An introduction to Point-Set-Topology Part-II
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Lecture 24
Complex and Extended Stone-Weierstrass Theorem

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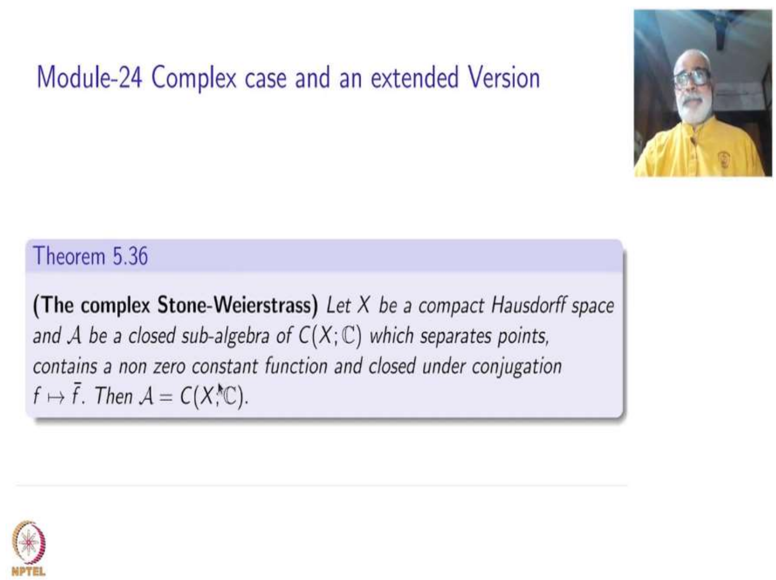

Hello. Welcome to Module 24 of NPTL NOC on Introduction to Point-Set Topology Part II. So we shall continue the study of Stone-Weierstrass theorem. Today, we will take the complex case and then we will also study the extended version, namely for locally compact spaces.

So theorem 5.36: Complex Stone-Weierstrass. Let X be a compact Hausdroff space, A be a closed sub-algebra of $C(X;\mathbb{C})$, the space of all complex value functions continuous on X. Let A be a closed subalgebra of $C(X;\mathbb{R})$ which separates points, (this hypothesis should always be there) and contains a non-zero constant (this is condition is optional, but this time, we are discussing results only under this hypothesis. There is one more condition). Further assume that A is closed under conjugation. (This hypothesis is very important in this case, viz., we are now studying complex valued functions).

The conjugation operation on elements of $C(X;\mathbb{C})$ is coming from the complex conjugation in $\mathbb{C}$. A functions taking values in $\mathbb{C}$, you know what is the conjugate of that.

So that is f going to $\bar{f}$. So if f is there, $\bar{f}$ should be also there is the extra condition, the meaning of closed under conjugation.)

Then this A is the whole of $C(X;\mathbb{C})$. This is the new statement.

If you remove this condition `closed under conjugation' and take $\mathbb{R}$ instead of $\mathbb{C}$, then this is the real version of Stone-Weierstrass theorem that we saw. Corresponding to the theorem of Gaddy, there is a version here that I will leave to you as an exercise. So we will only concentrate on the main version here.

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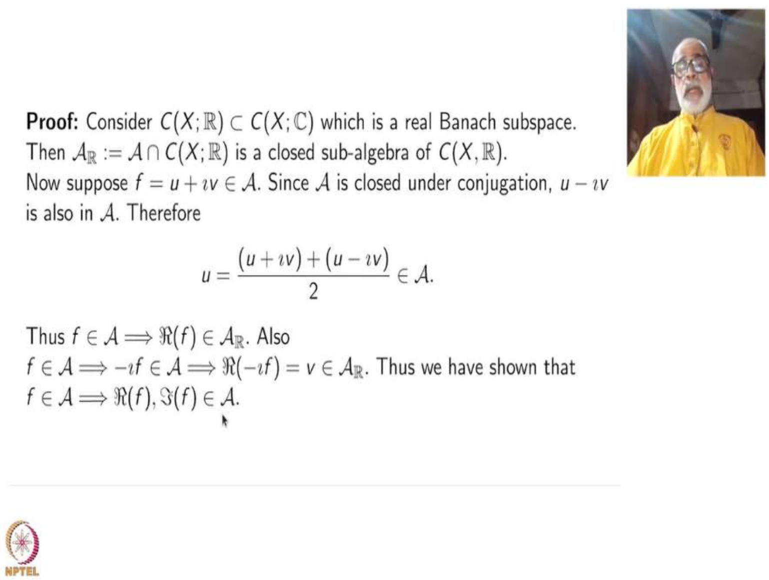

Consider $C(X;\mathbb{R})$ as a sub-algebra of $C(X;\mathbb{C})$ over the reals. This is like a real vector subspace of $C(X;\mathbb{C})$, which is a real Banach subspace. Let us put $A_{\mathbb{R}}$ here equal to A intersection $C(X;\mathbb{R})$. $C(X;\mathbb{R})$ is a real sub-algebra of $C(X;\mathbb{C})$. A is a complex sub-algebra but it is also real sub-algebra. So when you take the intersection, this will be the closed sub-algebra of $C(X;\mathbb{R})$, as an $\mathbb{R}$ algebra, $\mathbb{R}$ vector space, as a real vector space.

So this $A_{\mathbb{R}}$ is a sub-algebra, a closed sub-algebra real subalgebra. Now, something nice happens. Because A is closed under conjugation. So that is what we have to use. If I write an element of $C(X;\mathbb{C})$ as f equal to $u+iv$, any complex valued function can be written uniquely as a sum of its real part and the imaginary part. You can always write $u+iv$ where u and v are now inside $C(X;\mathbb{R})$. They are continuous also.

Since A is closed under conjugation, $u - iv$ will be also inside A. Therefore, their sum will be inside $A/2$ will be inside A, that is nothing but u. Similarly iv this will be also inside A. When you add the two you get $iv/2$ will be just divided by $2i$, you have to take that will be also v. So you see that f belongs to A implies real part of f is inside $A_{\mathbb{R}}$. Since it is inside A as well as inside $C(X;\mathbb{R})$, and therefore, it is in $A_{\mathbb{R}}$.

Similarly, the imaginary part also inside A, because once f is in A, if is in A, its real part is nothing but $-v$. Thus you have shown that f belongs to A implies both its real and imagined parts are inside $A_{\mathbb{R}}$.

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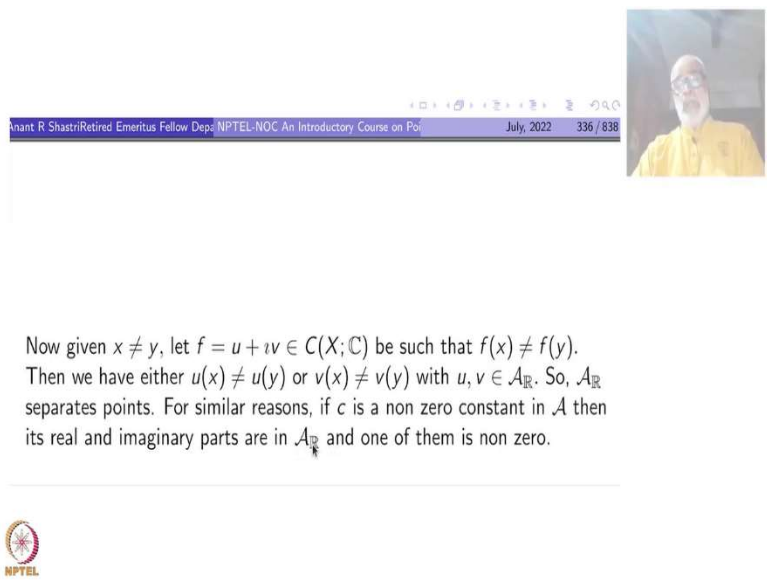

Now, given x not equal to y, let f belong to $C(X;\mathbb{C})$ be such that $f(x) \neq f(y)$. As usual you can write f as $u + iv$. Then $f(x) \neq f(y)$ implies either the real parts are distinct or the imaginary parts are distinct (or both). When two complex numbers are different, it can happen only if the real parts are different or imaginary parts are different.

Now, u and v are inside $A_{\mathbb{R}}$, so $A_{\mathbb{R}}$ also separates points. Either I can take the imaginary part or if I can give me a real part or imaginary part. So A separates points, implies $A_{\mathbb{R}}$ also separates points, thanks to closed under conjugation.

Without that, you could not have concluded this one.

For similar reasons, if c is a non-zero constant inside A, a non-zero constant could be what? Could be a complex number, one, either real part or imaginary part must be non-zero. So the same hypothesis will be true for $A_{\mathbb{R}}$ also. $A_{\mathbb{R}}$ also contains a non-zero constant. This time, non-zero constant has to be real. So you have to take real part or the imaginary part, whichever is non-zero. They will be there inside $A_{\mathbb{R}}$. So all the hypotheses are satisfied by $A_{\mathbb{R}}$.

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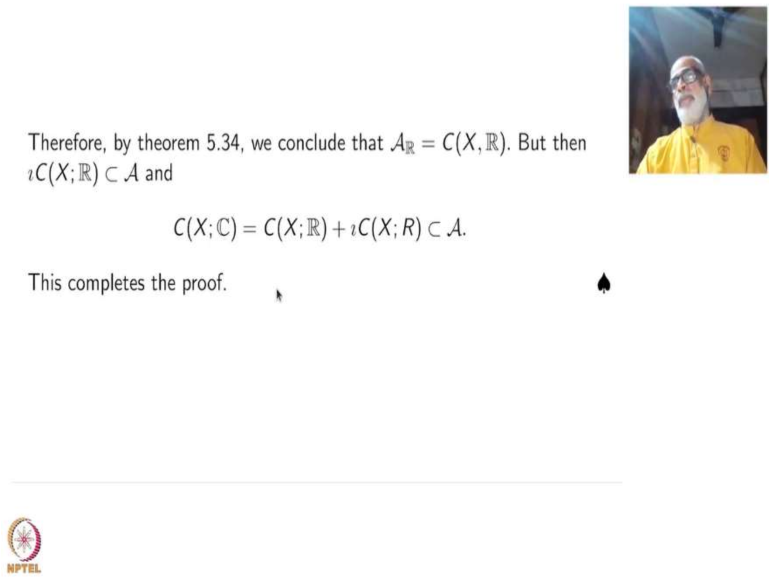

Therefore, by our real part theorem, we conclude that $A_{\mathbb{R}}$ must be equal to the whole of $C(X;\mathbb{R})$. But then $iC(X;\mathbb{R})$ will be also inside A because after the entire $A_{\mathbb{R}}$ is in A. A is a complex vector space. So i times this will be also inside A. Therefore $C(X;\mathbb{C})$ which is nothing but $C(X;\mathbb{R}) + iC(X;\mathbb{R})$, the sum is inside A. Therefore $A = C(X;\mathbb{C})$.

So that is the end of the proof for complex case. Alright.

Let us go to the extensions of this one, to the case of locally compact spaes. Once again, Alexandroff's one point compactification plays an important role here. Our job will be quite simple when you pass on to the Alexandroff's compactification. And then we can apply these theorems.

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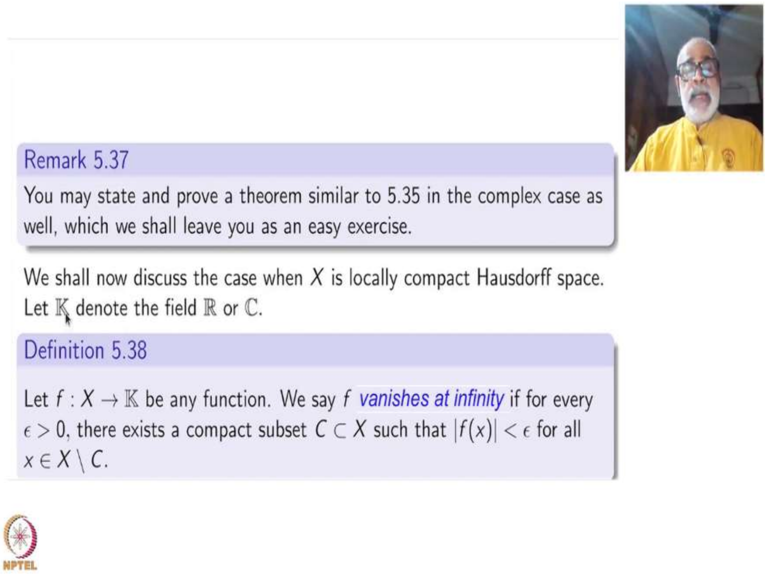

So I have told you, I repeat, you may state and prove a theorem similar to this 5.35 due to Gaddy in the complex case, which we shall leave you as an exercise. We shall now discuss the case when X is locally compact and Hausdroff. As usual, we will now combine both real and complex cases. This K denotes either $\mathbb{R}$ or $\mathbb{C}$.

So take a function f from X to $\mathbb{K}$. We say f `vanishes at infinity', (this is a phrase which I am defining. I am not defining infinity here. There is no infinity as such yet but we are defining what is the meaning of `f vanishes at infinity'. Understand? What is that?) If for every epsilon positive, there exists a compact subset C of X such that $|f(x)| < \epsilon$ for all $x \in X \setminus C$, x in the complement of C.

Away from a compact set, I should be able to control the value of $f(x)$. $|f(x)|$ should be arbitrarily small. Whatever ϵ, I have been given, correspondingly, I should be able to choose C. C will depend upon C as well as f, of course, such that away from C, $|f(x)|$ will be less than ϵ. That is the meaning of vanishing at infinity.

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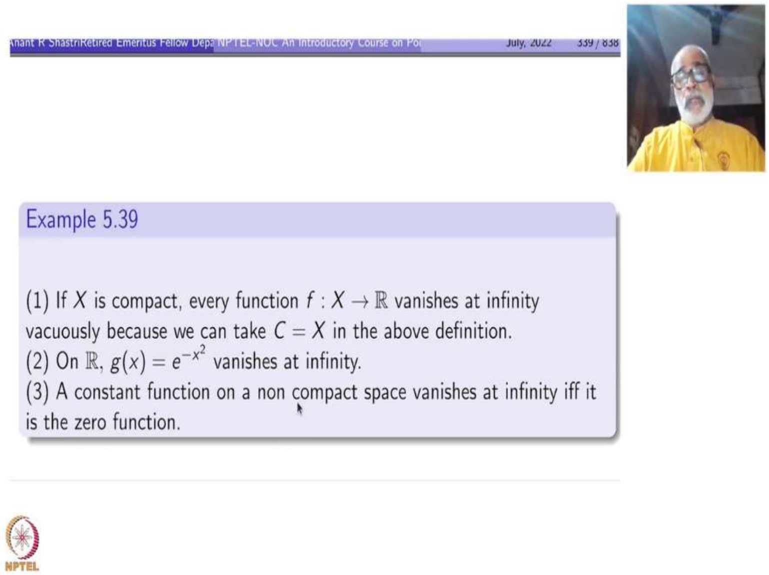

Just let me give you some examples.

If X is compact, of course, every function from X to $\mathbb{R}$ vanishes at infinity, vacuously. Why? Because I can take C equal to the whole of X and $X \setminus C$ is empty. So there is no condition. That is all.

On $\mathbb{R}$, look at this function given by $g(x)$ equal to e^{-x^2}. At $x = 0$, this is 1. And then it just tapers down and as x goes to infinity, this tends to 0.

The above example is a model. How are we going to use these word as x goes to infinity this function tends to 0, in the gereal case? So we have converted that into arbitrary topological space by this definition here, vanishing at infinity. Outside a compact set, it will become less than epsilon. Given any epsilon, you can choose your compact set such that that condition is alright. That is the meaning of that. So this example has motivate the definition above.

A constant function on a non compact space vanishes at infinity if and only if it is the 0-function. Take any constant function, take your epsilon to be such that it is less than modulus of that constant. That is possible if the constant is not zero. Then, that condition will not be satisfied at all. So it will not vanish at infinity.

This is a strong conclusion here, you see, because we are interested in subalgebras, having non-zero constants. With non-zero constants, then this condition will definitely fail now. So these two conditions are somewhat opposite of each other.

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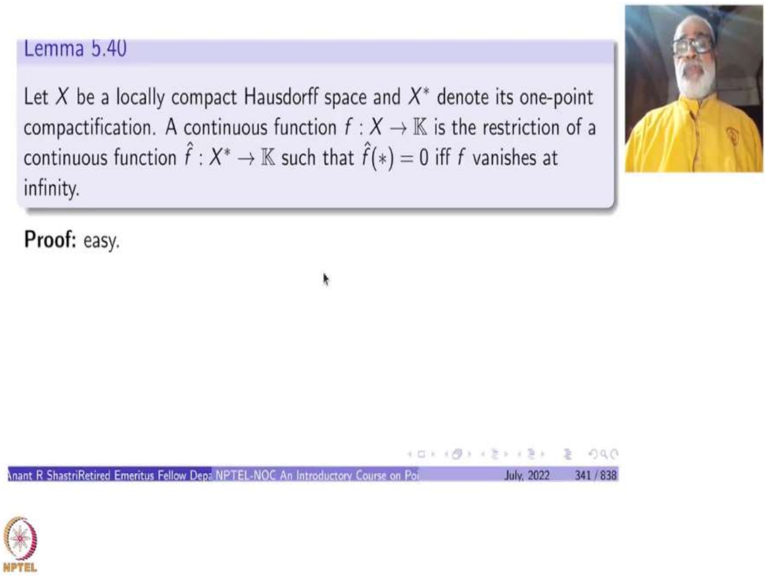

So let X be a locally compact Hausdroff space. (By the way, this condition is there for all spaces below). Now we are only interested in locally compact Hausdroff spaces.) Let X^star denote is one-point compactification. I want to specifically say that its one-point compactification when I mean the, so I am referring to Alexandroff's compactification.

A continuous function f from X to $\mathbb{K}$ is the restriction of a continuous function $\hat{f}$ from X^* to $\mathbb{K}$ such that $\hat{f}(\star)$ is 0, if and only if f vanishes at infinity.

So the definition of vanishing at infinity has been given a different meaning here. Take a $\mathbb{K}$-valued continuous function which vanishes at infinity on a locally compact Hausdroff space X.

Then, you can extend it to a continuous function on the one-point compactification by sending the infinity or the star to 0. And conversely.

You can always define $\hat{f}(\star)$ equal to 0, but then $\hat{f}$ may not be continuous. After defining this way, if it is continuous, then this f must be vanishing at infinity. So this is just a consequence of the definition of the topolpogy X^*, namely, what are the open subsets of

X^* which contain the point infinity. What are the neighborhoods of infinity? They are nothing but the complements of compact and closed subsets of X. So that is the hypothesis. So that will automatically give you this one. So I have, I am not going to explain anything more than that here. The proof is easy.

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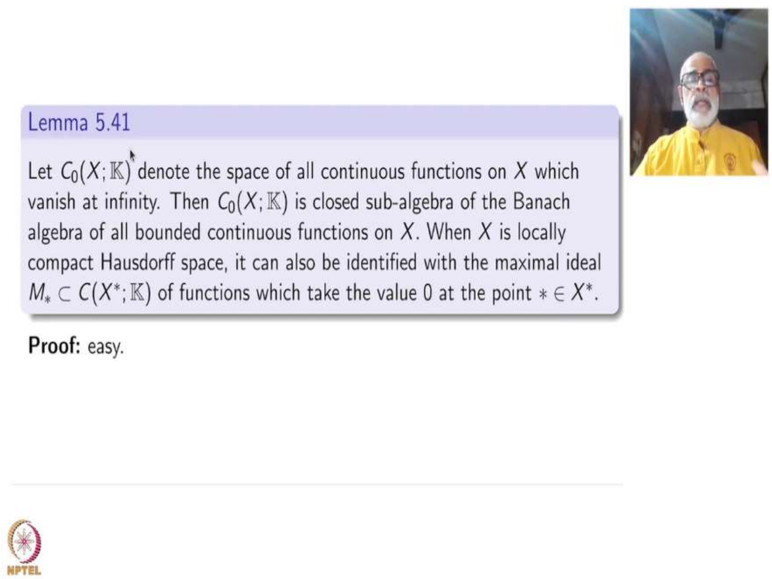

So now comes this extra notation we have here. Let $C_0(X;\mathbb{K})$ denote the space of all continuous functions on X which vanish at infinity. Then $C_0(X;\mathbb{K})$ is a closed sub-algebra of the Banach algebra of all bounded continuous functions on X. When X is locally compact Hausdroff space, it can also be identified with the maximal ideal M_* of the algebra $C(X^*;\mathbb{C})$.

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Theorem 5.42

(Extended Stone-Weierstrass) *Let X be a locally compact Hausdorff space and $\mathcal{A} \subset C_0(X, \mathbb{K})$ be a closed sub algebra which separates points of X and is closed under conjugation, if $\mathbb{K} = \mathbb{C}$. Then either $\mathcal{A} = C_0(X; \mathbb{K})$ or there exists a point $x_0 \in X$ such that*

$$\mathcal{A} = \{f \in C_0(X; \mathbb{K}) \ : \ f(x_0) = 0\} =: M_{x_0,0}$$

Now, we can state the Extended Stone-Weierstrass theorem. Start with any locally compact Hausdroff space. Let A contained inside $C_0(X; \mathbb{K})$ be a closed subalgebra, which separates points of X and is a closed under conjugation, if $\mathbb{K} = \mathbb{C}$. (So this closed under conjugation is not necessary or even if you put it, it is harmless, whenever $\mathbb{K}$ is $\mathbb{R}$. That is the only case, I have to mention these two results separately. Otherwise, the proofs are all the same. Together, I can handle the case $\mathbb{R}$ or $\mathbb{C}$ together. As soon as $\mathbb{K} = \mathbb{C}$, you should assume this extra condition. That is all.)

Then the conclusion is that, either A is $C_0(X; \mathbb{K})$, the entire algebra or you have a unique point $x_0 \in X$ such that this A is $M_{x_0,0} := M_{x_0} \cap C_0(X; \mathbb{K})$. I am using this notation because we should not confuse it with, the M_{x_0}, which has a different meaning in $C(X; \mathbb{K})$. This is the subspace of all f in $C(X; \mathbb{K})$ which vanish at x_0 as well as vanish at infinity. So, you could use the notation $M_{x_0,\infty}$ if you like. Fine. Of course, this is also equal to the set of all continuous functions $\hat{f}$ from X^* to $\mathbb{K}$ which take the value zero at x_0 as well as infinity.

So the sub-algebra A can be this one or it should be the whole of it. So this is the conclusion of the extended Stone-Weierstrass theorem. So we are not going to study $C(X; \mathbb{R})$ in particular here, the whole space. We are only going to study those which vanish at infinity. So that is the key for us. So that we can use the compactification of X.

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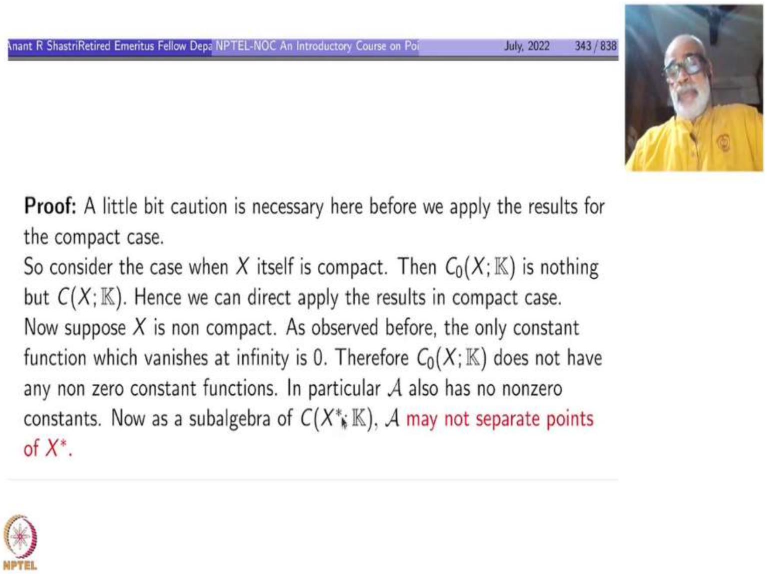

A little bit caution is necessary here before we apply the result for compact case. I told you that we want to convert the problem into studying the space of continuous functions on a compact space, by going to one-point compactification. But you have to be a bit cautious here.

Namely, consider the case when X itself is compact. Then $C_0(X;\mathbb{K})$ is nothing but $C(X;\mathbb{K})$, which we have seen. Every point, every function now vanishes at infinity. Hence, we can directly apply the result in the compact case. So, we do not have to prove this one, but we have to consider this one because a compact Hausdroff space is also a locally compact Hausdroff space.

That is why we have to check that the statement is correct in that sense that is all. We are not proving that. So that part is taken care of. Therefore, we can now come to the case when X is locally compact. So I repeat, consider the case when X is compact, then $C_0(X;\mathbb{K})$ is nothing but $C(X;\mathbb{K})$. Hence we can directly apply the earlier theorem.

Now, come to the case when X is non compact. As observed before, the only constant function which vanishes at infinity is 0. Therefore, $C_0(X;\mathbb{K})$ does not have any non zero constants. In particular, the subalgebras of $C_0(X;\mathbb{K})$ also have no nonzero constants. Therefore, the regular Stone Weierstrass Theorem cannot be applied here. So you have to take Gaddy's version, the other part.

Now, as a subalgebra of $C_0(X^*; \mathbb{K})$, A may not separate points of X^*. So that is also a problem. We have assumed that A separates points of X but when you pass to X^*, and think of A as a subalgebra of $C_0(X^*; \mathbb{K})$ this may fail to separate points, because there is an extra point now iX^*. So we have to be careful.

So let us study these things case by case.

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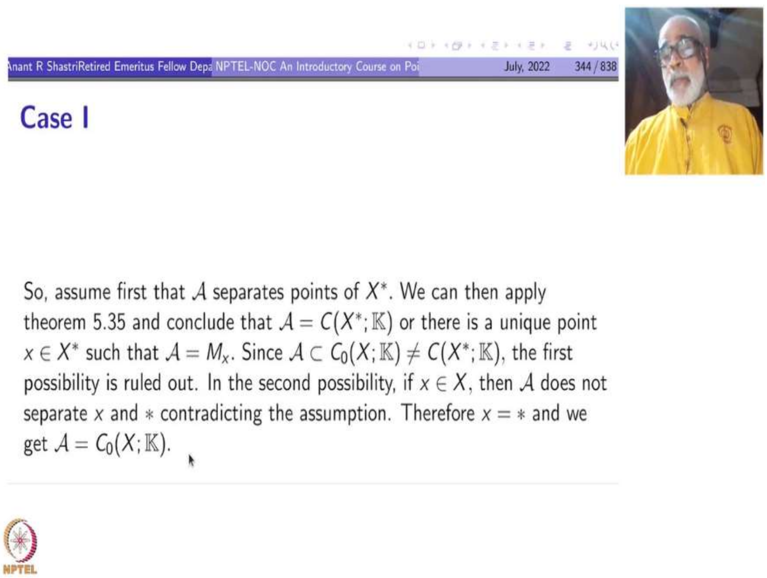

So assume first that A separates points of X^* also. Remember, A is a sub-algebra of $C_0(X, \mathbb{K})$. Therefore, for each $f \in A$, you can put $\hat{f}$ and think of that as a function from X^* to $\mathbb{K}$. There is a unique such $\hat{f}$. What is $\hat{f}$? $\hat{f}(\star)$ is going to 0, that is the only way you can extend f. So you can think of A as a sub-algebra of $C_0(X^*; \mathbb{K})$.

We can then apply the standard theorems 5.35 Gaddy's version to conclude that A is $C(X^*, \mathbb{K})$ or there is a unique point $x \in X^*$ such that $A = M_x$. So these are the two parts of that. But A is already in $C_0(X, \mathbb{K})$, which is not the whole of $C_0(X^*; \mathbb{K})$. So, the first possibility is ruled out.

Therefore, it is the second possibility. But then this unique point x cannot be inside X because then A does not separate the two points x and $\star$, contradicting the assumption. So this x must be equal to $\star$. Therefore, A is nothing but $C_0(X, \mathbb{K})$. Wonderful.

Now comes the little more complicated case. Suppose A does not separate points of X^*. Then, what I am going to do?

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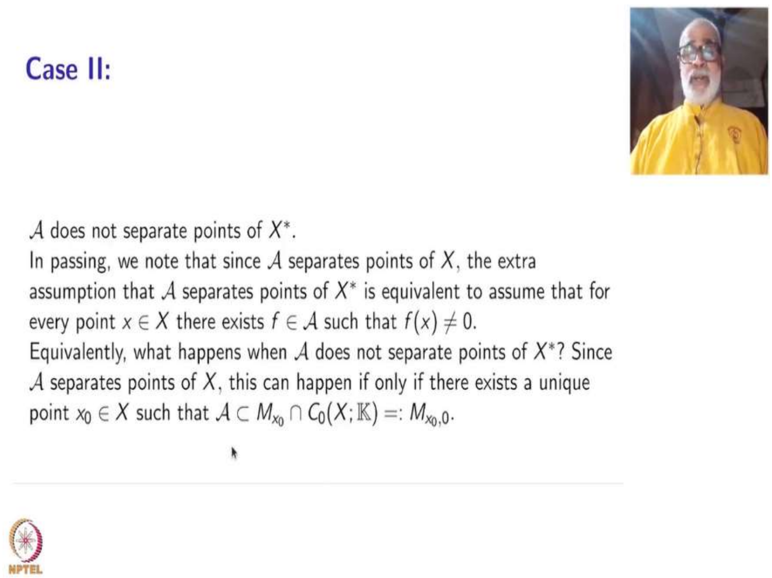

In passing, we note that since A separates points of X, the extra assumption that A separates points of X^* is the same thing as assuming that for every point $x \in X$, there must be a function f_x belong to A such that $f_x(x)$ is not equal to zero. This is just the same as saying that A is not contained in any maximal ideal M_x.

If this is not true, means equivalently what happens when A does not separate points of X^*? Since A separates points of X but does not separate points of X^*, you have to analyze this correctly, this can happen only if there exists a unique point $x_0 \in X$ such that A is inside $M_{x_0} \cap C_0(X; \mathbb{K})$ which we have denoted by $M_{x_0,0}$. Now you have to show that equalityholds. So our task is not yet over.

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We now consider the subspace $Y = X \setminus \{x_0\}$ which is also locally compact and Hausdorff.
Subcase (a): When x_0 is not an isolated point of X.
The inclusion map $i : Y \to X$ induces an algebra homomorphism of algebras $i^* : C(X, \mathbb{K}) \to C(Y, \mathbb{K})$ which is injective, because Y is dense in X. Check that

$$i^*(M_{x_0} \cap C_0(X; \mathbb{K})) = C_0(Y; \mathbb{K}).$$

Now consider the one-pt compactification Y^* of Y. Since x_0 is the unique point such that $\mathcal{A} \subset M_{x_0}$, it follows that $\mathcal{A}$ separates points of Y^*.
Therefore, from Case -I, we conclude that
$\mathcal{A} = C_0(Y; \mathbb{K}) = M_{x_0} \cap C_0(X, \mathbb{K}) = M_{x_0,0}$.

Now consider subspace $Y = X \setminus \{x_0\}$. Throw away the point x_0. You have got an open subset Y of X, that is also locally compact Hausdroff. So that hypothesis is not yet changed. Now, I have to make some more cases here.

Suppose x_0 is not an isolated point. (That is the easier case, perhaps? Indeed, the other one is too easy, that is why I am considering it later.)

The inclusion map i from Y to X induces an algebra homomorphism, which I write as i^* from $C(X; \mathbb{K})$ to $C(Y; \mathbb{K})$. Y is a subspace. So what is this? Take Y to X the inclusion map and then follow it by the function f from X to $\mathbb{K}$. So it is $i^*(f)$. So it is like restriction map actually. But the restriction map is injective now. This is always there, this restriction algebra homomorphism is always there. In general it may not be injective.

It is injective here, because x_0 is not an isolated point in X, which just means that the subspace Y is dense in X. On a dense subset, if two continuous functions agree, they agree on the whole space. Therefore, this i^* is an injective mapping. Now look at the image of $M_{x_0} \cap C_0(X; \mathbb{K})$ under i^*.

$C_0(X, \mathbb{K})$ is what? Those things which vanish at infinity. Intersect with M_{x_0} gives those which vanish at x_0 also. So i^* of that is nothing but $C_0(Y; \mathbb{K})$. What is meaning of $C_0(Y; \mathbb{K})$? Again, those continuous functions on Y which vanish at infinity. As F varies over compact subsets of Y, its complement will be also a neghbourhood of $x_0 \in X$ as well as `neighbourhood of infinity for X.

So this is an important point. Without referring to any extraneous points and so on, this image under i^* is just the set of all continuous functions on Y which vanish at infinity.

Now, consider the one-point compactification Y^* of Y. Since x_0 is the unique point such that A is contained inside M_{x_0}, it follows that A separates points of Y^*. Uniqueness of x_0 is important here.

Now, you see, A did not separate points of this X^*. That was the starting assumption. But after throwing away x_0 and taking the one point compactification, in a sense merges the bad point x_0 along with the point at infinity to be $\star$. That is the whole idea here. It follows that this A now separates points of Y^*.

Therefore, we can conclude from Case I, that A must be $C_0(Y;\mathbb{K})$, which is why we have defined, this image of this one. I am just writing M_{x_0} intersection this itself because i^* is injective mapping.

So that is the end of one subcase. What is the remaining subcase? When x_0 is an isolated point of X.

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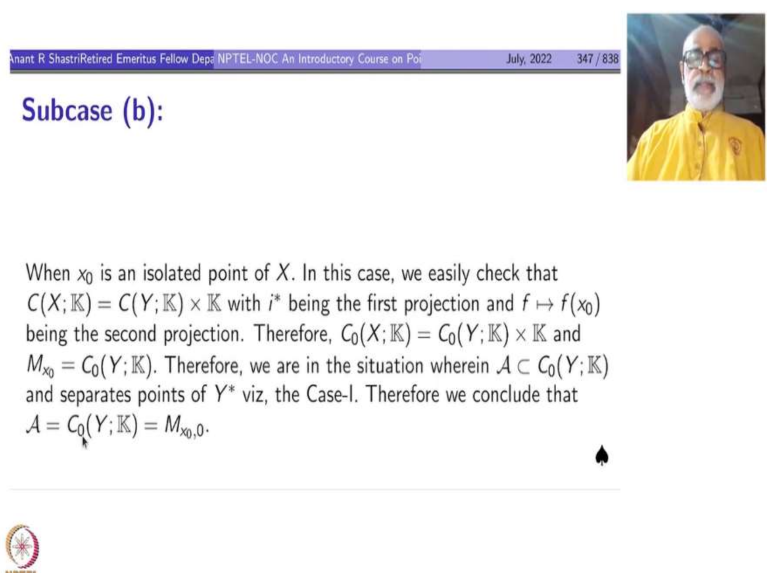

Here you do not need much topology, it is just algebra now. How does $C(X, \mathbb{K})$ look like? At the isolated point you are free to define the function whichever way you like. So that is a free point. So it can be assigned any real or complex number, any number in $\mathbb{K}$.

Therefore, what happens is $C(X, \mathbb{K})$ is nothing but $C(Y; \mathbb{K}) \times \mathbb{K}$. This second factor corresponds to arbitrary values taken at the point x_0. With, what is this product structure, I want to tell you. Namely, take the inclusion map i from Y to X, which we have taken earlier. Then i^* is the first projection. The second projection is the evaluation of the function at the point x_0. So if I know what are the two projection maps here, I know the product structure. In other words, every function element f of $C(X, \mathbb{K})$ can be written as the sum of $i^*(f) + f(x_0)$. That expression will be unique.

Therefore, $C_0(X; \mathbb{K})$ is nothing but $C_0(Y; \mathbb{K}) \times \mathbb{K}$. See, `vanishing at infinity' does not have any effect values of the function at the isolated point x_0. Because whenever you take a compact set, you can include the single point x_0 also inside this compact set. Away from that compact set vanishing is same thing. So $C_0(X; \mathbb{K})$ is $C_0(Y; \mathbb{K}) \times \mathbb{K}$.

And M_{x_0}, the ideal, is nothing but $C_0(Y, \mathbb{K})$ because this $\mathbb{K}$ factor will go away. Things which vanish, then $\mathbb{K}$ factor will go away. Therefore, we are in the situation wherein A is contained inside $C_0(Y, \mathbb{K})$, and separates points of Y^*, namely, Case 1. So subcase (b) is reduced to Case 1. Therefore, we conclude that this A is nothing but $C_0(Y, \mathbb{K})$ which is $M_{x_0,0}$. This was Case 1, if you should recall here.

So this completes the proof of the theorem that for locally compact spaces. How does a closed sub-algebra look like for locally compact Hausdroff spaces. Take a closed sub algebra of the algebra of all continuous functions which vanish at infinity. For that, you have this, this is one extension. I do not say this is the only extension. There can be many other possibilities here. This is one of the popular results. You can also have a look at Siemon's book for such versions. So thank you.

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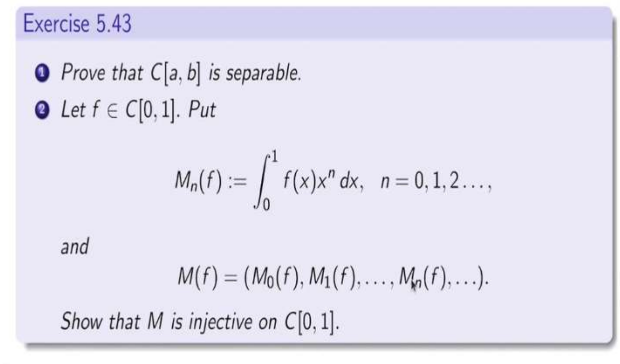

Maybe, I will let you know a few of the exercises here before closing up. They are not directly related to this one, but since we are studying function spaces, I think these are relevant.

(1) Prove that the space of all continuous functions on $[a, b]$, this Banach algebra, is separable. Separable Banach algebras are more rarer, and they are very important.

(2) Let f belong to $C[0, 1]$. Instead of $[a, b]$, I put $[0, 1]$ just for writing down. This result is actually applicable to $C[a, b]$ also. Put $M_n(f)$ equal to the integral over $[0, 1]$ of f multiplied by x^n. So this x^n is a weight function. For each $n = 0, 1, 2, 3, \ldots$ with one function f, you have these $M_n(f)$ which are these constants here. Put M_f equal to the sequence $\{M_0(f), M_1(f), M_2(f), \ldots\}$ and so on so. This is the definition.

So I have taken f and then I have produced a sequence here, which can be thought of as function M from $C[0, 1]$ to R^∞. Show that this map M is injective. What does it mean? If f and g are different, at least one of them, $M_n(f)$ and $M_n(g)$ will be different for some n. That is what you have to show.

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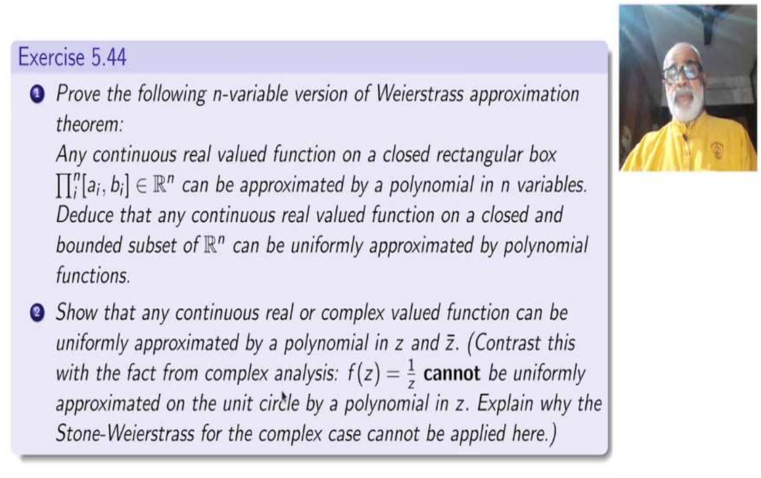

Now this exercise is directly related to the Stone-Weierstrass theorem. Through the following n-variable version of Weierstrass approximation theorem. Weierstrass approximation theorem was only for the interval, closed interval to $\mathbb{R}$. Now, you have to prove it for n variables. What is that? I have given the version also here in the next exercise.

(1) Any continuous real valued function on a closed rectangular box inside $\mathbb{R}^n$ can be approximated by a polynomial in n variables. Deduce from this, that any continuous real valued function on a closed and bounded subset of $\mathbb{R}^n$, (instead of the box), can be uniformly approximated by polynomial functions. Here also you can put the `uniformly approximated' word.

(2) Show that any real valued or complex valued continuous function can be uniformly approximated by a polynomial in a complex variables z and $\bar{z}$. A polynomial in z and $\bar{z}$ is different from just being polynomial in z and taking some conjugation. For example x can be written as $(z + \bar{z})/2$. So that is the whole idea here.

Contrast this with the fact from complex analysis. (If you do not know this, you may learn it from somewhere). Look at the simple function $f(z) = 1/z$. This function cannot be approximated uniformly on the unit circle by a polynomial in z.

The unit circle is closed and compact. Only unit circle you take. Do not take the whole disk. Whole disk does not make sense because $1/z$ is not defined on the whole disk; at 0 it is not defined. Try to find a sequence of polynomials which converges to $1/z$. It is not possible.

But why it should be true, I mean because why cannot we apply Stone-Weierstrass theorem for complex case to this function? After all, this is a nice function defined on $\mathbb{C} \setminus 0$. And then I am taking a compact subset there, the circle. So what is wrong? So you have to explain that. Do not make this mistake that Weierstrass theorem can be applied here. Nor Stone-Weierstrass theorem. Why? Just tell me why. There is a one line answer. That is this exercise.

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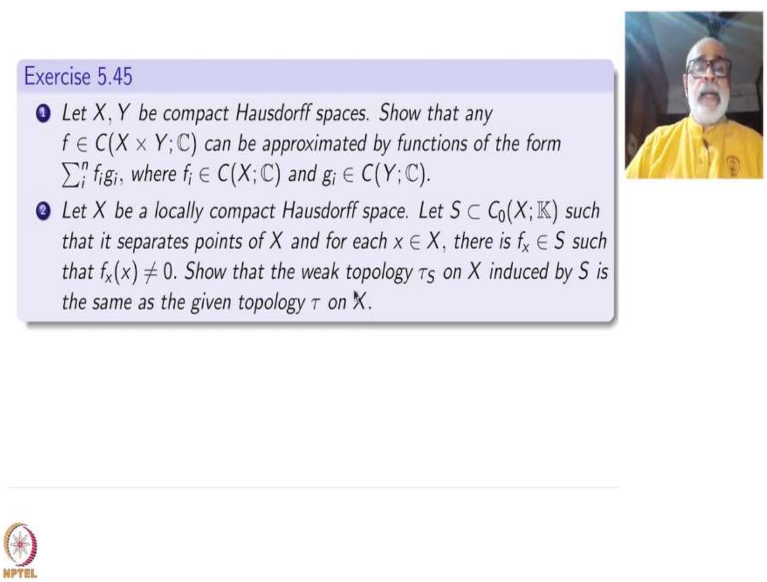

There are some more exercises. These exercises will be there in the pdf file I am going to give you anyway. So you do not have to depend upon the slide. But I would like to show it in the slide. That is all.

Alright. Thank you. So let us meet next time with a new topic.