An Introduction to Point-Set-Topology (Part II)
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Lecture 23
Real Stone-Weierstrass Theorem

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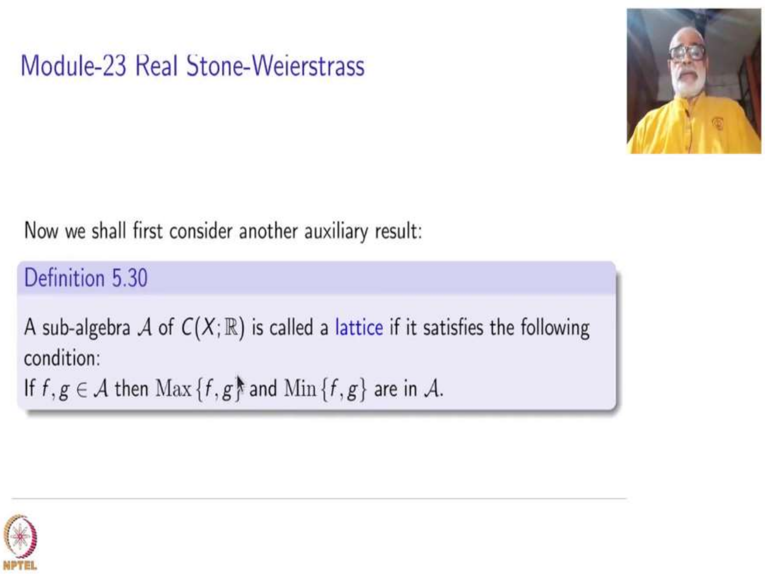

Hello, welcome to Module 23 of NPTEL NOC course on Point-Set-Topology Part II. So, we started with Stone-Weierstrass theorems, yesterday we have proved Weierstrass approximation theorem and prepared partly some ground for Stone-Weierstrass theorem. So, today we will continue. We will need one more concept which so far we have not introduced. We will introduce this concept only for directly applying it here in this context. So, we are not going to treat this on its own in general context.

So, this is the concept of a lattice. A subalgebra A of $C(X;\mathbb{R})$ is called a lattice if it satisfies the following condition: Given any two elements inside A, the maximum of f and g must be also inside A. Remember, maximum of any two continuous functions is again a continuous function, first of all. So, it is an element of $C(X;\mathbb{R})$. What we want is starting with f and g inside A, the maximum must be also inside A. Similarly, minimum must be also inside A. So, then we call it a lattice.

In particular what happens if you take instead of just two elements, suppose you take finitely many of them? Then also maximum and minimum make sense and they will also be there in A

by induction. f_1, f_2 first you take the maximum, then take f_3 and the maximum of these two, that is the same as taking the maximum of f_1, f_2, f_3 together. So, you can apply that. So, take finitely many elements of A then the maximum and minimum will be also inside A. So, that is the meaning of a lattice. So, let us continue the study.

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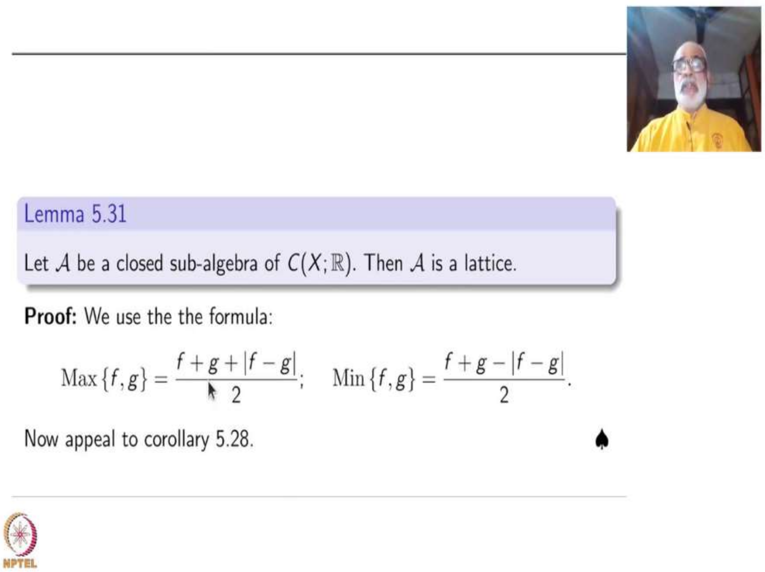

Let A be a closed subalgebra. This concept you have already introduced earlier. A closed subalgebra of $C(X;\mathbb{R})$ is a lattice. Therefore, this is comes to us free of charge, we want to start with closed subalgebras after all, and they are automatically lattices. So, this is the only additional property of a closed subalgebra that is what we are going to exploit here now. That is the content of this lemma.

So, first let us see that a closed subalgebra is a lattice. So, that is a one-line proof here. Because maximum of f and g can be written as $(f+g+|f-g|)/2$ and Minimum of f and g is also similarly equal to $(f+g-|f-g|)/2$.

So, this you must have seen in many other places like in measure theory and so on. So, if you have not seen it, spend some time and verify that these two are correct formulas. So, if f and g are inside A, A is an algebra, therefore, $f+g$ is there, and $f+g$ is also there. And its modulus is there why that is what we have proved earlier, there we have used to closedness of the subalgebra A.

Using Weierstrass theorem we have proved that if some function is there, then its modulus is also there provided the algebra is a closed subalgebra. Therefore, $f - g$ is there, $|f - g|$ is there, the sum total is there divided by 2 is also there. Similarly, minimum is also there. So, a closed subalgebra is a lattice is an easy consequence of what? Weierstrass theorem has been used here, namely, once f is there $|f|$ is there in a closed subalgebra. Let us go ahead now.

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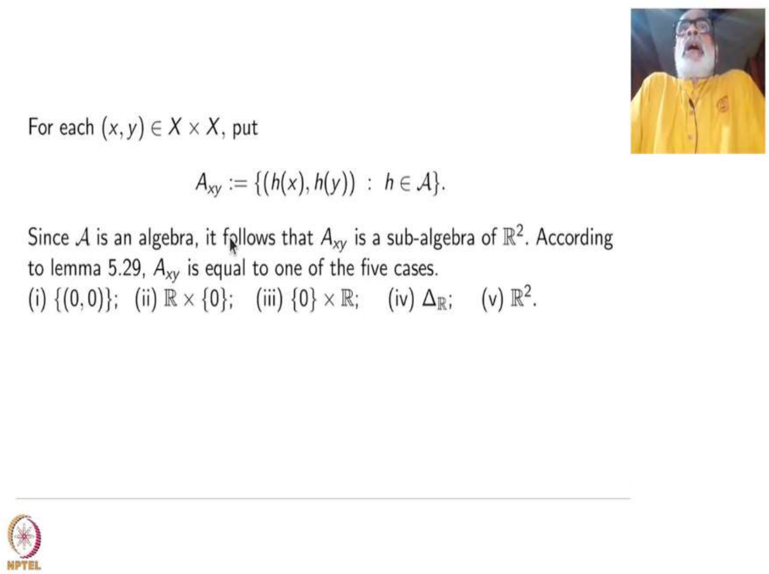

Last time we introduced this algebra $\mathbb{R} \times \mathbb{R}$, and we studied all possible subalgebras of $\mathbb{R} \times \mathbb{R}$. So, now, we are going to explore that. For each $(x, y) \in X \times X$, let us make this notation, A_{xy} is the set of all ordered pair $(h(x), h(y))$ of real numbers, it is a subset of $\mathbb{R}^2$ now, where h ranges over this $\mathcal{A}$, that $\mathcal{A}$ is the closed subalgebra that we are studying.

So, A_{xy} is subset of $\mathbb{R}^2$. Indeed, it is a vector subspace, it is easy to check that. Indeed, it is a subalgebra as well. If A is in algebra, it follows that A_{xy} is subalgebra of $\mathbb{R}^2$. Why? Because if you multiply two such elements, say $(h_1(x), h_1(y)) \cdot (h_2(x), h_2(y))$. It is $(h_1 h_2(x), h_1 h_2(y))$. So, that is the multiplication of two functions inside this algebra $C(X, \mathbb{R})$ and inside A also and $h_1 h_2$ is in A.

So, similarly addition scalar multiplication all these things you can check. So, this A_{xy} drops down to just a subalgebra $\mathbb{R}^2$ for every pair of elements (x, y) of $X \times X$. So, whatever is happening here, will tell you the entire story about the algebra A. That is the beauty of this approach here.

According to this lemma that we have studied last time, A_{xy} being a subalgebra of $\mathbb{R}^2$, is equal to one of these five things: 0, or $\mathbb{R} \times 0$, or $0 \times \mathbb{R}$ or the diagonal or the whole of $\mathbb{R}^2$. So, let us go ahead now.

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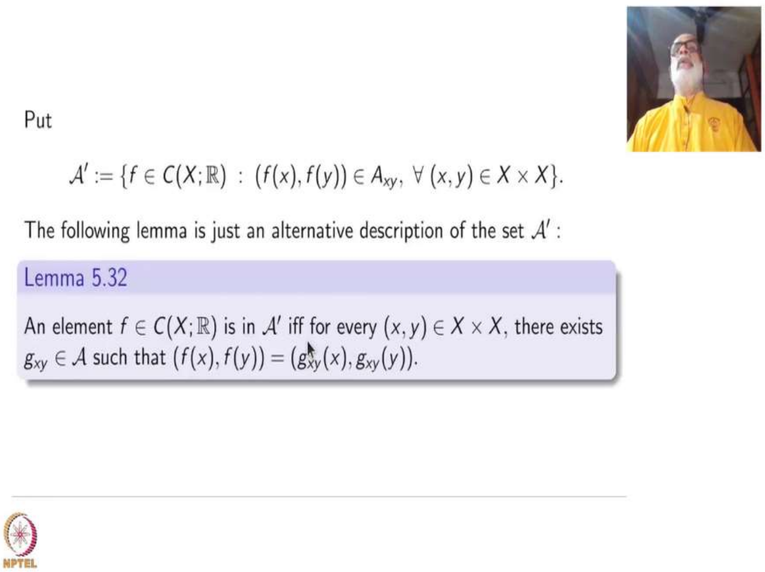

Let us put A' equal to all those $f \in C(X;\mathbb{R})$ such that when you take the ordered pair $(f(x), f(y))$, (it is a condition f), is inside A_{xy}, for all $(x,y) \in X \times X$. It is a non trivial condition on f, suppose f is already inside A, then this condition is automatic because the definition of A_{xy} itself is like that. So, A' automatically contains A. So, this A' looks like a fattening of A, something larger than A.

The following lemma is just an alternative description of what is this A'. So, here we are bringing the point- separation property of a subfamily of $C(X;\mathbb{R})$, in a strange way you have to just watch it out. Anyway, first of all I will give you an alternate description of this A'.

An element f inside $C(X;\mathbb{R})$ is inside A' if and only if, for every (x,y), there exists a g_{xy} inside A such that $(f(x), f(y))$ is equal to $(g_{xy}(x), gxy(y))$. This g_{xy} is inside A.

For each pair (x, y), there is a g_{xy} in A, with the above property. This is the strong way of separation of points. To begin with x and y may be chosen distinct points and f may be any continuous function which separates them. Then we want a member of A viz., g_{xy} to do that job, in a very stringent way namely, $g_{xy}(x) = f(x)$ and $g_{xy}(y) = f(y)$.

So, in some way this is trying to encode the point-separation property. But just do not bother about that one if you do not understand it now.

Just check that this description fits this definition of A'. Take an element f inside A'. That means what? $(f(x)f(y))$ is inside A_{xy}. What is A_{xy}? A_{xy} is nothing but some $(g(x), g(y))$, where g is an element of A. And that g, I am calling it g_{xy}, because it depends upon x and y. For different x and y, it could be different g's, that is why the notation g_{xy}. So, this description is obvious.

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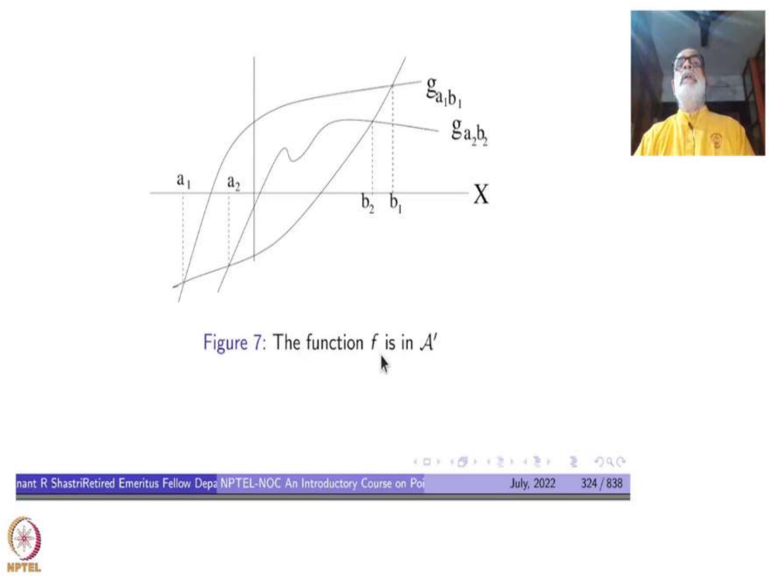

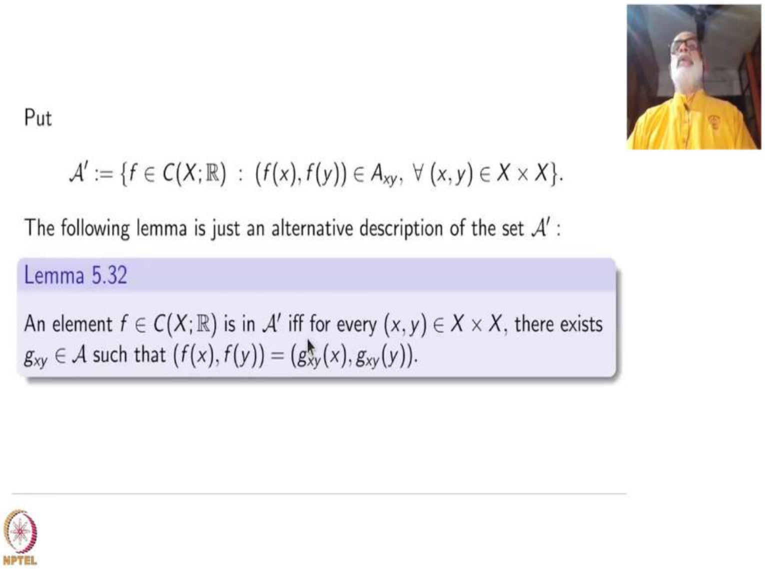

So, now, here is a picture of the same property that I have told. So, the function f is inside A' means what? This function is f here, some continuous function, take any pair of points $(a_1, b_1), (a_2, b_2), (a_3, b_3)$ whatever. Look at $f(a_1)$ and $f(b_1)$ it can be captured by a function $g_{a_1 b_1}$, f is just any element in the larger algebra $C(X;\mathbb{R})$.

There is an element g of A such that $f(b_1) = g(b_1)$ and $f(a_1) = g(a_1)$ so that is the meaning of this. Similarly, for any other pair (a_2, b_2). So, that is the meaning of capturing. So, somehow you know if you choose a lot of points fitting the curve then can guess the actual function approximately.

In all approximations, say for instance, Newton's approximations and so on, you choose a lot of points and then you choose something going through them that is an approximation for a function. So, that idea is coming here now, in a hidden way, in a very simple way, without our noticing it at all. All that you have to do is study the subalgebra of $\mathbb{R}^2$. So, we will see that how to do this one.

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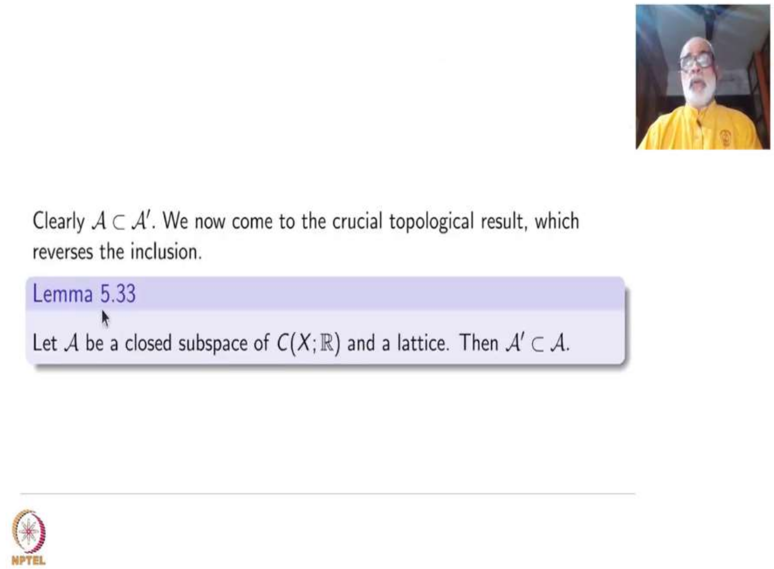

Clearly, I told you earlier that A is contained in A'. The crucial topological result in which we reverse this inclusion is the following: Let A be a closed subspace of $C(X;\mathbb{R})$ and a lattice. (In particular, for a closed subalgebra, these two conditions are satisfied.) Then this A' is contained inside A. In other words, these two are equal. So, this is the crucial lemma here. You have to do a little more combinatorial topology. Let us see, how you show that every element of A' is inside A.

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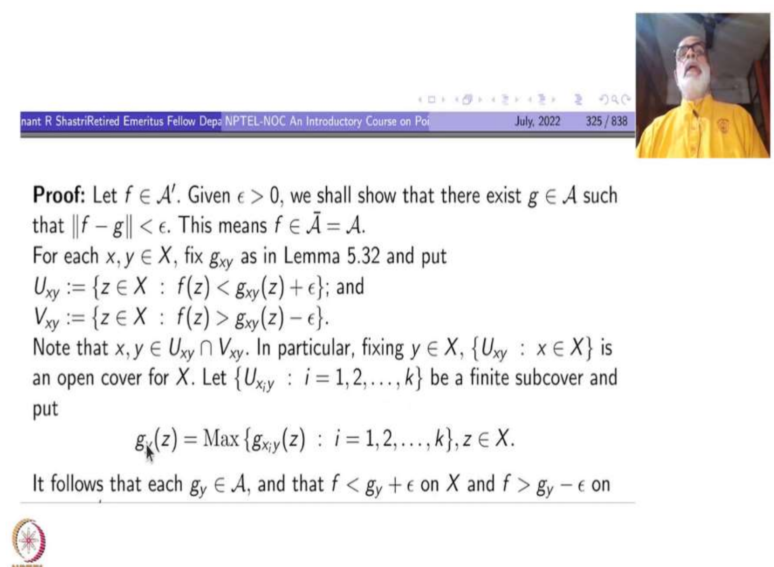

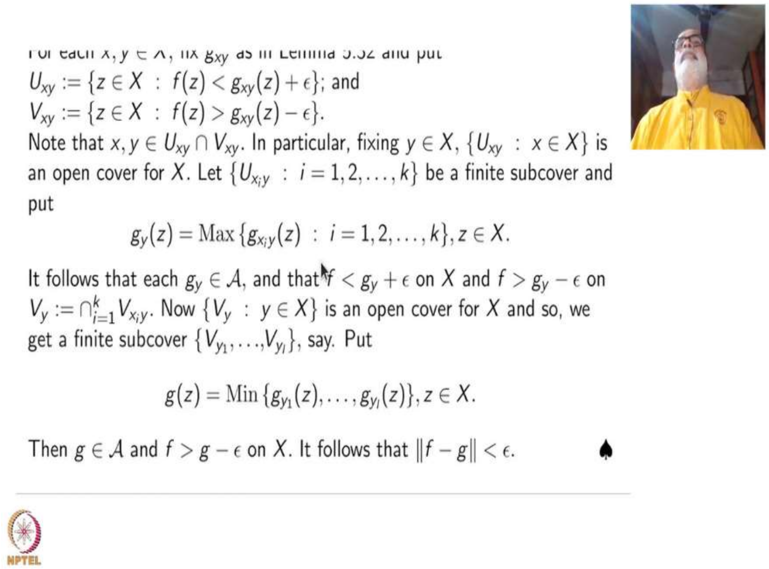

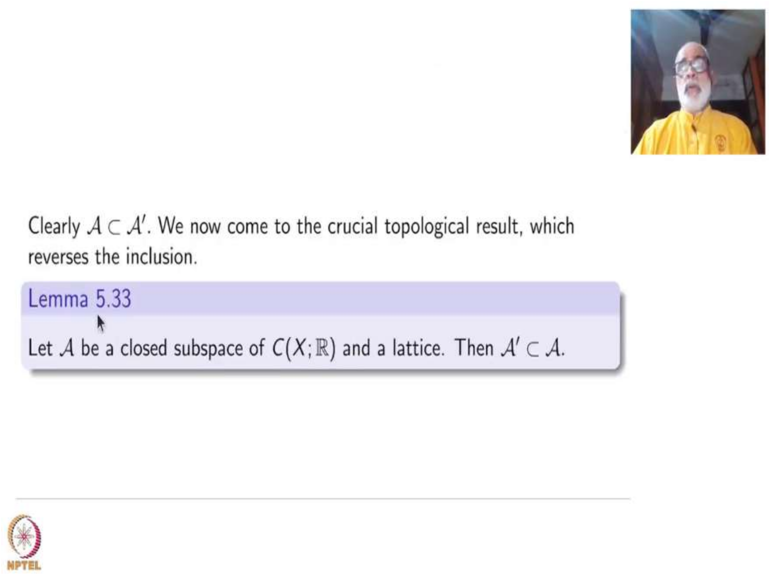

Let f be inside A'. Given $\epsilon > 0$, we shall show that there exists a $g \in A$ such that $\|f - g\| < \epsilon$. What does that mean? This means that f is approximated by functions from A that means f is inside $\overline{A}$, but $\overline{A}$ is A because we started with a closed subspace. So, this is our idea. How to construct the g such that g is inside A? So, for each $(x, y) \in X \times X$, fix g_{xy} as in Lemma 5.32.

This is what we have seen. Now, put U_{xy} equal to the set of all $z \in X$ such that $f(z)$ is less than $g_{xy}(z) + \epsilon$. So, this is the left ray. Similarly, let V_{xy} be the right ray, the set of all $z \in X$ such that $f(z) > g_{xy} - epsilon$. These two rays are around the point $g_{xy}(z)$, one is from $-\infty$ to $g_{xy}(z) + \epsilon$, and this one is from $g_{xy}(z) - \epsilon$ to ∞. So, these two rays I have taken. Of course, I have taken the inverse image all $z \in X$ such that $f(z)$ is inside this ray, which means it is the inverse image of the ray under f. So, this is U_{xy} and V_{xy}, these are open subsets of X obviously. Note that both x, y are inside U_{xy} as well as V_{xy}. Why? Because $f(x) = g_{xy}(x)$ and so $f(x) < g_{xy}(x) + \epsilon$, $f(y) = g_{xy}(y)$ and is bigger that $g_{xy}(y) - \epsilon$.

In particular, fixing y, if you vary x, then U_{xy}'s will be a cover for X. Similarly, I can fix x and vary y also. So, I will get another cover, but X is what? A compact Hausdorff space. Compactness allows you a finite subcover.

So, you get $U_{x_1 y}, U_{x_2 y}, \ldots, U_{x_k y}$ is an open cover for X. I put here. Correspondingly, you have finitely many $g_{x_i y} \in A$, you can take the maximum of $g_{x_i y}$, as i range from 1 up to k. That is my g_y, so that is inside A, because A is a lattice.

Now what happens to g_y? It follows the g_y inside A and $f < g_y + \epsilon$ on the whole of X, because f is less than each $g_{x_i y} + \epsilon$, and this is g_y is the maximum. So, f is less than $g_y + \epsilon$ on the whole of X.

On the other end, if you go through the other inequality, f is bigger than $g_y - \epsilon$ only on the intersection of $V_{x_i y}$'s. If it is bigger than all of them, then only $f(z)$ will be bigger than the maximum. Therefore, it should be true for all the i equal to $1, 2, 3$ up to k which means that I have to take the intersection. If I take intersection of V_y's, namely the intersection of the corresponding $V_{x_i y}$'s and call it V_y, on this V_y, f is bigger than $g_y - \epsilon$. Is that clear?

What are these V_y's. For each y, V_y is a neighborhood of y. So, they cover the whole of X, as y ranges over X. Therefore, I can get a finite sub cover of this one now say $V_{y_1}, V_{y_2}, \ldots, V_{y_\ell}$, say. Correspondingly, I get $g_{y_i}] \in A$, So, I take the minimum of g_{y_i}'s and call it g. So, first maximum and now minimum. So, this minimum is again an element of A because $g_{y_1],g_{y_2},\ldots}$ are inside A and A is a lattice.

Now, what happens? f is bigger than $g_y - \epsilon$ on whole of X now, because f is bigger than each of them it should be bigger than minimum of course, so, it will be valid for the entire of X. On the other hand, we also have f less than $g_y + \epsilon$ for all y and hence it is less than the minimum of these, namely f is less than $g + \epsilon$. So, it follows that combine these two inequalities, $\|f - g\| < \epsilon$

now, on the whole of X. So, the proof is over. So, it is like a min-max principle here, compactness is heavily used, compactness of the domain space.

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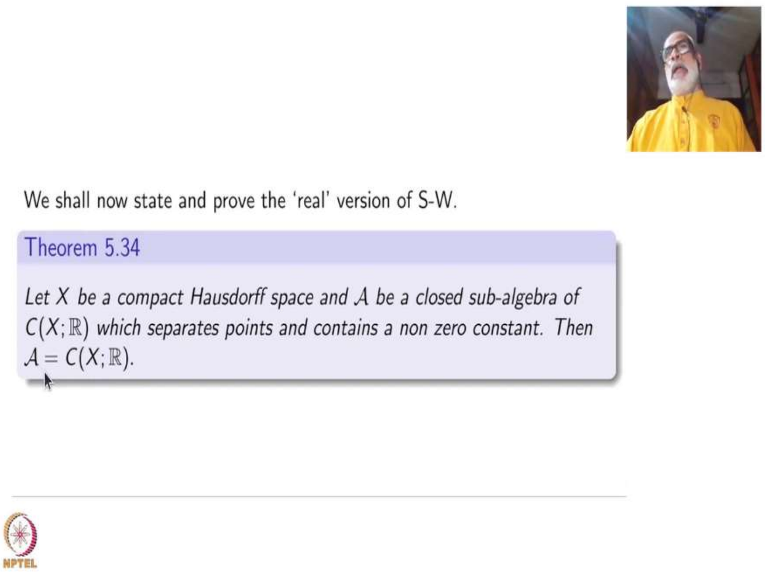

So, now, let us complete the proof of the real version of Stone-Weierstrass. Let X be a compact Hausdorff space, A be a closed subalgebra of $C(X;\mathbb{R})$ which separates points and contains a non-zero constant. Then A is the entire $C(X;\mathbb{R})$.

`Contains a non-zero constant' is the same thing as saying contains the subalgebra $\mathbb{R}$ itself. The real numbers considered as constant functions, they form a subalgebra of $C(X;\mathbb{R})$. So, that is the assumption here that this subalgebra is contained in A.

Note that given an arbitrary subalgebra, it may not contain any non zero constant. However, even if one non-zero constant is there, then entire $\mathbb{R}$ will be there.

So, this is the standard version of Stone-Weierstrass theorem. What we are going to prove is a slight modification of this one, a slightly stronger version.

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We shall actually prove a slightly more elaborate version of this:

Theorem 5.35

(P. Gaddy 2016) *Let X be a compact Hausdorff space and $\mathcal{A}$ be a closed sub-algebra of $C(X;\mathbb{R})$ which separates points. Then $\mathcal{A}$ is*
(i) $C(X;\mathbb{R})$ OR is equal to
(ii) the maximal ideal $M_{x_0} := \{f \in C(X;\mathbb{R}) \ : \ f(x_0) = 0\}$ for a unique point $x_0 \in X$.

We shall now state and prove the 'real' version of S-W.

Theorem 5.34

Let X be a compact Hausdorff space and $\mathcal{A}$ be a closed sub-algebra of $C(X;\mathbb{R})$ which separates points and contains a non zero constant. Then $\mathcal{A} = C(X;\mathbb{R})$.

And this is due to Gaddy, it is not very old; 2016. Without any extra effort we are going to prove this one. So, I will help with this one. So, let X be a compact Hausdorff space, A be a closed subalgebra of $C(X;\mathbb{R})$ which separates points. (So, I am not assuming that it contains non-zero constant.) So, I have two different conclusions here.

Then A is $C(X;\mathbb{R})$ or, or what, or equal to the maximal ideal M_{x_0}, set of all f belonging to $C(X;\mathbb{R})$ which vanish at x_0 for a unique point $x_0 \in X$. So, indeed, this part is much more revealing--what is happening in the subalgebra, than just studying them under the assumption that there is non-zero constant.

Let us take this one for granted. Then the standard version follows automatically, because as soon as there are non-zero constants it cannot be this M_{x_0} for any x_0 that is all. There is a non-zero constant that will not vanish at any point. So, M_{x_0} is much smaller after all, all those which vanish at a point.

So, this part will not occur. So it will to be always be the whole space $C(X;\mathbb{R})$ under the this additional assumption.

So, what should we prove? If we assume that there are no non-zero constants in A, then we must be able to conclude that this A is nothing but M_{x_0} namely, we must find out such a point x_0, a unique point in X. It sounds like a contraction happening, there is only one point such that all elements of A vanish at that point. And A will be precisely equal to M_{x_0}. So, A will be an ideal. this is an ideal in $C(X;\mathbb{R})$. Indeed, it is a maximal ideal because the moment you take anything which is not 0 at x_0 you can produce a non-zero constant. Once you do that, the constant 1 will be there in A. So, A will be the whole of the $C(X;\mathbb{R})$. So, let us go towards proving this.

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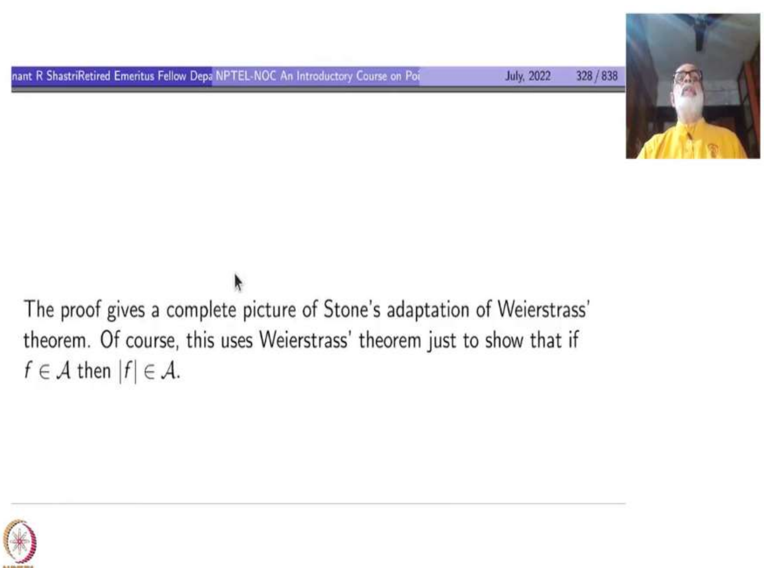

The proof given here gives a complete picture of Stone's adaption of Weierstrass theorem. Of course, this uses Weierstrass theorem just to show that $|f|$ is there as soon as f is there in A.

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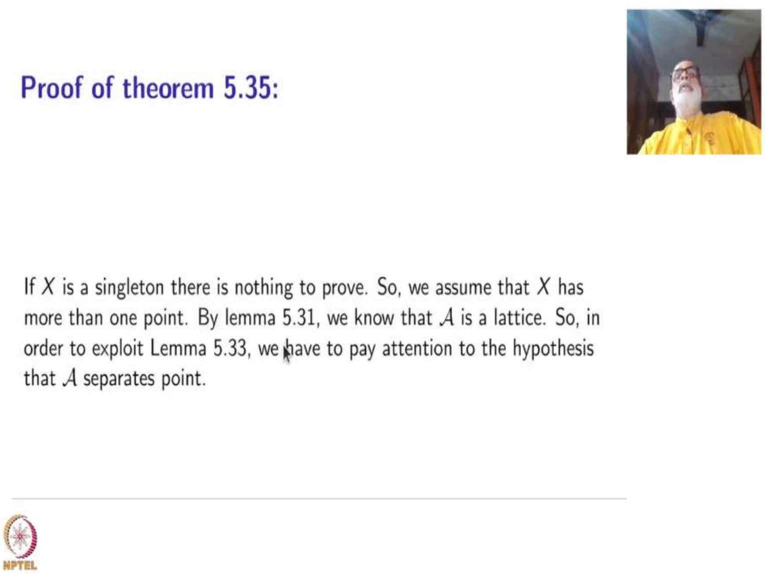

If X is a singleton there is nothing to prove. What is $C(X;\mathbb{R})$? It is just $\mathbb{R}$. So, any subalgebra of $\mathbb{R}$ is either 0 or whole of $\mathbb{R}$. So, there is nothing to prove. So, there is no problem. So, we assume that X has more than one point. By Lemma 5.31, we know that is A is a lattice. So, in order to exploit this Lemma 5.33, namely the classification of subalgebra of $\mathbb{R}\times\mathbb{R}$ into 5 classes we have to pay attention to the hypothesis that A separate points. Namely, we have come down to this A_{xy} as I explained, that is the only thing that is left out. We have done all other preparations. So, let us do that.

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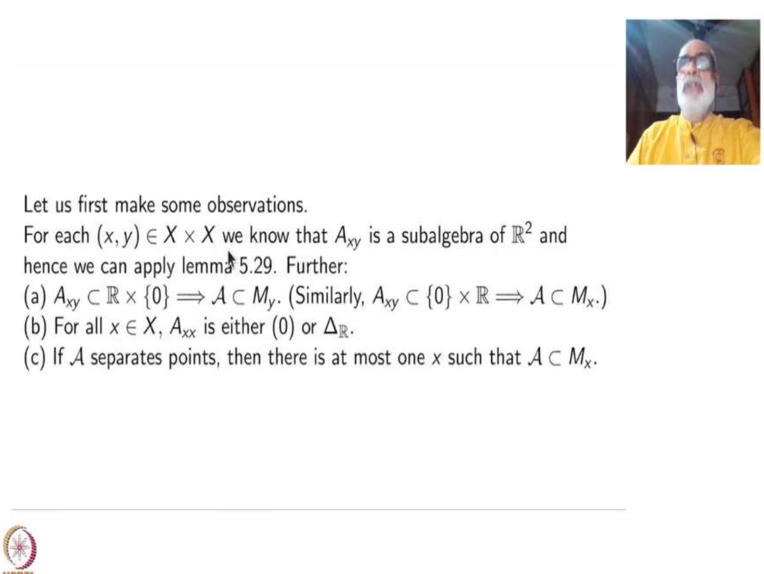

We shall actually prove a slightly more elaborate version of this:

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(i) $C(X;\mathbb{R})$ OR is equal to
(ii) the maximal ideal $M_{x_0} := \{f \in C(X;\mathbb{R}) \ : \ f(x_0) = 0\}$ for a unique point $x_0 \in X$.

Proof of theorem 5.35:

If X is a singleton there is nothing to prove. So, we assume that X has more than one point. By lemma 5.31, we know that $\mathcal{A}$ is a lattice. So, in order to exploit Lemma 5.33, we have to pay attention to the hypothesis that $\mathcal{A}$ separates point.

Let us first make some observations. All these, inside $\mathbb{R}^2$ now. Given (x, y) belongs to $X \times X$, we know that A_{xy} is subalgebra of $\mathbb{R}^2$. And hence, we can apply this lemma. And we know what A_{xy} is either 0 or $\mathbb{R} \times 0$ or $0 \times \mathbb{R}$ or the diagonal or the whole of X.

(a) Now, suppose $A_{xy} = \mathbb{R} \times 0$. What does it mean? Remember, A_{xy} consists of pairs $(f(x), f(y))$ etc., therefore, this just means that $f(y) = 0$ for all $f \in A$. That just means that A is contained n M_y. Similarly, $A_{xy} = 0 \times \mathbb{R}$ means that A is contained in M_x.

(b) Now, the second observation is: for all $x \in X$, A_{xx} is already the diagonal. So, it is the entire diagonal or 0. So, all these are elementary observations. But finally, you put all of them together you have a wonderful result.

(c) If finally, A separate points then there is at most one x such that A is contained inside M_x. See, A separates points. Suppose there are two points x, y such that A is contain inside M_x and M_y. What does it mean? All the elements of A vanish at both x and y. So, x and y cannot be separated, that is all. So, the uniqueness part comes only by the assumption that A separates points.

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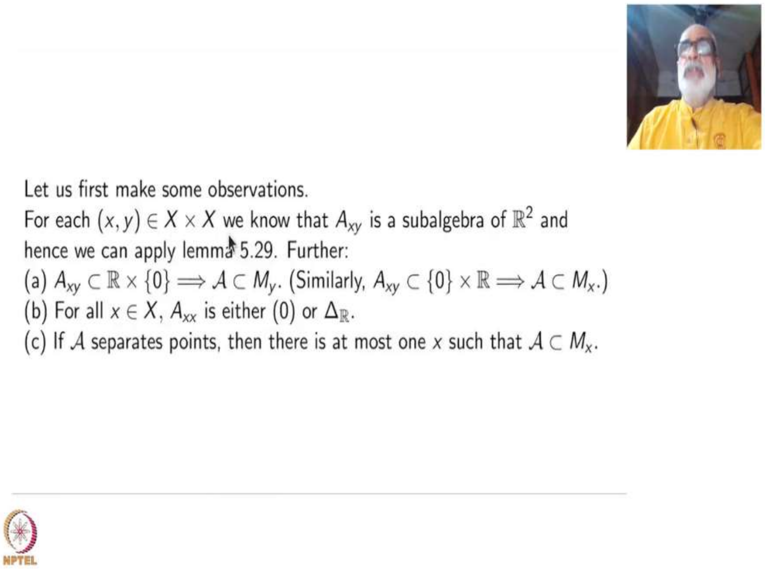
First consider the case when $A_{x_0x_0} = (0)$ for some $x_0 \in X$. In this case we claim that $M_{x_0} \subset \mathcal{A}$. This is achieved by showing that $M_{x_0} \subset \mathcal{A}'$.
From (a) above, we get $\mathcal{A} \subset M_{x_0}$. Since $\mathcal{A}$ separates points, from (c), we get $A_{xx} = \Delta_{\mathbb{R}}$ for all $x \neq x_0$. Also from (b) we get $A_{xx_0} = \mathbb{R} \times \{0\}$ and $A_{x_0x} = \{0\} \times \mathbb{R}$ for all $x \neq x_0$. Finally from (c), we also see that $A_{xy} = \mathbb{R}^2$ in all other cases, i.e.,when $x \neq x_0, y \neq x_0$.



So, now, consider the case when $A_{x_0x_0} = 0$. As we have observed A_{xx} is always 0 or Δ. Suppose for some x_0, I do not know whether it is there or not, suppose that this occurs, $A_{x_0x_0} = 0$. Automatically it implies that $f(x_0) = 0$ for all $f \in A$. In this case, M_{x_0} is contained inside A is the claim. A is contained in M_{x_0} is obvious.

If we show the M_{x_0} is contained inside A that would mean they are equal. And we are since the uniqueness part we have already shown here. So, we would get the conclusion in the part 2 of the theorem.

So, let us prove that in this case, M_{x_0} is inside A. So, this is achieved by showing that M_{x_0} is inside A'. Since A separate points, from (c) above, we get $A_{xx} = \Delta$ for all $x \neq x_0$. Also, from (b), we get A_{xx_0} is $\mathbb{R} \times 0$. Because if this were 0, that would that means A_{xx} is also 0. Similarly, A_{x_0x} is $0 \times \mathbb{R}$.

Finally, from (c) again we also see that $A_{xy} = \mathbb{R}^2$, whenever $x \neq x_0$ and $y \neq y_0$, and $x \neq y$. Because these distinct points, so there will be some $f \in A$, for which $f(x)$ and $f(y)$ will be different. Since the two coordinates of all points in $\Delta_{\mathbb{R}}$ are equal, this implies $A_{x,y}$ is not equal to Δ. So it must be for whole of $\mathbb{R}^2$.

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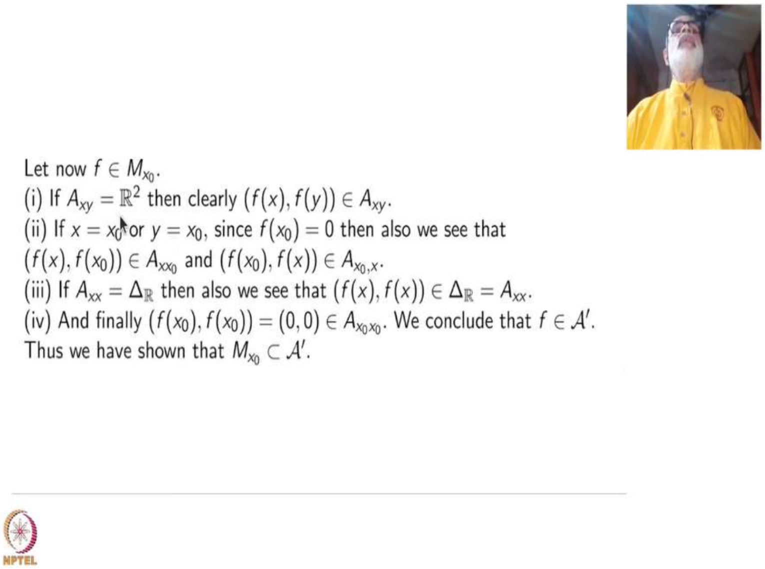

Now, let us look at any $f \in M_{x_0}$. I want to show that it is inside A. If A_{xy} is $\mathbb{R}^2$ then clearly $(f(x), f(y))$ will be inside A_{xy} because everything is in $\mathbb{R}^2$.. So, it is the full thing. If $x = x_0$, or $y = x_0$, since $f(x_0)$ is 0 then also we see that $(f(x), f(y)) = A_{x_0x}$ or a_{xx_0}, the other way around.

If A_{xx} is $\Delta_{\mathbb{R}}$, the diagonal, then also we see that $(f(x), f(x))$ is inside $\Delta_{\mathbb{R}}$, so it is in A_{xx}. And finally, the point $(f(x_0, f(x_0))$ is $(0,0)$ that is also $A_{x_0x_0}$. So, what we can conclude is that f is inside A', which was the criteria: namely, for each x, y, $(f(x), f(y))$ must be in A_{xy}, that is the description of A'. Thus, we have shown that M_{x_0} is contained in A'.

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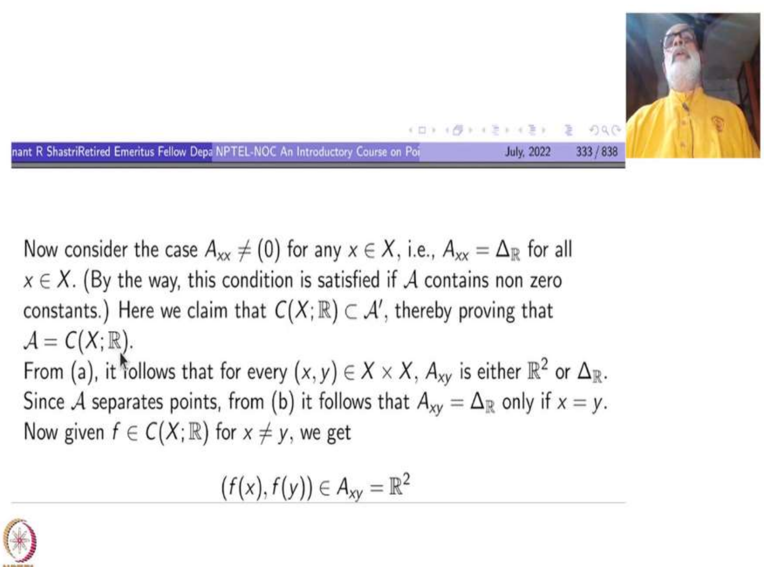

$x \in X$. (By the way, this condition is satisfied if $\mathcal{A}$ contains non zero constants.) Here we claim that $C(X;\mathbb{R}) \subset \mathcal{A}'$, thereby proving that $\mathcal{A} = C(X;\mathbb{R})$.
From (a), it follows that for every $(x,y) \in X \times X$, A_{xy} is either $\mathbb{R}^2$ or $\Delta_{\mathbb{R}}$.
Since $\mathcal{A}$ separates points, from (b) it follows that $A_{xy} = \Delta_{\mathbb{R}}$ only if $x = y$.
Now given $f \in C(X;\mathbb{R})$ for $x \neq y$, we get

$$(f(x), f(y)) \in A_{xy} = \mathbb{R}^2$$

and if $x = y$ then

$$(f(x), f(y)) = (f(x), f(x)) \in \Delta_{\mathbb{R}} = A_{xx}.$$

This verifies that $f \in \mathcal{A}'$. ♠

Let us first make some observations.
For each $(x,y) \in X \times X$ we know that A_{xy} is a subalgebra of $\mathbb{R}^2$ and hence we can apply lemma 5.29. Further:
(a) $A_{xy} \subset \mathbb{R} \times \{0\} \Longrightarrow \mathcal{A} \subset M_y$. (Similarly, $A_{xy} \subset \{0\} \times \mathbb{R} \Longrightarrow \mathcal{A} \subset M_x$.)
(b) For all $x \in X$, A_{xx} is either (0) or $\Delta_{\mathbb{R}}$.
(c) If $\mathcal{A}$ separates points, then there is at most one x such that $\mathcal{A} \subset M_x$.

Now, consider the case where A_{xx} is not equal to 0 for any x. We just finished this proof for the case when there is one x_0 for which $A_{x_0x_0}$ is 0. That is first assumption. Now, we are in the assumption that A_{xx} is never 0 for any x. Then what is the choice for A_{xx}? It must be $\Delta_{\mathbb{R}}$. (By the way, this condition is satisfied if A contains a non-zero constant. So, this is just in passing we are telling when this will satisfy. The earlier case does not occur if A contains a non-zero constant that is all.)

So, here we claim that this $C(X;\mathbb{R})$ itself is contained in A'. Combined with lemma 5.34 this will complete the proof that $A = C(X;\mathbb{R})$.

So, how do you show that $C(X;\mathbb{R})$ is contained in A'?

So, now, look at what we have in (a). Again I will just recall it. This says A_{xy} is $\mathbb{R}\times 0$ implies A contained M_y. Similarly, A_{xy} is in $0\times\mathbb{R}$ implies A is contained in M_x. That is all I am using here. So, from this it follows that for every $(x,y)\in X\times X, A_{xy}$ is either $\mathbb{R}^2$ or Δ. The possibilities $0\times\mathbb{R},\mathbb{R}\times 0$ or 0 do not occur. Since A separate points from (b) it follows that A_{xy} is $\Delta_{\mathbb{R}}$ if only if $x=y$. You see points (α,β) where $\alpha\neq\beta$, are in $\mathbb{R}^2$ but not in the diagonal Δ.

Now given f belonging to $C(X;\mathbb{R})$, for $x\neq y$ what we have, $(f(x),f(y))$ is in A_{xy}, which is the whole of $\mathbb{R}^2$. And if $x=y$, then $(f(x),f(y))=(f(x),f(x))$. So it is in Δ which is A_{xx}.

So f verifies the condition of the lemma and hence is in A'.

So, that completes the proof.

So, I will just recall the important steps here. So, where was the definition of A'?

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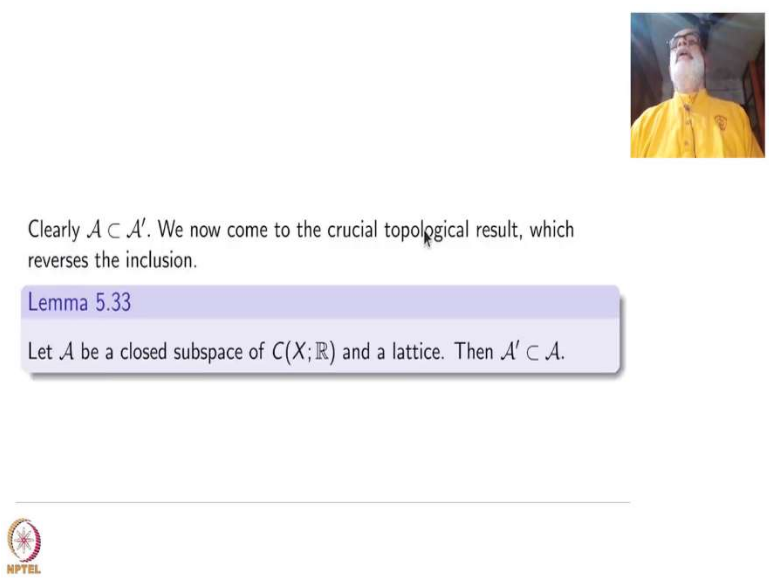

Put

$$\mathcal{A}' := \{f \in C(X;\mathbb{R}) \ : \ (f(x), f(y)) \in A_{xy}, \ \forall \ (x,y) \in X \times X\}.$$

The following lemma is just an alternative description of the set $\mathcal{A}'$:

Lemma 5.32

An element $f \in C(X;\mathbb{R})$ is in $\mathcal{A}'$ iff for every $(x,y) \in X \times X$, there exists $g_{xy} \in \mathcal{A}$ such that $(f(x), f(y)) = (g_{xy}(x), g_{xy}(y))$.

NPTEL

We shall now state and prove the 'real' version of S-W.

Theorem 5.34

Let X be a compact Hausdorff space and $\mathcal{A}$ be a closed sub-algebra of $C(X;\mathbb{R})$ which separates points and contains a non zero constant. Then $\mathcal{A} = C(X;\mathbb{R})$.

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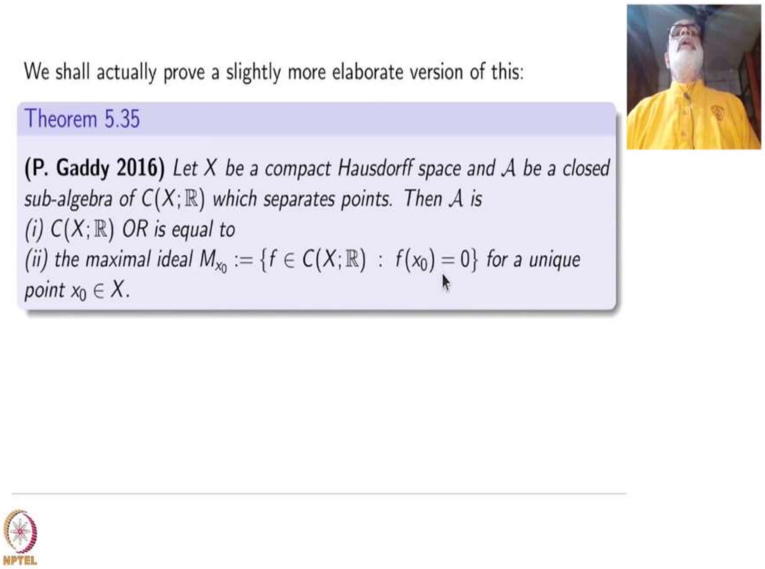

Because this A' which was obviously larger than A in general. So, we have proved that A' is equal to A. The description of A'. Given f is inside $C(X;\mathbb{R})$ if for all pairs (xy), $(f(x)f(y))$ is in A_{xy}, then f is inside A'. This is the simple description. So, this has been exploited here.

So, Stone-Weierstrass theorem for real valued continuous functions is proved now. A slightly better version of that one has been proved here. So, next time we shall do the complex case as well as some other extensions wherein we do not assume X is compact, but only locally compact. Thank you.