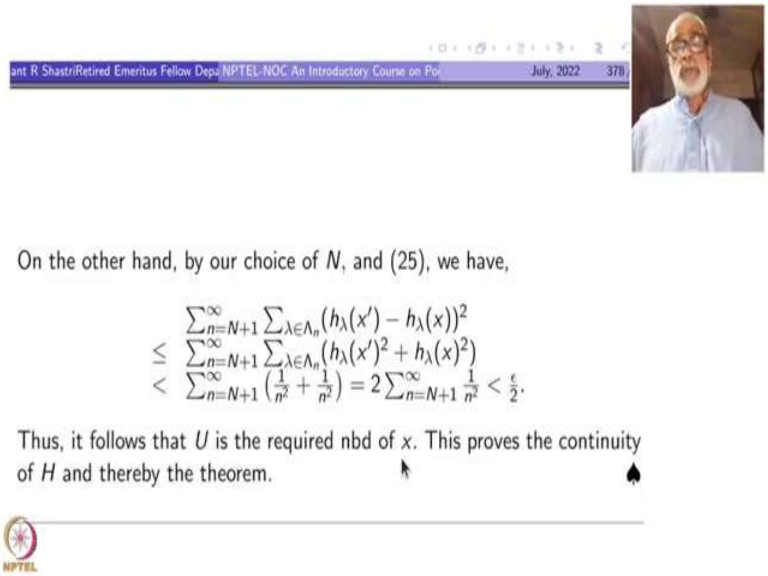
This yields, for all $x' \in U$,

$$\sum_{n=1}^{N} \sum_{\lambda \in \Lambda_n} |h_\lambda(x') - h_\lambda(x)|^2 < \frac{k\epsilon}{2k} = \frac{\epsilon}{2}.$$

On the other hand, by our choice of N, and (25), we have,

$$\begin{array}{rl} & \sum_{n=N+1}^{\infty} \sum_{\lambda \in \Lambda_n} (h_\lambda(x') - h_\lambda(x))^2 \\ \le & \sum_{n=N+1}^{\infty} \sum_{\lambda \in \Lambda_n} (h_\lambda(x')^2 + h_\lambda(x)^2) \\ < & \sum_{n=N+1}^{\infty} \left(\frac{1}{n^2} + \frac{1}{n^2}\right) = 2\sum_{n=N+1}^{\infty} \frac{1}{n^2} < \frac{\epsilon}{2}. \end{array}$$

Thus, it follows that U is the required nbd of x. This proves the continuity



Now, let us check that H is continuous. Now, you see the full force of all these construction. Given x belong to X, and $\epsilon > 0$, we have to produce some open subset U around x, such that x' belongs U implies norm of $H(x') \setminus H(x)$ is less than epsilon. So, this is the continuity of H at the point x. I am using the norm on the codomain. X is the given topological space. So, I have to use open subsets in X.

As soon as the $\epsilon > 0$ is given, choose some integer $\mathbb{N}$ such that the sum of $1/n^2$, from $n = N + 1$ to infinity is less that $\epsilon/4$. (Earlier I had used the notation k in this place now I am using $\mathbb{N}$, no problem.) $N + 1$ to infinity $1/n^2$ is less than $\epsilon/4$.

By local finiteness of all these $\mathcal{B}_n$'s, we get an open set V around x such that this open set V meets only finitely many members of $\mathcal{B}_n$ for all $n \leq N$. For each n, you will get an open set V_n and then you take V to be the intersection of these finitely many V_n's. That is all. This V is the neighborhood of x, which will meet finitely many members of $\mathcal{B}_n$, n ranging from $1 + N$. Let us denote these members by B_{λ_i} for $i = 1, 2, \ldots, k$. They will belong to one of the $\mathcal{B}_n$ for $n = 1, \ldots, N$ and in all they are finitely many as k of them.

Now, choose a neighborhood U of x such that U is inside this V, a smaller neighborhood, I am going to choose such that $|h_{\lambda_i}(x') - h_{\lambda_i}(x)|$ is less than, (I have to the RHS carefully, some number, say $\sqrt{\epsilon}/\sqrt{2k}$. This is possible because all h_λ are continuous and we have to handle finitely many of them.

This yields, for all $x' \in U$, if you take the summation of the squares over $i = 1, \dots, k$, is less than $\epsilon/2$, since there are k such terms. It follows that sum of the $(H(x)(\lambda) - H(x')(\lambda))^2$ for all $\lambda \in \Lambda_n$ where n itself ranges over 1 to N is less than $\epsilon/2$.

So, that is a very rough estimate actually, but that is enough. Because this summation here, all are non negative terms. Though this is maybe infinite sum, for each fixed n the sum over all $\lambda \in \Lambda_n$ is a finite sum and is actually less than $2/n^2$, and hence when you range n from N to infinity it is less than summation $2/n^2$ from N to infinity which is less than $\epsilon/2$. This is true for ax' in U. Therefore this proves the continuity of H and thereby the theorem is proved.

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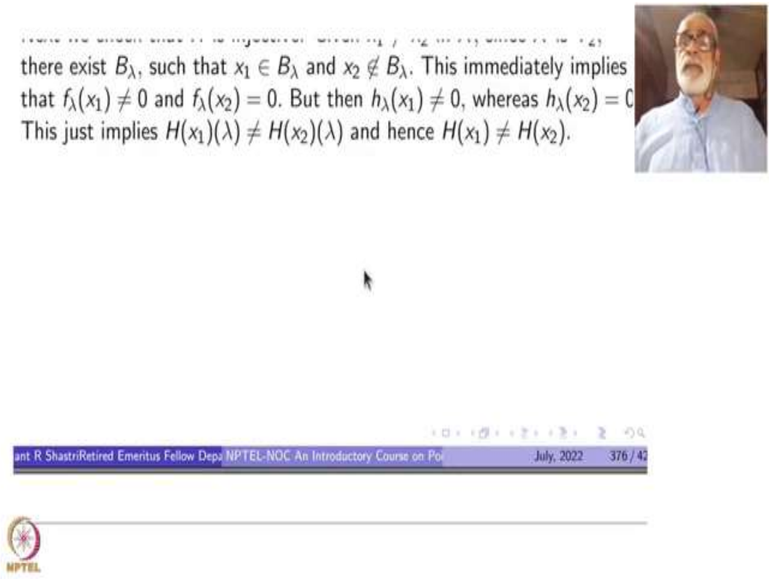

So, here are a few exercises for you. First one here is not very difficult. Every metric space is has a σ-locally finite base. I have used this result and mentioned it as an exercise separately earlier.

There are many problems in metrization, you may like to talk about. For example, when can a topological space be given a metric which is complete. Such things are called complete metrization problems. We shall not be able to discuss such things which are too special. Thank you.