**An Introduction to Point-Set-Topology (Part II)**
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**Lecture 22**
**Stone-Weierstrass Theorems**

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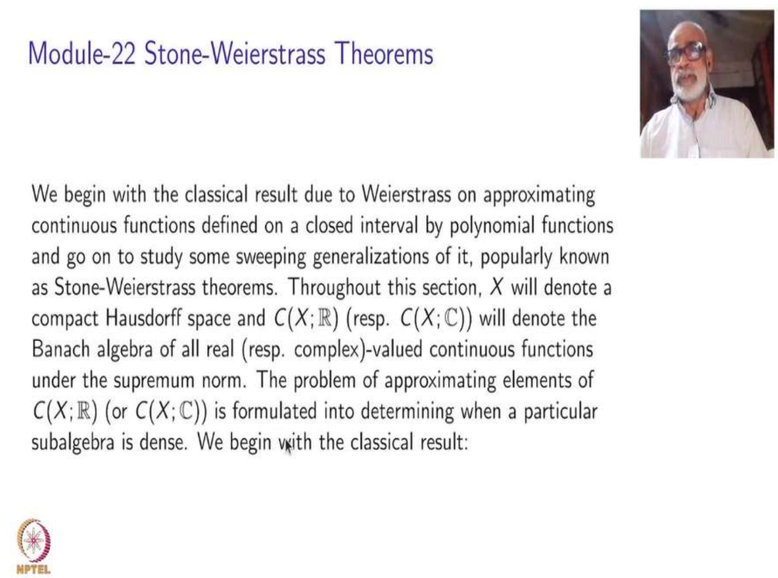


Welcome to NPTEL NOC on Point Set Topology Part II. So, today we will take Module 22 Stone-Weierstrass theorems. We begin with the classical result due to Weierstrass on approximating continuous functions defined on a closed interval by polynomial functions and then go on to study some sweeping generalization of it, popularly known as Stone-Weierstrass theorems.

Throughout this section, $X$ will denote a compact Hausdorff space in the beginning and later on, a locally compact Hausdorff space. That I will tell you again. $C(X;\mathbb{R})$ and $C(X;\mathbb{C})$ will denote respectively the Banach algebra of all real (or respectively complex-valued continuous functions on $X$. And we take the supremum norm which makes sense because $X$ is compact. The problem of approximating elements of $C(X;\mathbb{R})$ or $C(X;\mathbb{C})$ is formulated into determining when a particular subalgebra is dense.

So, density of certain thing just implies that when you take points in the closure, they are the approximated functions from the set which is dense. So, that is the whole terminology.

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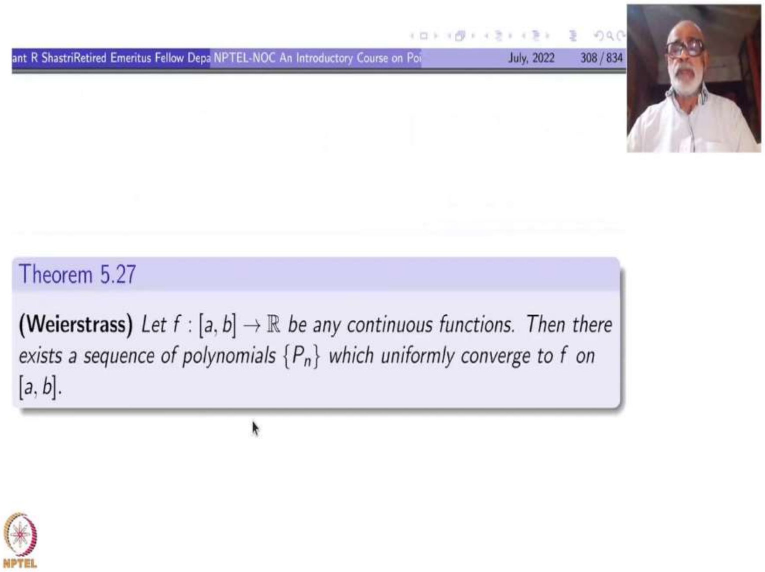


We begin with the classical result due to Weierstrass. Let $f$ from $[a, b]$ to $\mathbb{R}$ be any continuous function. Then there exists a sequence of polynomials which uniformly converge to $f$ on the interval $[a, b]$. See each term here is very important. $[a, b]$ must be a closed interval. And you start with a continuous function $f$. You can approximate it by polynomial functions. That is the way to remember it. But what is exactly the meaning of approximating here? That a sequence of polynomials uniformly converges to the function $f$ on the closed interval $[a, b]$.

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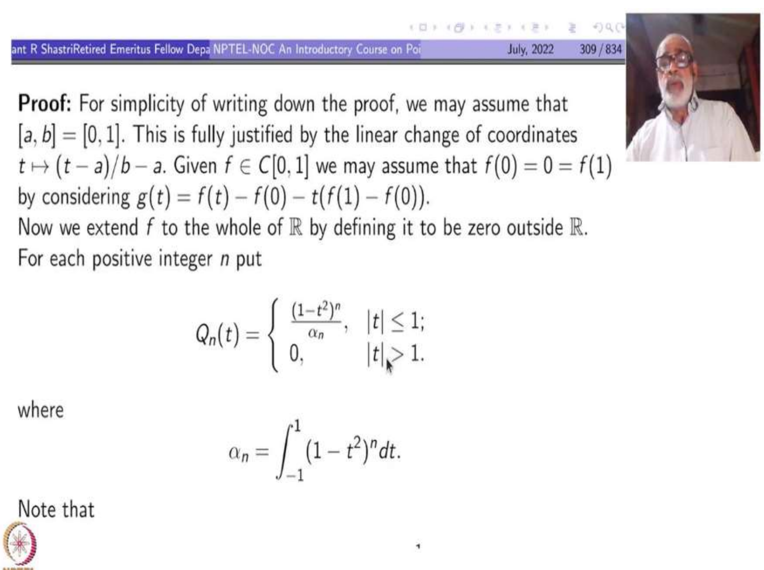

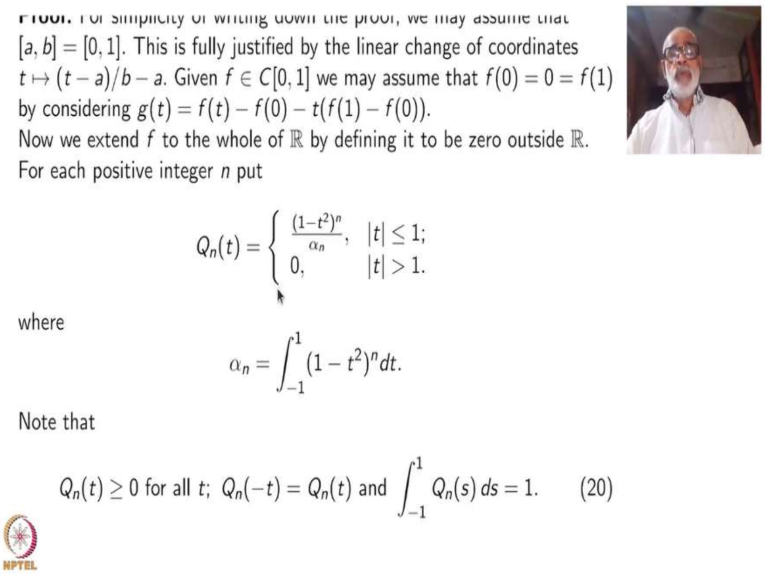


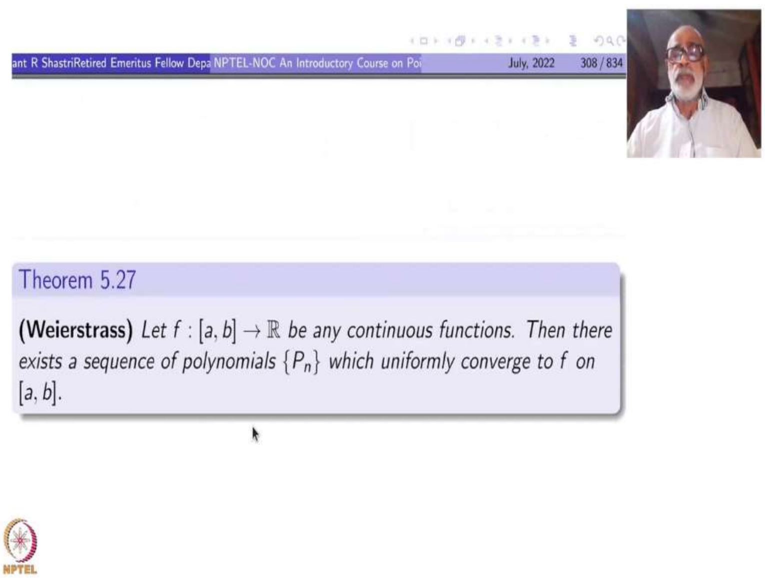


For simplicity of writing down the proof, I will assume that the interval is $[0, 1]$. But, there is no loss of generality because what you can do is, you can make a linear change of variable in the domain itself, namely, by taking $t$ going to $(t-a)/(b-a)$. When $t = a$, this will be $0$. When $t = b$, this is $(b-a)/(b-a) = 1$. So, interval $[a, b]$ will go to $[0, 1]$. So, this way you can change the coordinates in the domain.

If you have a polynomial in $t$ and if you substitute $(t-a)/(b-a)$ instead of $t$, that will be again a polynomial in $t$. So, there is no loss of generality. So, from now onwards, we will look at the closed interval $[0, 1]$. So, start with a continuous function defined on $[0, 1]$. So, I am assuming that this is a real valued function here.

So, we may make a second assumption, viz., $f(0) = 0$ and $f(1)$ is also $0$. So, how do we do justify that? By a linear change of variable in the codomain. By taking $g(t) = f(t) - f(0) - t(f(1) - f(0))$. Look at this function $g$. At $0$, it is $f(0) - f(0) = 0$. And $g(1)$ is $f(1) - f(0)$ minus the same thing and so that is again $0$.

So, this is again a linear change of coordinates. Your given polynomial is being added to a fixed polynomial of degree one. So, right in the beginning we are making two such assumptions with out any harm. The advantage of these assumption is that immediately we can extend $f$ to the whole of $\mathbb{R}$ continuously by defining it to be $0$ outside of the interval $[0, 1]$.

Next, for each positive integer $n$, put $Q_n(t) = (1 - t^2)^n/\alpha_n$ for all $t$ such that $|t| \leq 1$. So, $t$ ranges from $-1$ to $1$. I am taking this function and then I am putting $0$ outside this $[-1, 1]$. So, look at this one. When $t$ equal to $1$ or $-1$, this term is $0$, therefore, $Q_n$ is a continuous function on the whole of $\mathbb{R}$. This $\alpha_n$ is some positive constant here, what is that constant? It is just the integral of the numerator, $(1 - t^2)^n$ from $-1$ to $1$.

So, this dividing factor is some kind of normalizing factor here. So, it is reflected in this property (20), namely.: $Q_n(t)$ is, first of all non-negative, $Q_n(-t)$ is $Q_n(t)$, because it is a polynomial function in $t^2$. Finally, if you integrate this $Q_n$ from $-1$ to $1$, the constant $\alpha_n$ in the denominator, will come out, and cancel out with the integral of the numerator and the result is equal to $1$. That is why $\alpha_n$ is chosen in that way. These $Q_n(t)$ are auxiliary functions that are going to help us in the approximation.

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Put

$$P_n(t) = \int_0^1 f(s) Q_n(t-s)\, ds \tag{21}$$

Note that $P_n$ is a polynomial function. We claim that this sequence converges uniformly to $f$ on $[0,1]$. Since for all $t \in [0,1]$, we have $[0,1] \subset [t-1, t+1]$, and since $f(s) = 0$ outside $[0,1]$,

$$P_n(t) = \int_{t-1}^{t+1} f(s) Q_n(t-s)\, ds.$$

Now, by substituting $t - s = u$, we get

$$P_n(t) = \int_{-1}^{1} f(t-u) Q_n(u)\, du. \tag{22}$$

**Proof.** For simplicity of writing down the proof, we may assume that $[a,b] = [0,1]$. This is fully justified by the linear change of coordinates $t \mapsto (t-a)/b-a$. Given $f \in C[0,1]$ we may assume that $f(0) = 0 = f(1)$ by considering $g(t) = f(t) - f(0) - t(f(1) - f(0))$.

Now we extend $f$ to the whole of $\mathbb{R}$ by defining it to be zero outside $\mathbb{R}$.
For each positive integer $n$ put

$$Q_n(t) = \begin{cases} \frac{(1-t^2)^n}{\alpha_n}, & |t| \le 1; \\ 0, & |t| > 1. \end{cases}$$

where

$$\alpha_n = \int_{-1}^{1} (1-t^2)^n dt.$$

Note that

$$Q_n(t) \ge 0 \text{ for all } t;\ Q_n(-t) = Q_n(t) \text{ and } \int_{-1}^{1} Q_n(s)\, ds = 1. \tag{20}$$

Now, immediately, I am going to define the sequence that we are interested in. Namely, $P_n(t)$ is defined to be the intergal from $0$ to $1$ of $f(s)Q_n(t-s)ds$. So, what I have done is: $f$ is the given continuous function, these $Q_n$ are auxiliary functions which I have defined here, and I am convoluting $f$ with these $Q_n$'s. This idea, all the way goes back to Euler. So, Weierstrass also has used it.

Now, what I have to observe is that this formula will tell you that $P_n(t)$ is actually a function of $t$ first of all, because the variable $s$ is getting integrated here. It is actually a polynomial function, why? Because the integrand is a polynomial function of $t$ and taking integral with respect to $s$ is a linear operator. Suppose this polynomial were a constant, after the integration, this will still be a constant. If there is a $t$-term here that $t$ would have come out and get multiplied by the integral $0$ to $1$ of $f(s)ds$. If there is a $t^n$, the $t^n$ will come out and then there is a function of $s$ left out here.

So, when you expand $Q_n(t-s)$ in terms of $t$ and $s$, it will be a polynomial in the two variables $t$ and $s$, but under the integration with respect to $s$, powers of $t$ would come what is left out is some other linear combination of the same powers of $t$. So, $P_n(t)$ is actually a polynomial function in $t$ for each $n$.

Now we claim that this sequence $P_n(t)$ converges uniformly to $f$ on $[0,1]$. So, statement is very easy and clear. So, only thing is that you have to bring these $Q_n(t)$, which you may not remember. So, you may have to remember this one. In fact, there are many other auxiliary functions also which will do this job, giving different sequences. There is no uniqueness here. So, this is my personal choice you may say, but not exactly. I mean, many other people also used it and at this time I will tell you that there are many proofs of the Weierstrass' theorem, none of them go beyond the ideas of Weierstrass, in computational simplicity. Something else, some other things, these alternative proofs may achieve, some different things. For example, I also like the proof which is there in Rudin's book on Principles of Mathematical Analysis. So, you can have a look at that also.

So, now, we have to prove that $P_n$ converges to $f$ uniformly. So, first observation is:

For all $t \in [0,1]$, the interval $[0,1]$ is contained inside the interval $[t-1, t+1]$. If $t$ is between $0$ and $1$, for example, suppose $t=0$, then this is $[-1,1]$, so, it contains $[0,1]$. And if $t=1$, it will be $[0,2]$, so that also contains $[0,1]$. That is because the interval $[-1,1]$ is of length $2$ and you are shifting it a little bit.

Next observe that $f(s)$ is $0$ outside $[0,1]$. Therefore, when you take the integral from $t=-1$ to $t=1$ of $f(s)$ times something, since $f(s)$ is $0$ outside this interval $[0,1]$, it is as if we are taking the integration from $0$ to $1$. So, that is $P_n(t)$. So I can rewrite $P_n(t)$ as an integral from $t=-1$ to $t=1$ of the same function.

Now, you will see the advantage of writing this integral like this, namely, substitute $t-s=u$, typical thing to do when you are doing convolution, interchanging the variables here. So, substitute $t-s=u$, what happens to $s$? $s$ becomes $t-u$. And $t-s$ becomes $u$. So, what happens here is that $P_n(t)$ gets a new form, namely it will be integral $-1$ to $1$ of $f(t-u)Q_n(u)\,du$. $Q_n(t-s)$ is this, of course, and $ds = -du$. So, you have to interchange the lower and upper limits of the integral. So, this $t=1$ becomes $1$ and $t=1$ becomes $-1$. So, that is why you get $-1$ and $1$, along with the sign here $ds$ is $-du$.

Now, why I am doing all this? The point is now, there is symmetry, the $Q_n$ remember was a symmetric function. So, this property of $Q_n$, $Q_n$ is positive, and $Q_n(-u) = Q_n(u)$ for all $u$ in the interval. So, also the integral of $-1$ to $1$ of $Q_n$ is $1$. So, these properties can be useful now in this form. So, you will see, each of these statements will be useful now. So, that is all computational now, but it is interesting and quite entertaining.

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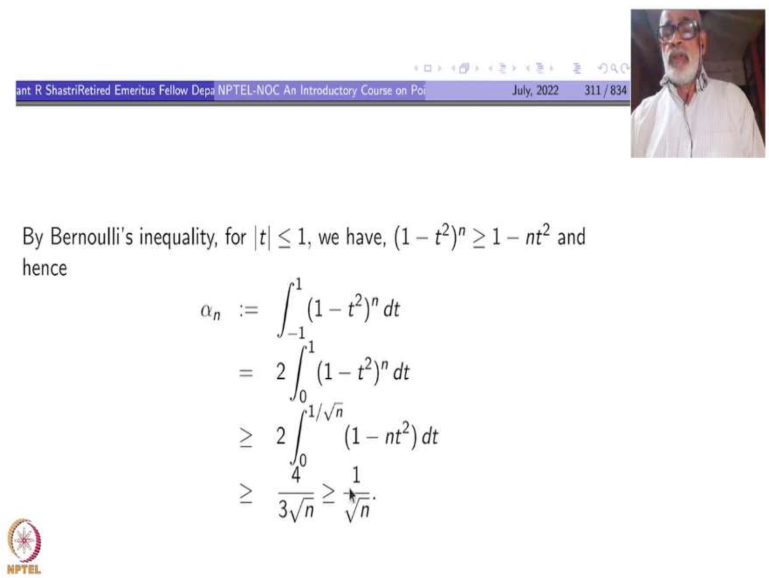


The simplest thing is the Bernoulli inequality: for $|t| \le 1$, we always have $(1-t^2)^n \ge (1-nt^2)$. When you take the binomial expansion, the first two terms are $1$ and $-nt^2$. You can ignore the rest of the terms, if you put inequality like this. This is easy to prove, there are several ways of proving this, this just an elementary calculus.

And hence, $\alpha_n$ which is integral $-1$ to $1$ of $(1-t^2)^n\,dt$ is nothing but, see by symmetry, this integral is equal to twice the integral from $0$ to $1$ of $(1-t^2)^n\,dt$, the same function, but that is now greater than or equal to twice integral of the same thing from $0$ only upto $1/\sqrt{n}$.

This function is smaller than this function, they are all both of them positive in this interval provided you take only up till here, there are other terms which you can ignore because you are taking only the inequality, bigger than equal to this one. But now, you integrate, what you get is this is bigger than or equal to $4/3\sqrt{n}$ and that itself is bigger than $1/\sqrt{n}$.

There are heavy, liberal, inequalities here, no economy is employed. You may be able to prove that $\alpha_n \geq 1/\sqrt{n}$ in many other ways, I do not care, I want one proof. So, $\alpha_n \geq 1/\sqrt{n}$, this all I wanted.

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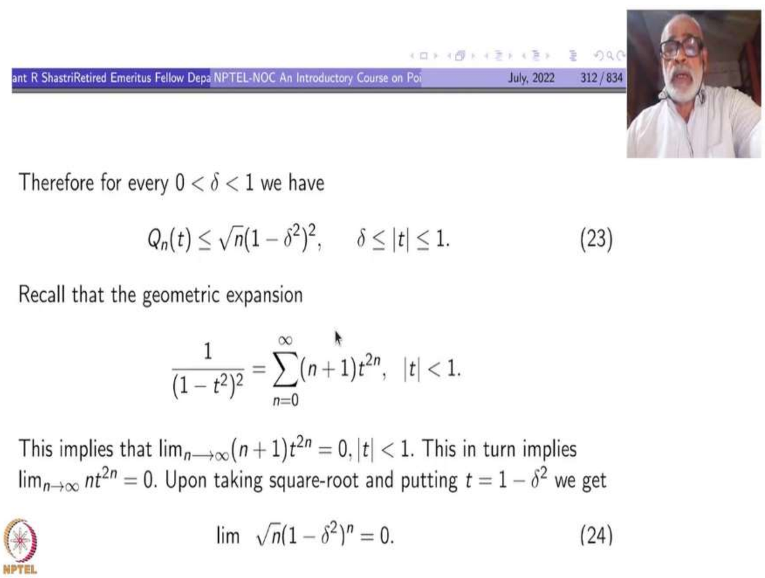


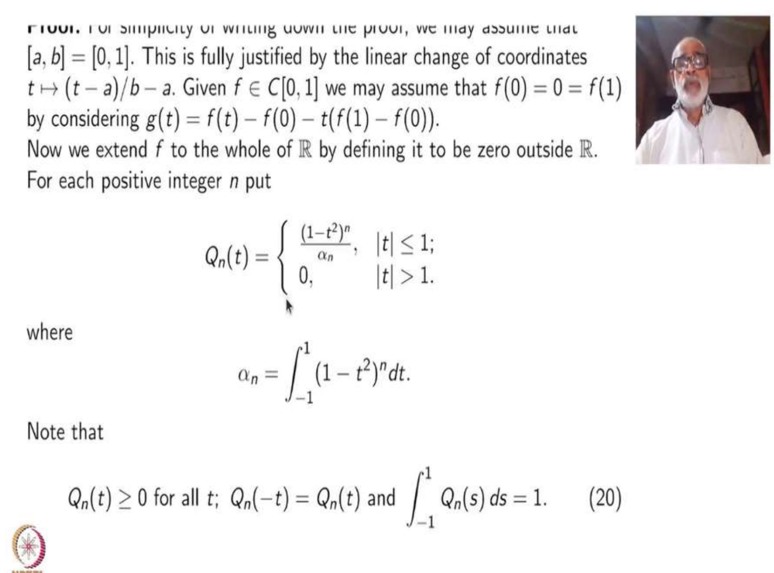

$^{\alpha}$Next thing is: for every $\delta$ between $0$ and $1$, we have $Q_n(t) \leq \sqrt{n}(1-\delta^2)^2$. Remember what was $Q_n(t)$. $Q_n(t)$ was $(1-t^2)^n/\alpha_n$, this $\alpha_n$ we have estimated is bigger than $1/\sqrt{n}$. Therefore, I can ignore this now. I can simplify, it becomes it comes in the numerator here $Q_n(t) \leq \sqrt{n}(1-\delta^2)^2$ provided $|t| \geq \delta$.

So, in this interval $(\delta, 1)$ this inequality holds. Now, recall the geometric expansion $1/(1-t^2)$, the whole square is nothing but $0$ to $\infty$ of $(n+1)t^{2n}$, (there is a square term here that is why, $(n+1)t^{2n}$). It is a power series in $t^2$. This is valid for $|t| \leq 1$, this is the geometric series. This implies that if you take $(n+1)^{th}$ term here, whatever it is, it must tend to $0$ as $n$ tends to $\infty$.

Limit as $n$ tends to infinity of $(n+1)t^{2n}$ is $0$, whenever $|t| \leq 1$. This in turn implies, of course, you can change $n+1$ to $n$, by Sandwich theorem that the limit of $nt^{2n}$ is $0$. Upon taking square root and putting $t = 1-\delta^2$, we get that limit of $\sqrt{n}(1-\delta^2)^n$ is $0$. The conclusion is that $Q_n(t)$ converges uniformly to $0$ in any interval $[\delta, 1)$ for all $\delta > 0$. This is one of the purpose for choosing these $Q_n(t)$'s.

If you can choose something simpler you will get a simpler profile no problem. So, what I have what is that you see this right-hand side here tends to $0$ as $n$ tends to infinity. Provided this is always true, now, if you use $\delta \leq t \leq 1$ then we can pass on to $Q_n$ here. So, let us see what happens.

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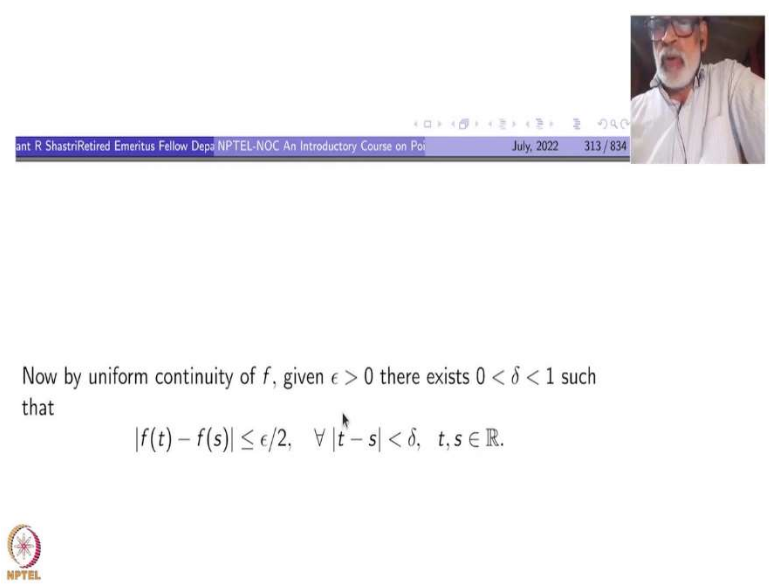

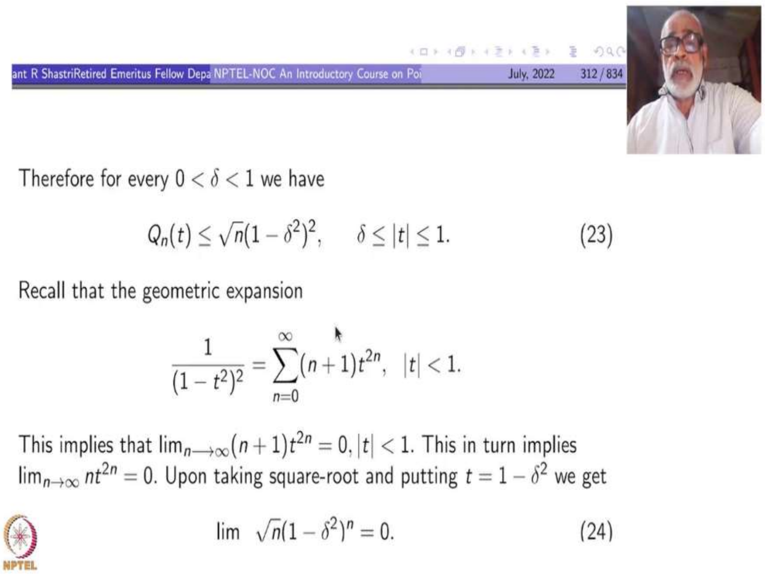


Now, by uniform continuity of $f$, $f$ is continuous and we are interested only in the closed interval $[0,1]$. So, therefore, it is uniformly continuous. Given $\epsilon > 0$ there will be a $\delta \in (0,1)$, I can always just choose this $\delta$ to be less than 1 such that $|f(t) - f(s)| < \epsilon/2$, whenever $|t-s| < \delta$, both $t$ and $s$ are inside $\mathbb{R}$. I have written inside $\mathbb{R}$. Clearly, we have this inside $[0,1]$ of course. but then because outside $[0,1]$ we have extended $f$ to be 0 remember that. Uniform continuity holds over the whole of $\mathbb{R}$. First we get it for the closed interval $[0,1]$ but on the rest of the space the function is 0.

Now, $\delta$ has been chosen and this statement was true for any $\delta$ between 0 and 1.

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Let $|f(t)| \leq M$ for some $M$. Combining (20), (21),(22), (23) and (24), we get

$$|P_n(t) - f(t)| = \left| \int_{-1}^{1} [f(t-s) - f(t)] Q_n(s)\, ds \right|$$

$$\leq \int_{-1}^{1} |f(t-s) - f(s)| Q_n(s)\, ds$$

$$\leq 2M \int_{-1}^{-\delta} Q_n(s)\, ds + \frac{\epsilon}{2} \int_{-\delta}^{\delta} Q_n(s)\, ds + 2M \int_{\delta}^{1} Q_n(s)\, ds$$

$$\leq 4M\sqrt{n}(1-\delta^2)^n + \frac{\epsilon}{2}.$$

This proves the theorem. ♠

**Proof:** For simplicity of writing down the proof, we may assume that $[a, b] = [0, 1]$. This is fully justified by the linear change of coordinates $t \mapsto (t-a)/b-a$. Given $f \in C[0,1]$ we may assume that $f(0) = 0 = f(1)$ by considering $g(t) = f(t) - f(0) - t(f(1) - f(0))$.
Now we extend $f$ to the whole of $\mathbb{R}$ by defining it to be zero outside $\mathbb{R}$.
For each positive integer $n$ put

$$Q_n(t) = \begin{cases} \frac{(1-t^2)^n}{\alpha_n}, & |t| \leq 1; \\ 0, & |t| > 1. \end{cases}$$

where

$$\alpha_n = \int_{-1}^{1} (1-t^2)^n dt.$$

Note that

$$Q_n(t) \geq 0 \text{ for all } t;\ Q_n(-t) = Q_n(t) \text{ and } \int_{-1}^{1} Q_n(s)\, ds = 1. \qquad (20)$$

Put

$$P_n(t) = \int_0^1 f(s) Q_n(t-s)\, ds \tag{21}$$

Note that $P_n$ is a polynomial function. We claim that this sequence converges uniformly to $f$ on $[0,1]$. Since for all $t \in [0,1]$, we have $[0,1] \subset [t-1, t+1]$, and since $f(s) = 0$ outside $[0,1]$,

$$P_n(t) = \int_{t-1}^{t+1} f(s) Q_n(t-s)\, ds.$$

Now, by substituting $t - s = u$, we get

$$P_n(t) = \int_{-1}^{1} f(t-u) Q_n(u)\, du. \tag{22}$$



Therefore for every $0 < \delta < 1$ we have

$$Q_n(t) \leq \sqrt{n}(1-\delta^2)^2, \qquad \delta \leq |t| \leq 1. \tag{23}$$

Recall that the geometric expansion

$$\frac{1}{(1-t^2)^2} = \sum_{n=0}^{\infty} (n+1) t^{2n}, \quad |t| < 1.$$

This implies that $\lim_{n \longrightarrow \infty} (n+1) t^{2n} = 0, |t| < 1$. This in turn implies $\lim_{n \to \infty} n t^{2n} = 0$. Upon taking square-root and putting $t = 1 - \delta^2$ we get

$$\lim \sqrt{n}(1-\delta^2)^n = 0. \tag{24}$$

Now by uniform continuity of $f$, given $\epsilon > 0$ there exists $0 < \delta < 1$ such that

$$|f(t) - f(s)| \leq \epsilon/2, \quad \forall\, |t - s| < \delta, \quad t, s \in \mathbb{R}.$$

So, now, we will use this data and combine all the various properties (20) (21) etc. What is (20)? Let us just recall, (20) is I told you gives these are 3 properties here symmetry of $Q_n$, non-negativity and integral $Q_n$. And (21) is a formula for $P_n(t)$. (22) is the new formula after change of variables. Then (23) is $Q_n(t)$ being dominated by this term which converges to $0$, that is (24).

So, if we combine all these things, what you get is as estimate for $|P_n(t) - f(t)|$. We have to estimate this one, we have to show that this is less than $\epsilon$, irrespective of what $t$ we take, provided $n$ is sufficiently large.

So, I am just writing down the formula for $P_n(t)$ and using the fact that $Q_n(s)\,ds$ integral is $1$. So, I can multiply by $f(t)$ which is a constant as far as the integration is concerned. So, $f(t-s)Q_n(t)$ is $P_n(t)$. This $f(t)Q_n(s)\,ds$ integral is just $f(t)$ because integral of $Q_n(s)\,ds$ is $1$. So, that has been used here. So, there is nothing else here.

Next, the modulus sign is taken in the inside the integral, modulus of integral is less than or equal to integral of the modulus is the elementary property of Riemann integral of real valued functions. When you take the modulus inside what you get is $f(t-s) - f(s)$ modulus of the whole thing into $Q_n(s)\,ds$, $Q_n(s)$ being non-negative, comes out of the modulus sign. All this is less than or equal to other input, because I cannot put equality here because this is only inequality.

Now, this integral from $-1$ to 1, I am breaking it into integral over threee parts: first one is $-1$ to $-\delta$, next is, $-\delta$ to $\delta$ and the last one is $\delta$ to $1$ of the same function. So, write down these three integrals.

Now, in the first interval $(-1, -\delta)$, what is happening? This integrand will be less than or equal to $M$, where $M$ is a constant chosen such that $|f(t)| \leq M$. So, this is a general bound, so I am using that. So, $|f(t-s) - f(s)|$, the modulus of this difference is less than or equal to sum of the moduli of each of them, less than or equal to $2M$. Then mutiplies by the integral of $Q(s)\,ds$ as it is, I do not know what it is, first I have got this much.

In the second term, I am using the fact that the same term is less than $\epsilon/2$, since $t$ ranges between $-\delta$ and $\delta$. So, this is where I have used this one now. So, this $|f(t-s) - f(s)|$, this is less than or equal to $\epsilon/2$ and that comes out of the intergal sign and integral of $Q_n(s)\,ds$ remains. In the the third part, again I am estimating this part as in the first part. This is less than or equal to $2M$ times integral of 0 to 1 of $Q_n(s)\,ds$. So, different estimates in 3 different parts.

Now, what happens? Combining the first and third term, the sum is less than or equal to 4 $M\sqrt{n}(1-\delta^2)\,ds$. So, this is where I used the fact that $Q_n$ is dominated by this term. This term is also dominated by that. And $-1$ to $\delta$ and $\delta$ to 1, these two integrals are the same. So, I can bring them together. So, $4M$ times this one, this is where the symmetry, $Q_n(-s)$ is $Q_n(s)$ is used, the symmetry is used. The middle term is less than or equal to $\epsilon/2$ times the integral, this integral is smaller than the integral $-1$ to plus 1, which is 1. So middle term is also less than $\epsilon/2$.

So, what we have shown is $|P_n(t) - f(t)|$ is less than equal to this one, $\sqrt{n}$ has come here. If you choose $n$ sufficiently large, this can be made less than $\epsilon/2$, $\epsilon/2 + \epsilon/2$ is less than $\epsilon$. So, in the choice of $n$, $t$ is not involved. Sufficiently large $n$, that will depend upon your $\epsilon$ only because this term goes to 0, as $n$ goes to infinity.

So, this proves Weierstrass's theorem.

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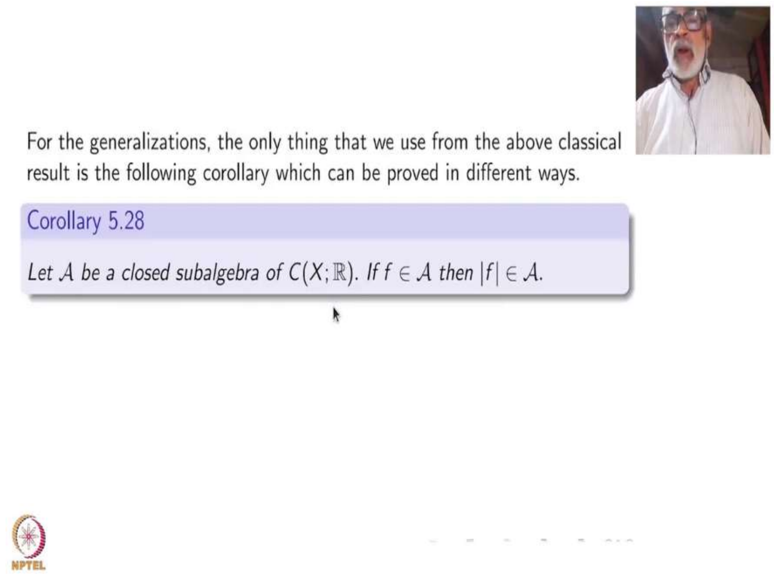


Let us take one small step before closing up today, towards a more general results now. For generalizations, the only thing that we use from above classical result is the following corollary, which can be proved in different ways. You do not have to prove Weierstrass theorem fully. So, what is the corollary, corollary is:

Let $A$ be a closed subalgebra of $C(X;\mathbb{R})$.

Remember subalgebra etc we have defined in the part I. Closed means that there is a topology on $C(X;\mathbb{R})$ and closeness is taken with respect to that topology. This $A$ is an algebra, so it is a closed subalgebra of $C(X;\mathbb{R})$. If $f$ is inside $A$, then $|f|$ is inside $A$.

So this is what we want to prove. Why this so so?  Here is a proof.

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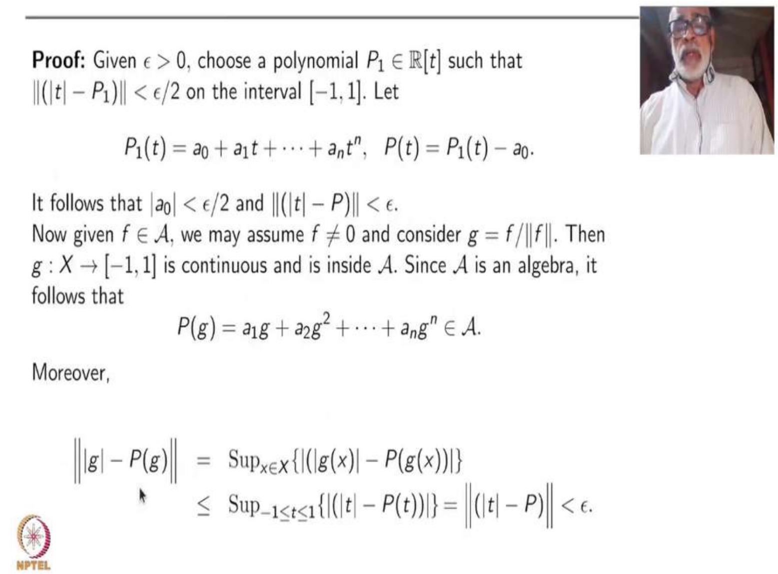


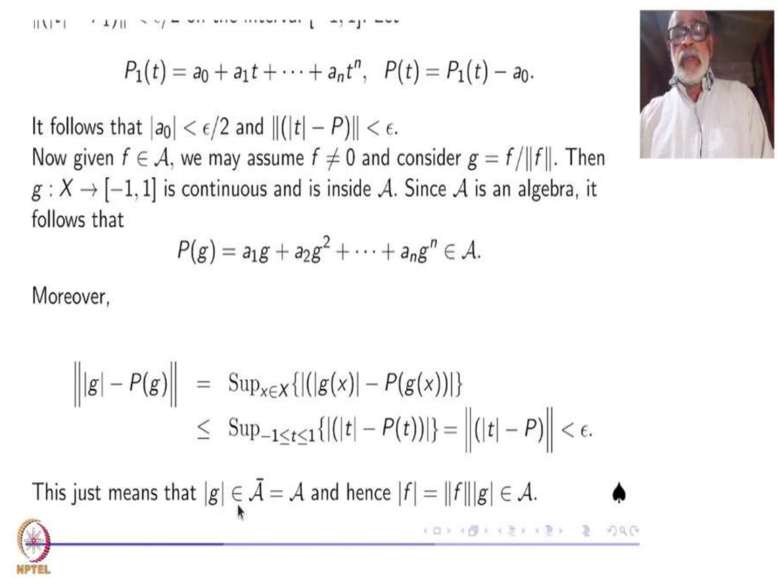


It is directly by Weierstrass approximation. Given $\epsilon$ positive, choose a polynomial $P_1$, a polynomial means, with real coefficients everything so far, $P_1$ is in $\mathbb{R}[t]$ such that the function $|t|$ is approximated by $P_1$, namely, $\||t| - P_1\| < \epsilon/2$, on the entire interval, this time the interval is $-1$ to 1, I am taking. Remember the Weierstrass theorem was proved for all closed intervals.

Let this polynomial $P_1(t)$ look like $a_0 + a_1 t + \ldots |a_n t^n$. Put $P(t) = P_1(t) - a_0$. This constant term $a_0$ is disturbing me, so I will throw away that and let us look at the rest of the terms in $P_1(t)$, call it $P(t)$, equal to $P_1(t) - a_0$. Then $|a_0| \leq \epsilon/2$, why, because you put $t = 0$, here in the above inequality, this is just $|a_0| < \epsilon/2$.

It follows that $\||t| - P\| < \epsilon$. This $a_0$ is missing, that will contribute another $\epsilon/2$; so this is $\epsilon$. That is why I have chosen here $\epsilon/2$ in the beginning. So, now what I have got is a polynomial $P$ without a constant term just like $t$ here, and $|t|$ is approximated by this polynomial. Now, given any $f \in A$, any element of $A$, but first of all, we may assume that $f$ is not the zero element.

So, what we want to prove is that $|f|$ belongs to $A$. If $f$ is zero function $|f|$ is also $0$ function there is no nothing to prove. So, we may assume $f$ is not zero identically, so that its norm is also not zero. Now, you divide by the norm, take $g = f/\|f\|$. Now, look at $g$. It is inside $A$.

Why? Because I have just divided by a scalar function. So, subalgebras are vector subspaces after all. Since $A$ is a subalgebra, it follows that if I take a polynomial in $f$ such as $a_1 f + a_2 f^2 + \cdots + a_n f^n$, that will be also inside. So, in particular, if you take $P(g)$, see the constant term is missing here that is important, that is also in A.

Moreover, now, look at $|g| - P(g)$, norm of this one. See, remember that Weierstrass theorem was only for continuous functions defined on a closed interval. Now, we have gone into arbitrary spaces, but now, the image of $g$ is in the interval $[-1, 1]$. So, everything is happening in the image.

So, $|g|$ where is it taking values, $[-1, 1]$. So, it is like $|g|$ itself being a variable that is precisely what I am thinking, $|g|$ is like a variable $t$, $t = g(x)$, so that is the function. The $|g| - P(g)$, the norm of that, this is same thing as supremum over all $x \in X$ of all $\big||g(x)| - P(g)(x)\big|$, take the modulus of the difference and take the supremum. That is the definition of the supremum norm.

But this supremum norm is less than of equal to the supremum over all $t$ varying between $[-1, 1]$, of $\||t| - |P(t)|\|$. As $x$ varies over $X, g(x)$ simply varies in the interval $[-1, 1]$. So, I take $t$ between $[-1, 1]$, $|t| - P(t)$, you take the supremum of all these elements. So, this may be larger because all $t$ inside $[-1, 1]$ may not look like $g(x)$ for some $x$. So, this is the larger set, the supremum over a larger set is larger. So, this is less than equal to this one.

But this is the same thing as norm of $|t| - P(t)$. See, if you do not like this symbol you should write something like $\lambda(t) = |t|$ and then this term here is the norm of $\lambda - P$. So that norm of $\lambda - P$ is less than $\epsilon$. So, $|g| - P(g) < \epsilon$, that is what we have got now.

This just means that given every $\epsilon$, there is a $g$ like this. $P(g)$ is an element of $A$ for all $P$. That means $|g|$ is in the $\overline{A}$, but the $\overline{A}$ is $A$ itself because we have assumed that $A$ is closed. Therefore, $|f|$ which is a scalar multiple of $|g|$ will be also inside $A$. Once $|g|$ is inside $A$, $\|f|g|\|$ will be also inside $A$ but that is $|f|$.

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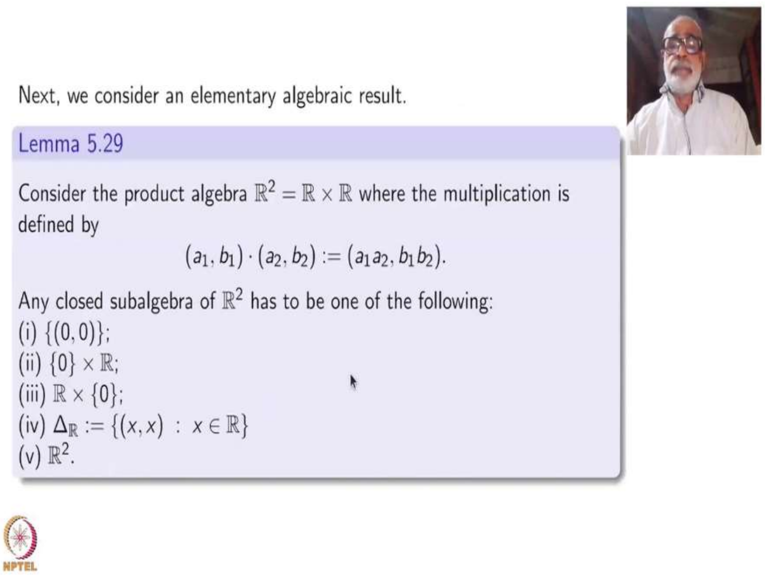


Next, we consider an elementary algebraic result. Namely, instead of studying the big algebra $C(X;\mathbb{R})$ we just study the algebra $\mathbb{R} \times \mathbb{R}$. Think of $\mathbb{R}$ as a ring, $\mathbb{R} \times \mathbb{R}$ has a ring structure. So, what is that ring structure? I am telling you: $(a_1, b_1) \cdot (a_2, b_2)$ is just coordinatewise multiplication $(a_1 a_2, b_1 b_2)$.

So, this is not like multiplication of complex numbers which is more complicated. So, this is the algebra, which is a product of $\mathbb{R}$ and $\mathbb{R}$. Look at this algebra, any closed subalgebra of $\mathbb{R} \times \mathbb{R}$ has to be one of the following: (i), (ii), (iii), (iv) or (v). It can be the $(0)$ subalgebra. It can be $\{0\} \times \mathbb{R}$. It can be $\mathbb{R} \times \{0\}$. It can be the diagonal $\Delta_{\mathbb{R}}$, or it can be the whole of $\mathbb{R} \times \mathbb{R}$. So, these are the only five possibilities of subalgebras of $\mathbb{R} \times \mathbb{R}$.

Why? Of course, 0 is there. Of course, $\mathbb{R}^2$ is also there. Similarly, the other three are subalgebras that is very easy to verify. But why they are the only ones? That is also easy. Because, a subalgebra is, first of all, a vector subspace, vector subspace of 2-dimensional vector space, ( $\mathbb{R} \times \mathbb{R}$ is 2-dimensional vector space over $\mathbb{R}$) has to be either 0-dimensional, 1-dimensional or 2-

dimensional, 0 and 2 are taken care of in (i) and (v). The rest correspond to 1-dimensional subspaces.

How do you show that there are only these three 1-dimensional subalgebras? A 1-dimensional subspace is spanned by a single vector. If that vector is of this form $(0, r)$, then it will be this one. Instead if it is of the form $(r, 0)$ then it will be of this. In the general case, I want to say that the generator is of the form $(x, x)$. Why this is true? So, this is one thing which bothers us. So, you have to do a little more algebra than vector subspaces. Otherwise, you are classifying all vector subspaces, when all the lines passing through origin had to be taken. Other lines do not come here is what you have to see.

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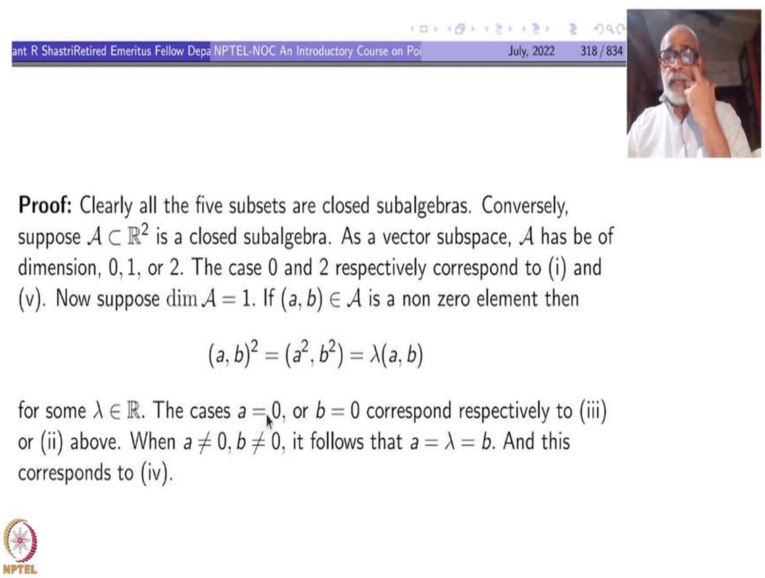

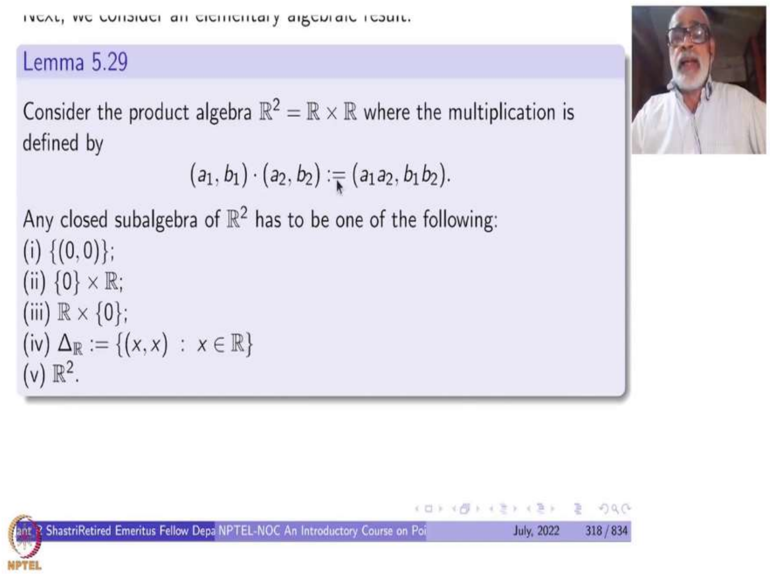


So, we will see that, that will be end of it. So, take $(a, b)$ belonging to $A$ which is nonzero element of this subalgebra. So, I am looking at this 1-dimensional case, but it is a subalgebra. Therefore, $(a, b)$ square must be also inside $A$. But $(a, b)$ square is by definition is $(a, b)^2 = (a, b).(a, b) = (a^2, b^2)$. This must be also inside this 1-dimensional space. So, it must be a scalar multiple of $(a, b)$.

Let $\lambda$ be some real number, so that $(a^2, b^2)$ is $\lambda(a, b)$. The case when $a = 0, b = 0$ correspond to $\{0\} \times \mathbb{R}$ or $\mathbb{R} \times \{0\}$. So, let us forget about that. Look at the other cases when $a$ and $b$ are non-zero. $a^2 = \lambda a$ and $a$ is non-zero implies $a = \lambda$. Similarly, $b^2 = \lambda b$ imply $\lambda = b$. So, we have got $a = \lambda = b$.

So, therefore, this $A$ is nothing but a diagonal of slope 1.

So, that is the lemma here. So, there are only 5 sub algebras. We will see that the entire thing will be reduced to this 2-dimensional case, the proof of Stone-Weierstrass theorem that we are going to study. So, that we will do next time. Thank you.