An introduction to Point-Set-Topology Part-II
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Lecture 26
Urysohn's Metrization Theorem

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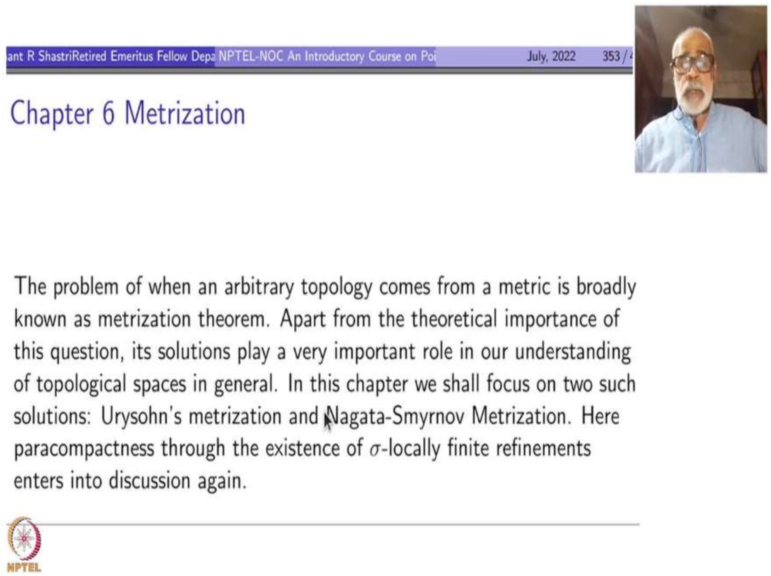

Hello. Welcome to NPTEL NOC, an introductory course on Point-Set-Topology Part-II. So we begin with a new chapter, Metrization. The problem of when an arbitrary topology comes from a metric is broadly known as metrization. Apart from the theoretical importance of this question, its solutions, let me say, there are many solutions, play a very important role in our understanding of topological spaces, in general.

In this chapter, we shall focus on two such solutions. One is Urysohn's Metrization, other one is Nagata-Smyrnov Metrization. The same kind of result was proved separately by Nagata as well as Smyrnov. So here, in the Nagata-Smyrnov metrization, paracompactness enters into picture, namely, with its property of admitting sigma-locally finite refinements for every open cover. So we are going to present the proof due to Nagata. For Smyrnov's proof, you can see the book of Willard, for example.

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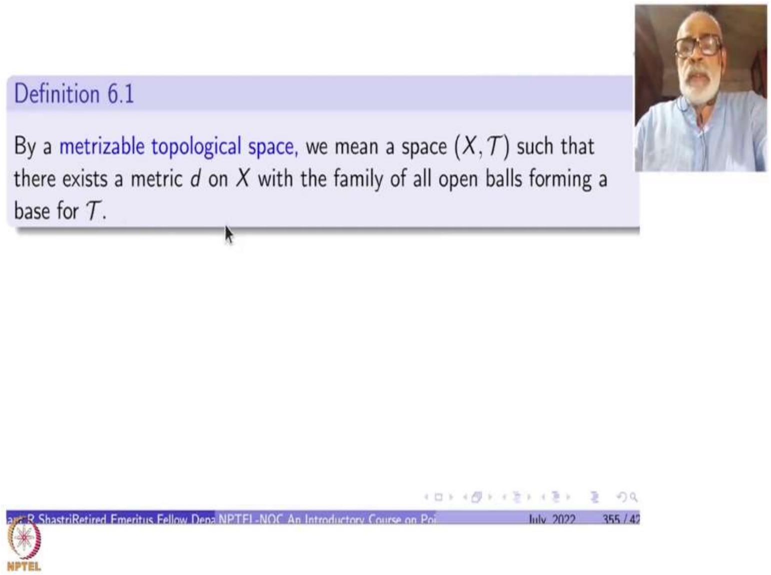

So let us make a formal definition. Take a topological space $(X, \mathcal{T})$. Then it is called metrizable topological space if there exists a metric on the underlying set X with the family of all open balls forming a base for tau. In other words, some set U is open if and only if for each point inside U, you have an open ball corresponding to the metric, centered at the point and contained inside given set U. That is the topology induced by the metric. If that coincides with tau, we will say $\mathcal{T}$ is metrizable.

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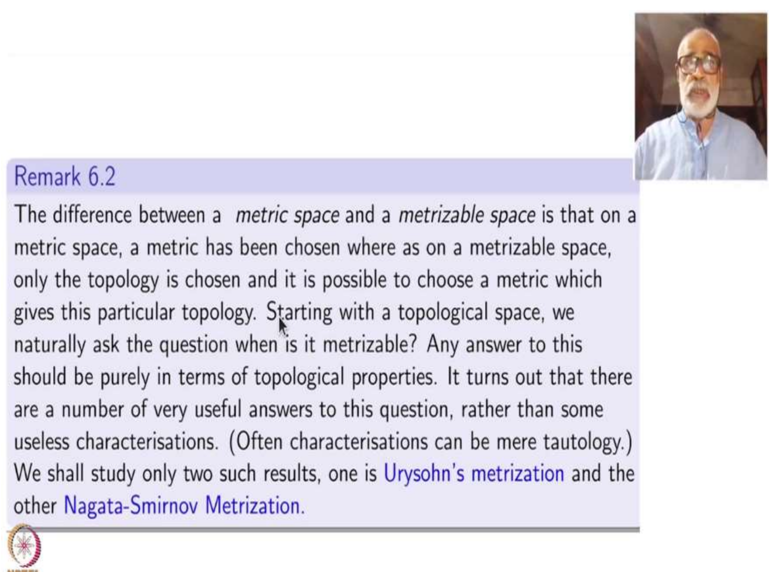

The difference between a metric space and a metrizable space is that on metric space, a metric has been chosen already, whereas on a metrizable space only the topology is chosen and it is possible to choose a metric which gives this particular topology. In fact, there may be several metrics which give the same topology but we will be interested only in the study of the topology there not the metric itself.

Starting with the topological space, we naturally ask the question when it is metrizable.

Any answer to this should be purely in terms of the topological properties. Turns out that there are number of very useful answers to this question, rather than some characterizations which may be useless. Often characterizations can be merely tautologies. That is why if-some-thing-then-this-happens kind of theorems are more valuable than if-and-only-if theorems. The alternative condition may be as difficult as the original. Mind you, it is not always the case. Several characterizations always help you to find things also in an easy way. So that is not a global remark anyway.

So we shall study only two such results here. One is Urysohn's Metrization, and the other one is due to Nagata-Smyrnov.

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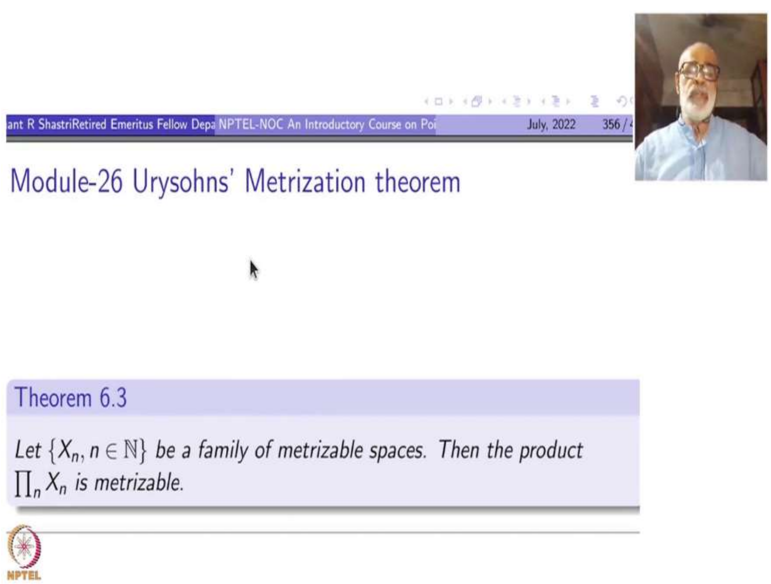

So today in module 26 let us concentrate on Urysohn's Metrization theorem. To begin with, we have this theorem about products which will be used in the proof of the Urysohn's metrization. What does it say?

Take a countable family of metrizable spaces. (This `countable' condition is important, It of course includes finite). Then the product is metrizable, with the product topology here. Whenever you take the product, the first thing you do is to take product topology.

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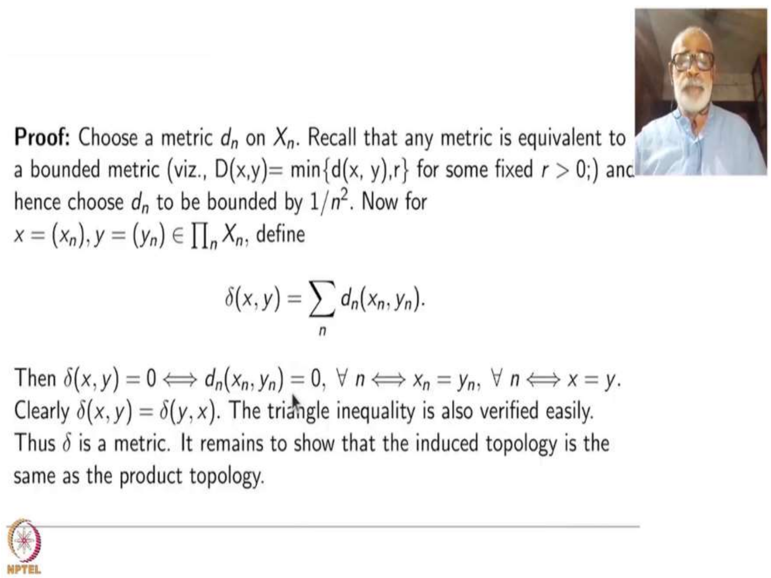

So how do you do that? To begin with, for each n, you can choose any metric on X_n, there is one, which gives you the corresponding $\mathcal{T}_n$, whatever topology it may be. But then, I would like to choose it carefully by choosing it to be bounded by some number, $r > 0$. You know how to do that. Namely, if d is a metric, you can define capital $D(x, y)$ to be the minimum of $d(x, y)$ and r.

In particular for each d_n, I will do the same thing taking $r = 1/n^2$. Maybe $1/n$ will also do. Something, some control for each n, the control should become stronger and stronger as n increases. That is all I need here. So you can take $r = 1/2^n$, for example.

Now, once I have chosen that, now for each $x = (x_n)$ and $y = (y_n)$, i.e., x and y inside the product of all X_n, we define $\delta(x, y)$ to be the sum of all $\mathrm{d}_n(x_n, y_n)$. So when I take this sum, you should know that this is convergent. Otherwise this will not make sense as an element of $\mathbb{R}$.

And that is precisely the role of this choosing these metrics properly. The n-th metric d_n is bounded by $1/n^2$. So each element here $d_n(x_m, y_n)$ will be less than or equal to $1/n^2$. Therefore, the sum total, less than or equal sum total $1/n^2$ which is convergent. So $\delta(x, y)$ makes sense, no problem.

Moreover, I want to claim that this delta is going to be a metric on the product space.

So first of all suppose, $d(x, y)$ is 0. That means look at this summation, this summation is sum of all non negative numbers. So if the total is 0, each of them must be 0. $d_n(x_n, y_n)$ is 0, implies $x_n = y_n$. This is true for every n which means x and y are the same elements.

Of course, if you interchange the slots for x and y this value δ does not change because each d_n is symmetric. Therefore $\delta(x, y) = \delta(y, x)$.

The triangle inequality is also valid because it is valid for d_n for all n. $d_n(x_n, y_n) + d_n(y_n z_n)$ is less than or equal to $d_n(x_n, z_n)$. So you can take the sum. That will give you $\delta(x, y) + \delta(y, z)$ is less than or equal to $\delta(x, z)$. So that is also easily verified.

So what remains now is to to prove that this metric gives you the topology that we have already chosen, namely, the product topology product set.

Giving a metric on a set, is not at all difficult. There are so many metrics. You can just take a discrete metric also and so on. The point is that the topology that the metric induces should be same as the one that we start with. That is the thing.

One thing is sure, namely, if this is a finite product. So that is the way we have become bold enough here to do this one, and our boldness pays here.

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Let $\epsilon > 0$, and $x \in \prod_n X_n$ be any point. Choose k such that $\sum_{n>k} 1/n^2 < \epsilon/2$. Choose $r > 0$ such that $r \sum_{n=1}^{k} \frac{1}{n^2} < \epsilon/2$. Consider the balls

$$B_n = \{y_n \in X_n \ : \ d_n(x_n, y_n) < r/n^2\}$$

in X_n around x_n. Let

$$U = \{y \in \prod_{n=1}^{\infty} X_n \ : \ y_n \in B_n, \ 1 \leq n \leq k\}.$$

So let $\epsilon > 0$ be given and let $x = (x_n)$ belong to product of X_n be any point. First, choose a number k such that summation $1/n^2$ for all $n \geq k$, is less than $\epsilon/2$. So this is the remainder term after k-terms in the convergent series summation $1/n^2$. That should be less than $\epsilon/2$.

Next, choose $r > 0$ such that the sum of $1/n^2$ where n ranges from 1 to k, viz., the sum of the first k terms multiplied by $r < \epsilon/2$. This sum be too large, so multiply it by r such that it becomes less than $\epsilon/2$. r has to be chosen that way. So, essentially, the sum of first k terms is controlled by r and the sum of the remaining terms are controlled, by the choice of k. So, that is the whole idea.

Now consider B_n to be the set of all y_n belonging to X_n such that $d_n(x_n, y_n) < r/n^2$. So B_n an open ball inside X_n centered around x_n. Now take U to be the subset of the product space consisting of points $y = (y_n)$ such that y_n is inside B_n only for $1 \leq n \leq k$. For the rest of the y_n's, there is no condition.

Clearly, it follows that x is inside U, because x_n is in B_n for $n = 1, 2, \ldots, k$. On the rest of the coordinates, there is no condition. That is clear. Also clear is that U is open in the product topology.

If y is inside U, what happens?

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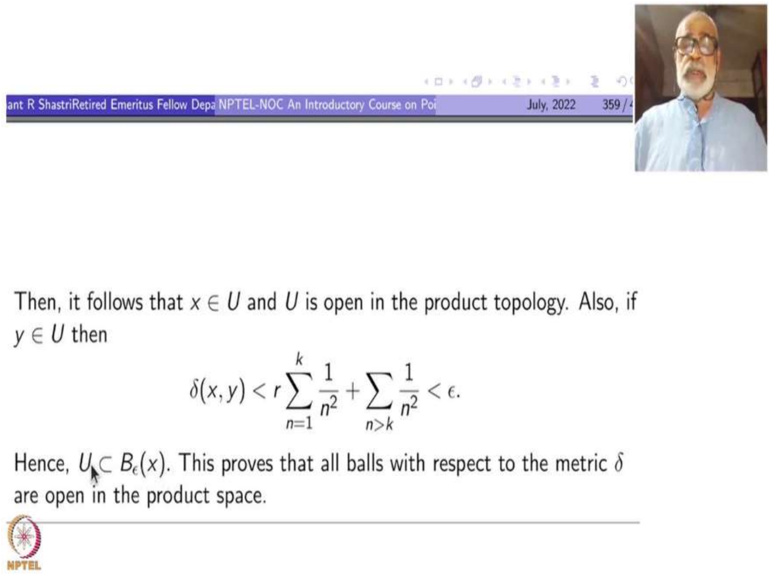

The first k coordinates of y_n are such that $d_n(x_n, y_n) \leq r/n^2$ and total is less than $\epsilon/2$. Also by the choice of k, the sum total of the rest of $d_n(x_n y_n)$ is less than $\epsilon/2$. This just mean that the distance δ between y and x is less than epsilon. That is, y is in the open ball of radius epsilon. Thus we have shown that U is contained in the open ball $B_\epsilon(x)$ of radius epsilon around x.

Since this is true for all x and all epsilon, this means every open ball in the metric topology is open in the product topology. So therefore, the metric topology is finer than the product topology.

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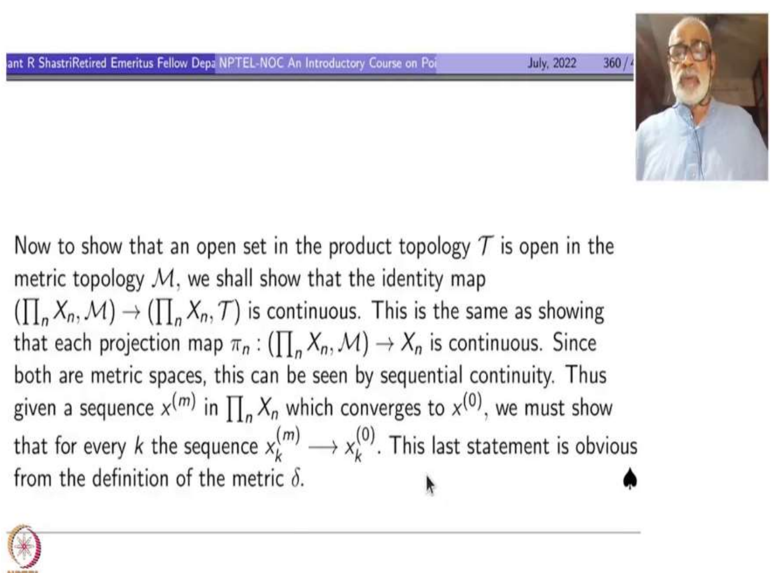

Now, to show that an open set in the product topology is open in the metric topology, we shall show that the identity map from the metric topology to the product topology is continuous. I have used the notation $\mathcal{M}$ to denote the metric topology on the product set and $\mathcal{T}$ to denote the product topology. Suppose I show that the identity map is continuous. What does that mean? Take any open set in the product topology, this is identity map, under the inverse image we get the same set which should be in $\mathcal{M}$.

So how to show any map into a product space is continuous? Any map into the product space is continuous if and only if all the coordinate projections of that map are continuous. So take projection map π_n from the product space into X_n, composed with the identity, it is just again the projection map π_n but the domain is now with the metric topology. What I get is, I have to show that product X_n with metric topology to X_n with the usual topology, whatever topology it comes, this is continuous. So that is what I have to show. Then, this identity map will be continuous.

Since both domain and codomain of π_n are metric spaces, because on $(X_n, \mathcal{T}_n)$ coincides with the topology given by d_n, continuity of π_n can be easily checked by sequential continuity. Suppose $\{x^{(m)}\}u$ is a sequence in the product space which converges to $x^{(0)}$, then we have to show that or every n, $\{x_n^{(m)}\}$ converges or $x_n^{(0)}$. That would prove the sequential continuity of π_n. But what is the meaning of this sequence converges to this point? The sequence $\delta(x^{(m)}, x^{(0)})$ converges to zero.

Note that this is a sequence of non negative terms, and we have $0 \leq d_n(x^{(m)}, x_n^{(0)}) \leq \delta(x^{(m}, x^{(0)})$. By sandwich theorem, this means that $d_n(x^{(m)}, x^{(0)})$ also converges to zero. This is the same as saying that $\{x_n^{(m)}\}$ converges or $x_n^{(0)}$.

So what we have proved here is that a countable product of metric spaces is metrizable.

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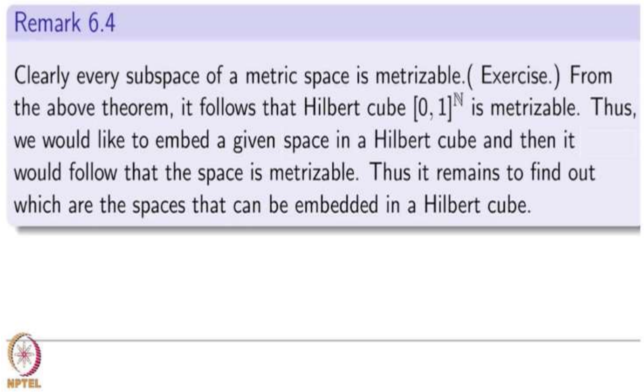

Now, every subspace of metric space is metrizable, there is no problem. You have seen this one in part I itself. You can take the restricted metric that will give you the subspace topology, that is all. From the above theorem, it follows that if you take $[0,1]^{\aleph_0}$, a countable product of copies of the closed interval $[0,1]$, that is metrizable.

In general, I can call it Hilbert cube when I am refering to it as a topological space. There is no problem. As a metric space, you will have to see what metric you give. Here, I am taking only product topology. It can be made into a metric sapce. Quite often people call it Hilbert cube only after choosing a specific metric.

So that is why I have said that this is metrizable. Thus, we would like to embed a given topological space in a Hilbert cube. Then it will follow that that the given topological space is also metrizable. Thus, it remains to find out which are the spaces that can be embedded in a Hilbert cube.

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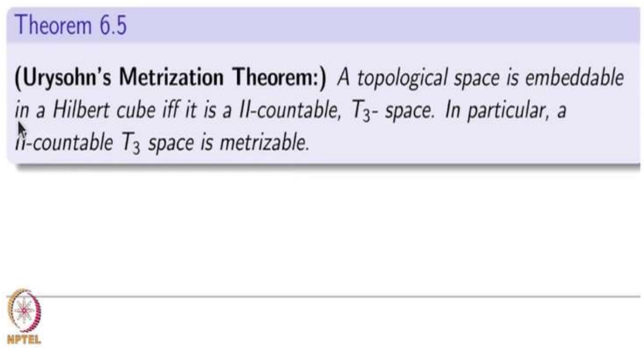

A topological space is embeddable in a Hilbert cube if and only if it is a second countable T_3 space. In particular, a second countable T_3 space is metrizable.

So this is the final theorem of Urysohn's Metrization. What we are going to do? Take a second countable T_3 space, i.e., regular and T_1 and we will show that it can be embedded in the Hilbert cube.

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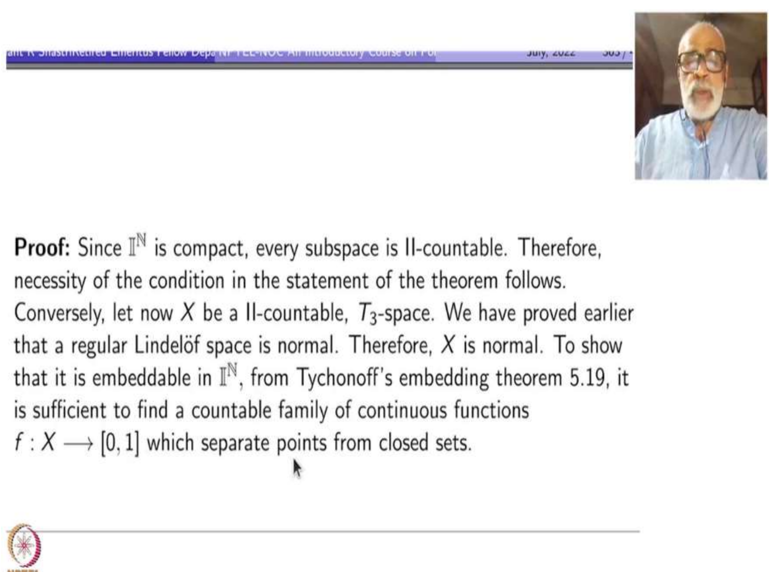

In fact, what it says is if you have a subspace of a Hilbert cube, it has to be second countable T_3 space. This is if and only if. There is no other choice. So Urysohn's

Metrization theorem is used with because of its `if' part. It is only a partial solution to our problem, the sense that there may be many other metric spaces which are not necessarily second countable. It does not answer those things. So it is only a partial answer but it is a very useful theorem.

Proof is very easy now. See $\mathbb{I}^{\mathbb{N}}$ is compact and every subspace of a compact metric space is second countable. Therefore, the necessary condition, in the statement of the theorem follows. Any subspace of a metric space is in fact T_5 also. So it will be automatically T_3.

Now conversely, let X be a second countable T_3 space. We have proved earlier that a regular Lindelof space is normal. Second countable implies Lindelof. T_3 includes regularity. Therefore, our space is automatically normal. To show that it is embeddable in $\mathbb{I}^{\mathbb{N}}$, we will use Tychonoff's embedding theorem.

So, 5.19. It is sufficient to find a countable family of continuous functions from X to $[0, 1]$ which separates points and closed sets. If it separates points, the corresponding embedding will be, corresponding function that we are getting will be injective. If separates closed sets and points, it will be an open mapping onto the image. That is why, it is an embedding that is how we have proved it.

So we will only prove this part now that there is a countable family of continuous functions which separates points as well as closed sets. So automatically, separate points because points are closed in our X because X is also Hausdroff space also, T_1 space also.

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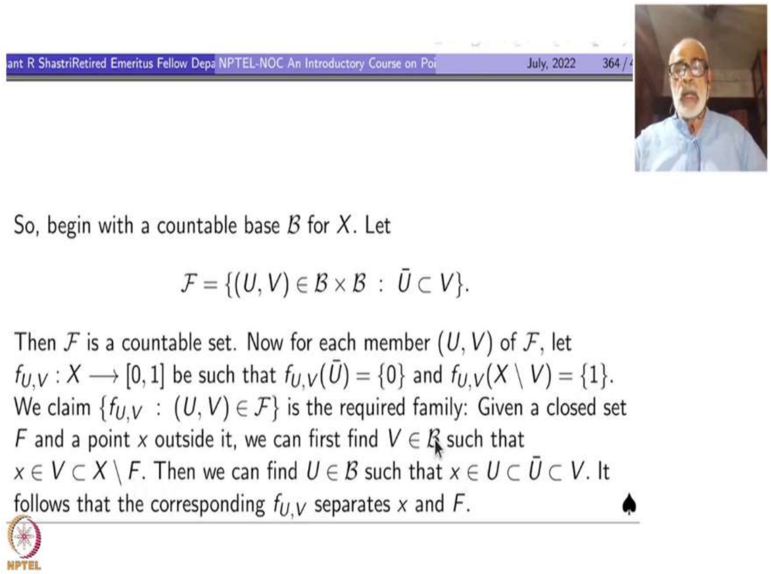

So begin with a countable base $\mathcal{B}$ for X. This countable family is now coming because of second countability of X. So start with the countable base for the topology of X. Put$\mathcal{F}$ equal the subset of $\mathcal{B} \times \mathcal{B}$ consisting of pairs (U, V) such that $\bar{U}$ is contained inside V. So this kind of things will make sense because of regularity. Otherwise, such things may be empty. That should not happen. In fact, there are plenty of members in $\mathcal{F}$ due to regularity of X.

Nevertheless, $\mathcal{F}$ is a countable family because $\mathcal{B} \times \mathcal{B}$ is countable. Now, for each member $(U, V) \in \mathcal{F}$, let $f_{U,V}$ from X to $[0, 1]$ be a continuous function such that this $f_{U,V}$ operating on $\bar{U}$ is singleton 0, and operating upon $X \setminus V$ is $\{1\}$.

So this is where we have used normality of X. You see $\bar{U}$ is closed, $X \setminus V$ is closed. They are disjoint because $\bar{U}$ is contained inside V. For each pair (U, V), you have an $f_{U,V}$, which takes $\bar{U}$ to 0 and $X \setminus V$ to 1. This family $f_{U,V}$ as (U, V) varies over $\mathcal{F}$ is the required family that you have to show that it separates points and close subsets.

Given a closed set F and a point x outside it, we can first find a V belonging to $\mathcal{B}$, in the base, such that x is inside V and V is contained inside $X \setminus F$. x belongs to $X \setminus F$, and $X \setminus F$ is open because F is closed. Therefore, you can find a basic element V, as above. Then, we can find U again inside V such that x is inside U contained inside $\bar{U}$ contained inside V because of regularity of X.

Now what we have got is this (U, V) is a member of $\mathcal{F}$. It follows that the corresponding $f_{U,V}$ separates x from F. Over. The whole of $\bar{U}$ goes to 0 and so x goes to 0 and the complement $X \setminus V$ goes to 1. And $X \setminus V$ contains F. So the proof of our theorem, Urysohn's metrization theorem is over.

Basically because we have already done Tychonoff's theorem and then we just finished the product theorem, this countable product of metrizable spaces is metrizable. So this is the way to remember. Tychonoff's theorem and this product metric for countable, metric. So let us do next time, Nagata-Smyrnov. Thank you.