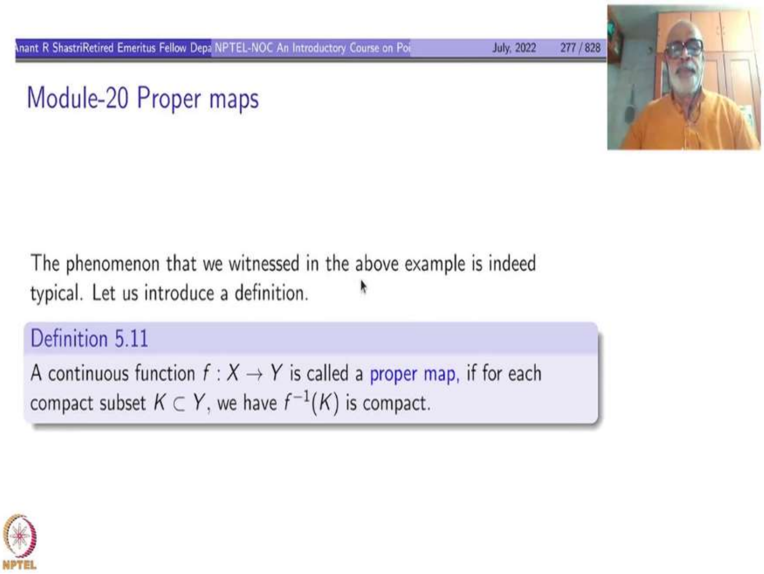
# An Introduction to Point-Set-Topology (Part II)
**Professor Anant R Shastri**
**Department of Mathematics**
**Indian Institute of Technology Bombay**
**Lecture 20**
**Proper Maps**

(Refer Slide Time: 00:16)

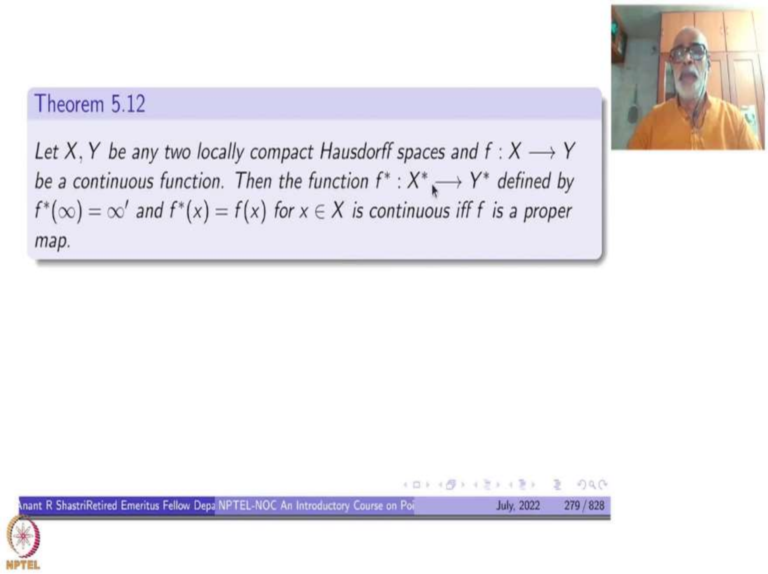



Welcome to NPTEL NOC an introductory course on Points Set Topology Part II. We continue our study of compactifications. As a special topic today we will study proper maps which are very closely related to, hand in glove with, compactifications. The phenomenon that we witnessed in the above example is indeed a typical one. So, let us introduce a definition.

A continuous function $f$ from $X$ to $Y$ is called proper if for each compact subset $K$ of $Y$, we have $f^{-1}(K)$ is compact. For example, quite often in a locally compact space, such as $\mathbb{R}^n$ and so on, if you have a finite-to-one map, not always, okay, a finite-to-one map will be compact. Infinite to one map may not be proper. I am talking about proper map. So, proper maps are a kind of tools to beat the non-compactness of domain. Study of continuous functions from one non-compact space to another non-compact space often involves proper maps.

So, I repeat the definition. It just says that inverse image of a compact set is compact. Remember if you have a continuous function image of a compact set is compact. That comes freely. Now, we want inverse image of compact sets to be compact. So, you may say okay you are talking about $f$ being a homeomorphism? No, $f$ can be an infinite- to- one map also, no problem, but I want inverse image of a compact set to be compact. I am not assuming that is one oneness. Inverse of $f$ as a function may not exist, but inverse of a subset of the codomain under $f$ makes sense. That is all we are using here.

(Refer Slide Time: 02:51)



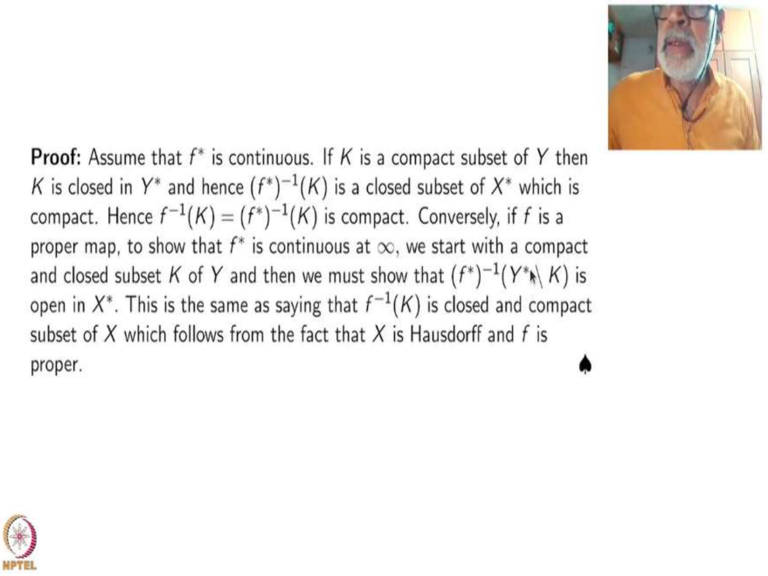

So, here is a theorem, which explains why we study the proper maps in the context of one point compactifications.

Let $X$ and $Y$ be any two locally compact Hausdorff spaces. Let $f$ from $X$ to $Y$ be a continuous function. Then the function $f^*$ from $X^*$ to $Y^*$ defined by $f^*(\infty)$ is $\infty'$ (I am denoting the two extra points by $\infty$ and $\infty'$ respectively) and $f^*(x)$ is equal to $f(x)$ for all $x \in X$, (i.e., $f^*$ is an extension of $f$) that function is continuous if and only if $f$ is a proper map.

So, this is the motivation of defining and studying properness here. Remember what we did in the example earlier, for the function $\phi$ which was defined from $\mathbb{S}^{n-1}$ minus the north pole into $\mathbb{R}^{n-1}$ sitting inside its one-point compactification on the other side. $\phi$ was extended to the whole of $\mathbb{S}^{n-1}$ by taking the north pole to $\infty$.

So, that was the model, so the extension became continuous. There we could verify it by just purely looking at the inverse image of neighbourhoods of the infinity there, that is all. (Indeed, this was left as an exercise.) So, here, we have to verify the continuity by pure logic, there is no geometry here, merely follow the definitions of these $X^*$ and $Y^*$, and the proper map. This is also not difficult, let us go through this. (In fact, once you prove this, the exercise that you are suppose to do i the above example will also get done!)

(Refer Slide Time: 05:07)




Assume that $f^*$ is continuous. If $K$ is a compact subset of $Y$, then $K$ is closed inside $Y^*$ also, because it is subspace of $Y^*$ and $Y^*$ is Hausdorff. And hence $(f^*)^{-1}(K)$ will be a closed subset of $X^*$. But closed subsets of compact space are compact. But you have $f^{-1}(K)$ is nothing but $(f^*)^{-1}(K)$, because $K$ is a subset of $Y$ and $f^*$ sends $\infty$ to $\infty'$.

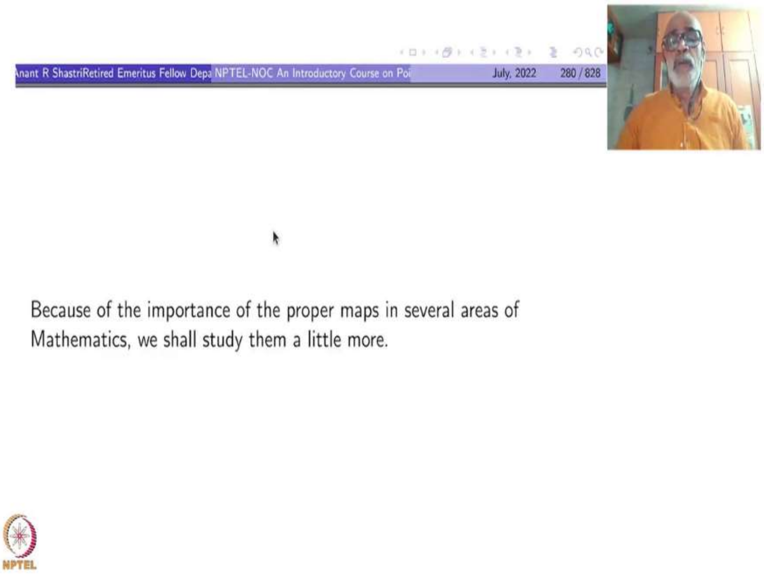
So, we have proved that inverse image of every compact subset of $Y$ is compact inside $X$ under $f$. That is $f$ is a proper map. Now, let us prove the converse. If $f$ is a proper map to show that $f^*$ is continued at $\infty$, this $\infty$ goes to $\infty'$. (Elsewhere it is continuous, because it is an extension of $f$.)

So, all that I have to do is take neighborhoods of $\infty'$ show that $f^*$ inverse of such neighborhoods are open inside $X^*$. That is what you must show. So, how are the neighborhoods of infinity prime described? They are nothing but $Y^* \setminus K$ where $K$ is a closed and compact subset of $Y$. You take a compact subset of $Y$, take the complement of that in $Y^*$ that is an open neighborhood of $\infty'$ by definition.

So, I have to show that inverse image of such a set is open in $X^*$. But just now, we have seen that $(f^*)^{-1}(Y^* \setminus K)$ is nothing but $X^*$ setminus $f^{-1}(K)$. So, its complement in $X^*$ is just f $f^{-1}(K)$. So, this is same thing as saying that $f^{-1}(K)$ is closed and compact subset of $X$.

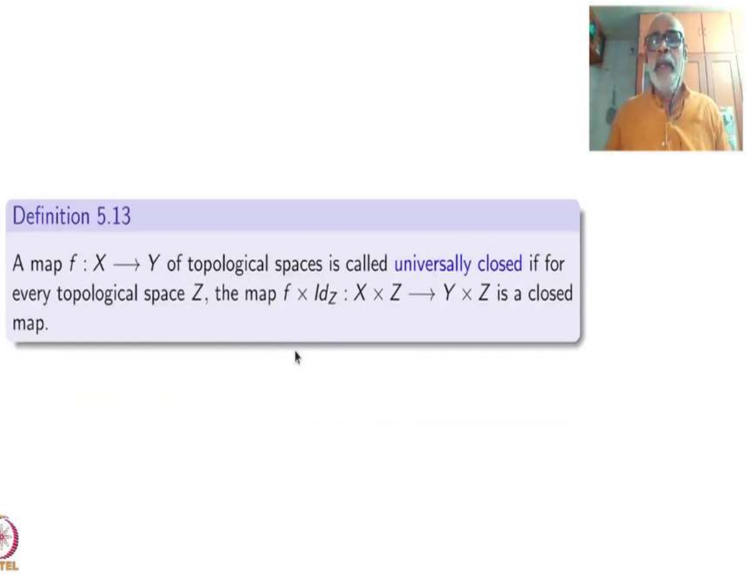
So, that is an open neighborhood by definition of infinity here in $X^*$. Since we have assumed that $f^{-1}(K)$ is compact whenever $K$ is compact, this will be a neighborhood. So, neighborhoods of $\infty'$ inverse image are neighborhoods of $\infty$ inside $X^*$. This completes the proof.

(Refer Slide Time: 08:15)




Because of the importance of proper maps in several areas of mathematics, we shall study them a little more. We have introduced the properness via this wonderful property of Alexandroff's compactification. But the proper maps have their own proper life other than just Alexandroff's compactifications.

(Refer Slide Time: 08:49)

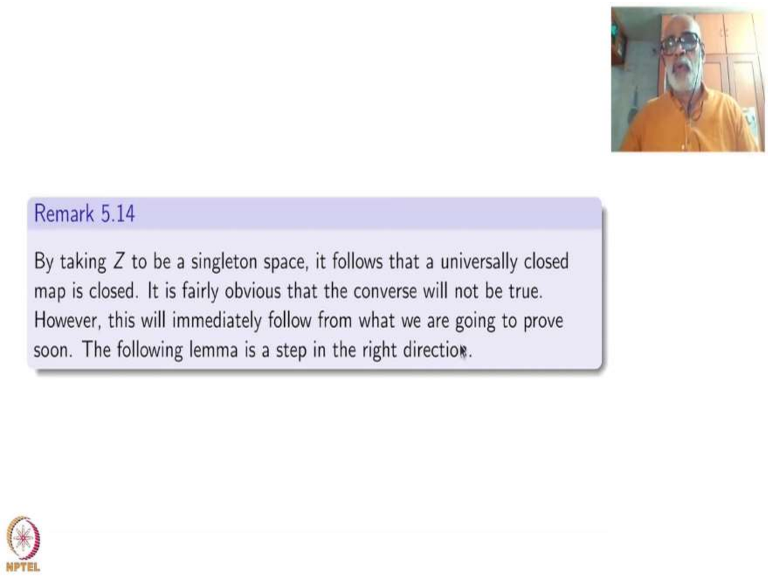



Let us do a little more of this one. Indeed, the concept of proper maps appropriately adopted in algebraic geometry, is very important because there you do not have Hausdorffness at all.

So, all these Alexandroff's compactifications do not make sense, but proper maps do make sense.

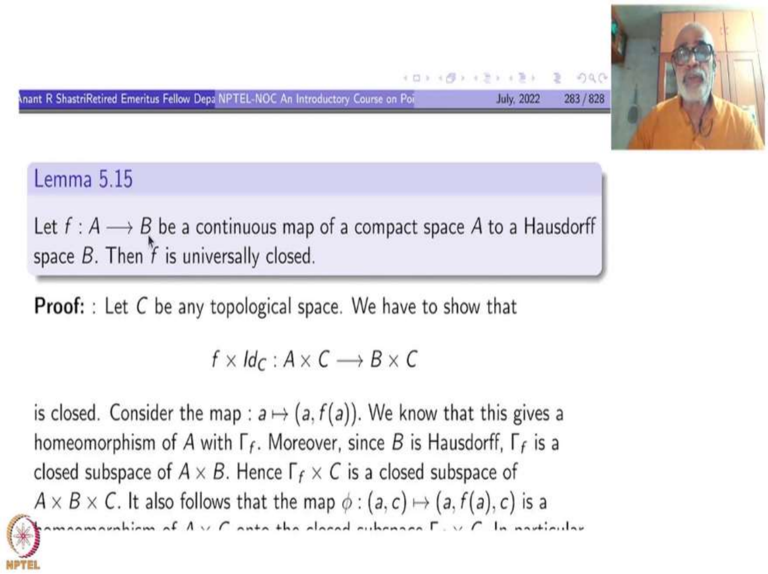
So, let us study them a little more. Start with any continuous function f of topological spaces; $f$ from $X$ to $Y$ is called universally closed if for every other topological space $Z$, no matter what it is, $f \times id_Z$, from $X \times Z$ to $Y \times Z$ is a closed mapping. See clearly $f \times id_Z$ is a continuous function. We wan it to be a closed mapping, which means that closed subsets of $X \times Z$ are taken to closed subset of $Y \times Z$. So, such a function is called universally closed. More generally, with reference to taking product with $Z$ etc., if a closed subset goes to closed subset by a continuous function such a function is called a closed map. For all $Z$ and for a product like this, it is closed, means universally closed; that is the definition.

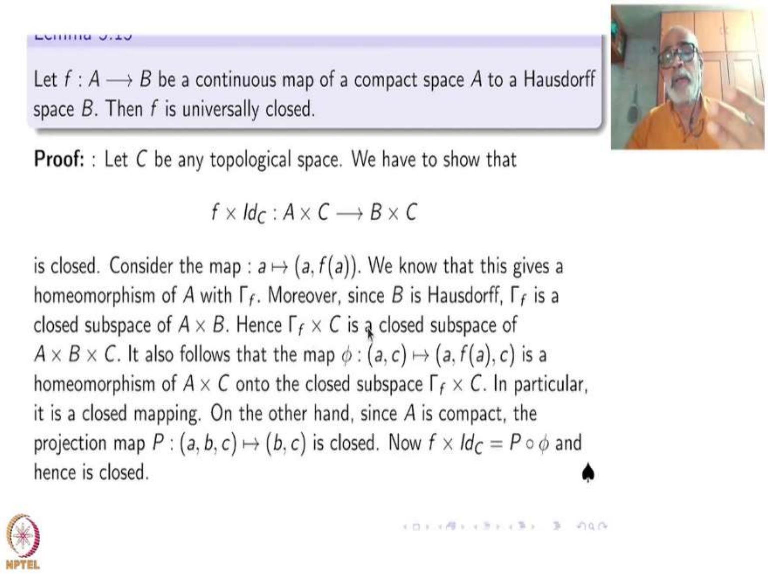
(Refer Slide Time: 10:24)




By taking $Z$ to be the singleton space, it follows that a universally closed map is a closed map also. There is no problem. It is fairly obvious that the converse will not be true in general. You look at examples on your own you can easily find some. However, this will immediately follow from what we are going to prove soon. Therefore, you do not have to worry. The following lemma is a step in the right direction.

(Refer Slide Time: 11:02)




Take any continuous function $f$ from $A$ to $B$, where $A$ is compact and $B$ is Hausdorff. Then that function $f$ is universally closed. Now, it must have rang a bell inside you, right from part I, we have been studying this kind of situation. We have proved that a continuous bijection from a compact space to a Hausdorff space is a homeomorphism. Then we also prove that any surjective map from a compact space to Hausdorff space is a quotient map. This is just one more step.

Now, the secret is open. This is precisely what it is. Not just a closed map, but it is what, it is universally closed. From a compact space to Hausdorff space, a continuous function is universally closed. So, let us prove this. What you have to do? Take any space $C$. I am now changing the notation: instead of $X, Y, Z$ etc., I have $A, B, C$ etc. deliberately.

So, take any topological space $C$, take $A \times C$ to $B \times C, f \times Id$. You must show that this is a closed map. So, go to the graph of this function $f$ inside $A \times B$ and consider the function a going to $(a, f(a))$. So, we know that this gives a homeomorphism of $A$ with $\gamma_f$. This map itself is a homeomorphism, the image is the graph $\gamma_f$. The domain is $A$. Moreover, since $B$ is Hausdorff, by criterion of Hausdorffness, $\gamma_f$ is a closed subset of $A \times B$. (So, this also you have seen in part I. Not very difficult.)

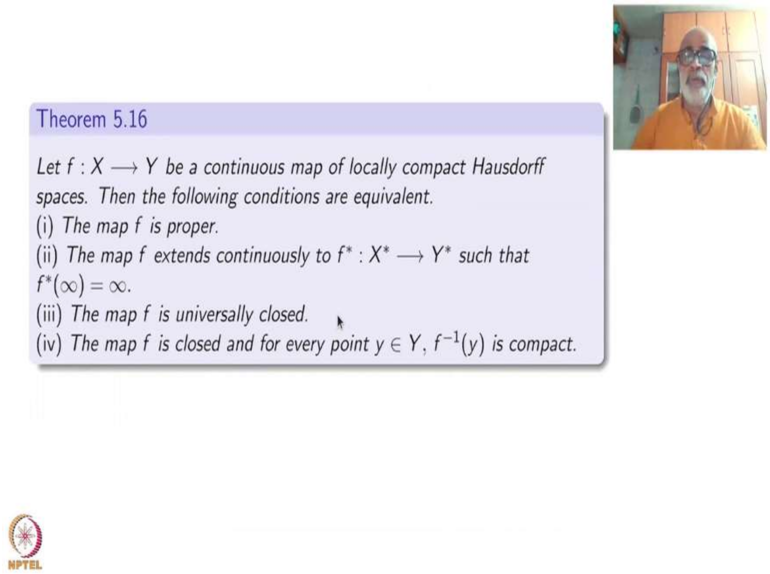
Hence, $\gamma_f \times C$ is a closed subspace of $A \times B \times C$. You see, there is no map now. Take a closed subset, crossing with the whole space here, which is another closed set, yields a closed subspace of the product. Also it follows that if you take the continuous function given by $(a, c)$ mapsto $(a, f(a), c)$, is a homeomorphism of $A \times C$ onto the graph of $f \times Id_C$. In particular, it is a closed mapping being a homeomorphism. A homeomorphism is a bijection which is continuous and closed.

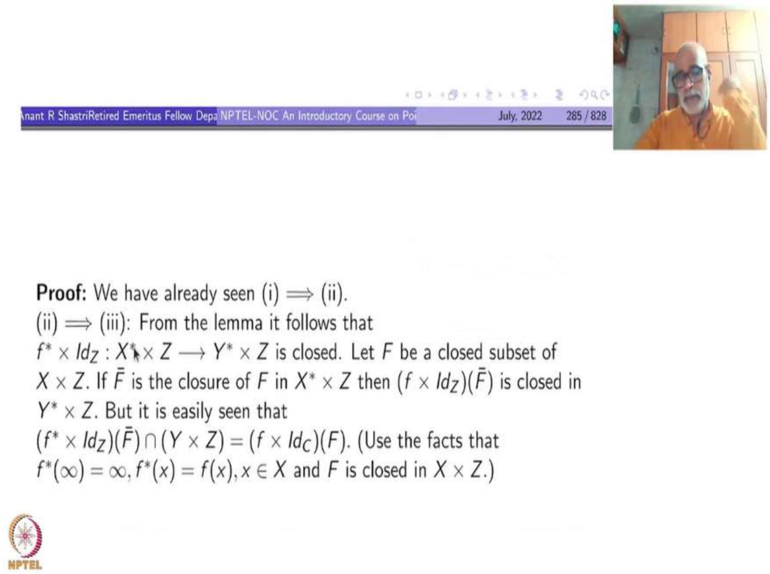
So, it is a closed mapping, from where to where, from $A \times C$ to $\gamma_f \times C$ and since $\gamma_f \times C$ is closed in $A \times B \times C$, it follows that the map $\phi$ with $(a, c)$ going into $(a, f(a), c)$ is a closed mapping of $A \times C$ into $A \times B \times C$.

Now, $f \times Id$ is nothing but $p \circ \phi$, the $A$-factor goes away. Since composite of closed mappings is a closed map, we are done.

I would request you to pay attention to the technique which we have used in this proof. This kind of technique is used in several topological proofs. It involve a very simple idea: statements about functions being converted into statements about sets and vice versa, by taking the graphs. So, pay attention to this.

(Refer Slide Time: 16:51)




So, now, we have come to the properness. Let $f$ from $X$ to $Y$ be a continuous map of locally compact Hausdorff spaces, both $X$ and $Y$ are locally compact Hausdorff. Then the following conditions are equivalent:

(i) $f$ is a proper map.

(ii) $f$ extends continuously to a function $f^*$ from $X^*$ to $Y^*$, the one-point compactifications, where $f^*(\infty) = \infty'$.

(iii) $f$ is universally closed.

(iv) The map $f$ is closed and for every point $y \in Y$, $f^{-1}(y)$ is compact.

See, if $f$ is a proper map, singleton $\{y\}$ are always compact and hence $f^{-1}(y)$ will be compact. But this property is much weaker. However, along with just closedness of the map, it gives properness. So, the last property is adopted as the definition of properness, quite often, rather than the other three conditions. So, when you do not want to mention compactness at all, you may put some stronger condition viz., $f^{-1}(y)$ is finite for every $y \in Y$.

So, such functions are studied. So, this property, pay attention to this, is a wonderful property which you can modify. So, (i), (ii), (iii), and (iv) are four different conditions. They are all equivalent for functions between locally compact spaces. Again, proofs are not difficult because we have already developed enough techniques here.

So, (i) implies (ii) we have already seen. That was our motivation to introduce the notion of properness. So, let us prove, (ii) implies (iii). From the lemma, it follows that $f^* \times Id$ from $X^* \times Z$ to $Y^* \times Z$ is closed, which is the same thing as saying $f^*$ is universally closed, because $X^*$ is a compact space, and $Y^*$ is Hausdorff space.

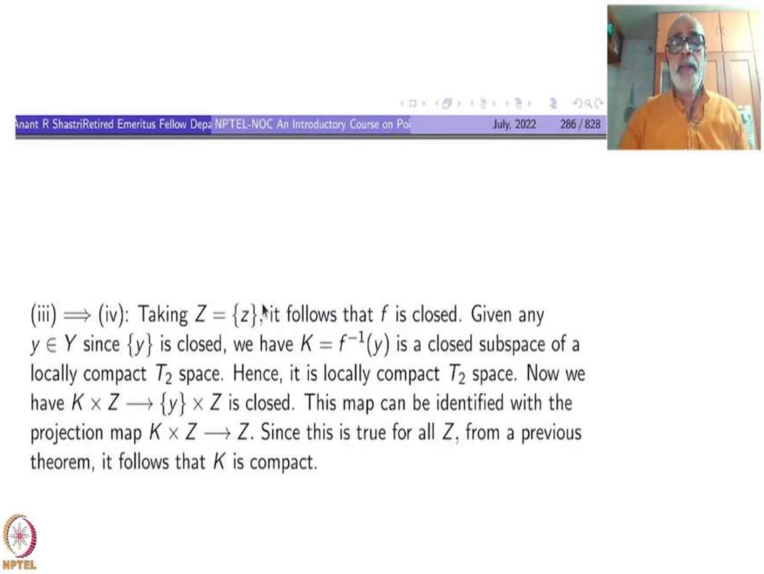
Any continuous function is universally closed from a compact space to a Hausdorff space. That was the lemma. Now, let $F$ be a closed subset of $X \times Z$. Note that we have to prove $f \times Id$ is a closed map, not just $f^* \times Id$. For $f^*$, it was ok because the domain was compact. So, now I have to come back to $X$ now.

So, let $F$ be a closed subset of $X \times Z$. Let $\bar{F}$ be the closure of $F$ inside $X^* \times Z$. So, be sure where you are taking the closure, because inside $X \times Z$ the closure of $F$ is $F$ itself, but $F$ may not be closed subset of $X^* \times Z$, so you have to take the closure here. So, take the closure of $F$. Then $f^* \times Id$ of $\bar{F}$ is a closed subset of $Y^* \times Z$.

But it is easily seen that $(f^* \times Id)(\bar{F}) \cap (Y \times Z)$ is nothing but $(f \times Id)(F)$. Because, the point $\infty$ goes to $\infty'$ under $f^*$. So, if you remove that $\infty'$, namely, by taking intersection with $Y \times Z$ here, consider only points which look $(y, z)$, then it is the image of $F$ under $f \times Id$. Use the fact that $f^*(\infty)$ is $\infty'$ and $f^*(x) = f(x)$ for $x \in X$. Any extra points in the closure of $F$ should have its first coordinate equal to infinity.

So, $f \times Id_Z(F)$ is equal to $(f^* \times Id_Z)(\bar{F}) \cap (Y \times Z)$. This thing is closed in $Y^* \times Z$, so this intersection with $Y \times Z$ is closed inside $Y \times Z$, because $Y \times Z$ is a subspace of $Y^* \times Z$. Ok, that proves (ii) implies (iii) namely we have proved that now $f$ is universally closed.

(Refer Slide Time: 16:51)




Now, I have to prove (iii) implies (iv). Of course, universally closed implies closed. So, only thing that is left out: inverse image of a single point must be compact. This is what remains to be proved. Taking $Z$ as singleton space $\{z\}$. It follows that $f$ is closed. That is what we have seen earlier.

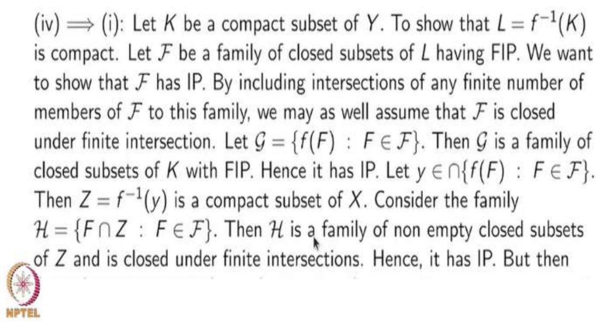
Now given any $y \in Y$, singleton $\{y\}$ is closed, because $Y$ is Hausdorff. Put $K = f^{-1}(y)$. That will be a closed subset of a locally compact Hausdorff space $X$. Hence, it is a locally compact Hausdorff space by itself. I want to prove that it is compact.

Since $K$ is a closed subset of $X$, it follows that $f|_K$ is also universally closed. That is, for every $Z$, $f \times Id$ from $K \times Z$ to $\{y\} \times Z$ is a closed mapping. But this can be identified with the projection map $p$ from $K \times Z$ to $Z$. Since, this is true for all $Z$. From a previous theorem, it follows that $K$ is compact.

See, this is what we have to use now. Projection map away from a space is a closed map then the space is compact. That is what we have proved, Michael's theorem.

So, (iii) implies (iv) is done.

(Refer Slide Time: 24:21)


Now, the proof of (iv) implies (i). So, what we have, we have a function $f$ which is closed and inverse image of each single point is compact. From that we have to show that the function is actually proper, namely, inverse image of any compact set is compact. So, start with a compact subset $K$ of $Y$, put $L$ equal to $f^{-1}(K)$. We have to show that $L$ is compact. So, take $\mathcal{F}$ be a family of all closed subsets of $L$ having the finite intersection property.

Remember that if we show that intersection members of $\mathcal{F}$ is non-empty, then it will follow that this $L$ is compact. That is the thing that we are going to use now. We want to show that $\mathcal{F}$ has intersection property under the assumption that it has FIP, finite intersection property, namely, any finite collection of members of $\mathcal{F}$ when you take their intersection that is non-empty.

By including intersections of any finite number of members of $\mathcal{F}$ to this family, we may assume that $\mathcal{F}$ is closed under finite intersection. To begin with any family you can take, but then you can expand it to include intersections of all finitely many members $F_1, F_2, \ldots, F_k$. It does not change our problem, it does not simplify our problem as such, but this assumption helps, that is all. All finite intersections of members of $\mathcal{F}$ are again inside $\mathcal{F}$, we can assume that.

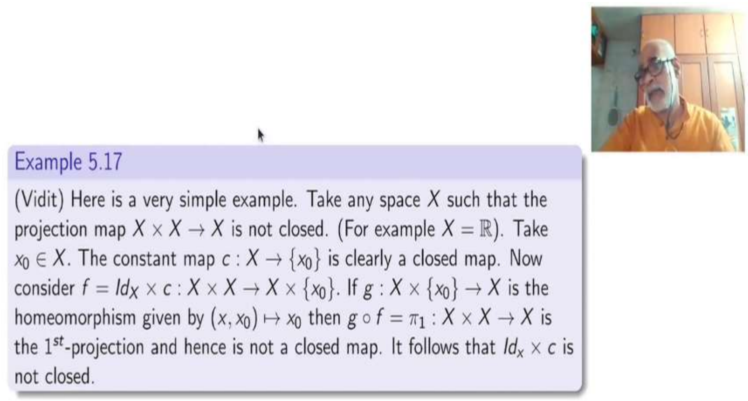
Now, let $\mathcal{G}$ be collection of all the images of members of $\mathcal{F}$ under $f$. So, these are now inside $Y$. Then $\mathcal{G}$ is a family of closed subsets of $K$, why? Because $f$ is a closed map. They are all closed subsets of $K$ because I started everything inside $L$ here. So, when you take $f$ of that they will go inside $K$. Subsets of $K$ with finite intersection property. So, finite intersection property is true for $\mathcal{G}$ also. Therefore, $\mathcal{G}$ has intersection property, i.e., intersection of all the members of $\mathcal{G}$ is non-empty.

Let $y$ belonging to this intersection of all $f(F)$ where $F \in \mathcal{F}$. Then put $Z$ equal to $f^{-1}(\{y\})$. That is a compact subset of $K$, this is the assumption on $f$. Actually $Z$ is a subset of $L$ because this $y$ is inside $K$. Now, consider the family $\mathcal{H}$ whose members are $F \cap Z$, where $F \in \mathcal{F}$.

So, we have this family $\mathcal{F}$ already, we are taking each member there and intersecting it with this $Z$. You see remember this $y$ was chosen in the image of $f(F)$ for all $F$. Therefore, $F \cap Z$ is non empty for all $F$. Clearly each is a closed subset. And finally, $\mathcal{H}$ is closed under finite intersections because we have assumed that $\mathcal{F}$ has that property.

Hence, $\mathcal{F}$ has IP because they are subsets of this compact subset $Z$. But then the problem is over, because this non-empty subset is a subset of all the intersections without taking $Z$ into consideration. This is a larger intersection. So, this must be non-empty.

(Refer Slide Time: 30:02)




So, I told you that any arbitrary closed map need not be universally closed. You can use the fact that if you have a non-compact set, then the projection map will not be closed and so on. So, here is a very simple example. By the way, a few years back, maybe 2 years back, I taught this course, to a couple of students and one of them came up with this idea, his name is Vidit. So, I put his name here. So, here is a very simple example.

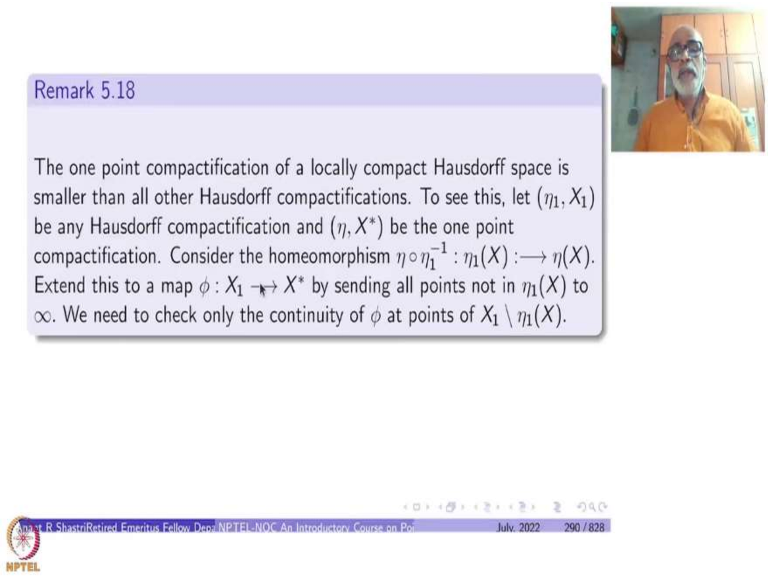
Take any space $X$ such that the projection map is not closed. For example, you can take $X$ equal to $\mathbb{R}$. (Indeed any non-compact space will do. If you are not sure of that, at least you know that the projection map from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$ is not closed.) Now, $x_0$ inside $X$ to be any point. Then constant map $c$ from $X$ to $\{x_0\}$ is clearly a closed map.

This we have used earlier. A function into a singleton space is always a closed map. Now, consider $f$ equal to identity cross the constant map from $X \times X$ to $X \times \{x_0\}$. I have taken $X \times X$ to $X \times \{x_0\}$, say, this is $Y$, $X$ to $Y$.

If $g$ from $X \times \{x_0\}$ to $X$ is the homeomorphism which ignores this $x_0$ factor, viz., $(x, x_0)$ going to $x$, then $g \circ f$ is nothing but the projection $\pi_1$ from $X \times X$ to $X$. Just look at $g \circ f$ on $(x, y)$. Under $f$, it goes to $(x, x_0)$ and then under $g$, it is mapped to $x$. So, whole thing is $(x, y)$ goes to $x$. So, this is the projection map, the first projection map. That is not a closed map.

It follows that identity cross $c$ is not a closed map. Because if $f$ where closed, since $g$ is closed, their composite would have been closed. The composite is not closed and one of them is closed therefore the other one cannot be closed, that is all. So, identity cross $c$ is not a closed map. All that I have taken is $X$ is a space which is a non-compact space. That is all.

(Refer Slide Time: 33:34)

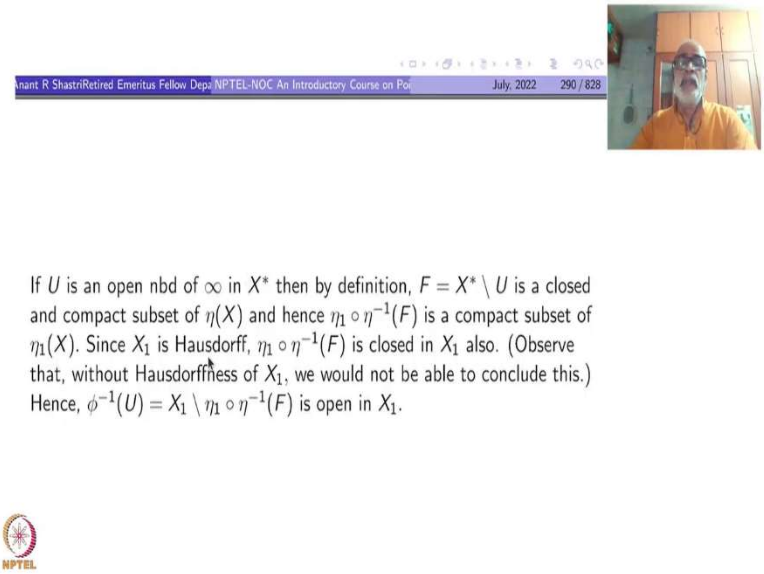



The one-point compactification of a locally compact Hausdorff space is smaller than all other Hausdorff compactifications. Remember, we have put a partial order on compactifications. So, given any two of them, they may not be comparable at all, but here is one conclusion namely the one-point compactification is smaller than all of them.

In particular, each one-point compactification is smaller than all other one point compactifications. But we have verified arbitrary one point compactification, one is smaller than the other, the other is smaller than the one, that need not imply they are equal. Equality, or what we actually want, equivalence is not always possible. So, be careful about that. But smaller makes sense.

So, what is the meaning of smaller? To see that, let $(\eta_1, X_1)$ be any Hausdorff compactification and let $(\eta, X^*)$ is a one point compactification. Consider the homeomorphism $\eta \circ \eta_1^{-1}$ from $\eta_1(X)$ to $\eta(X)$. You start with $\eta_1(X)$, come to $X$ and then go to $\eta(X)$. Extend this to map to a map $\phi$ from $X_1$ to $X^*$, by sending all the points not in $\eta_1(X)$ to the single point infinity in $X^*$. So, this is a nice way of extending it. Do not disturb the $X$ part at all. So, all the points which are outside of $X$, you collapse it to one single point. The only thing you have to check is that, why this $\phi$ is continuous. We need to check only the continuity of $\phi$ at points other than $\eta_1(X)$ in the compactification $X_1$. Once you do that, you have got the inequality, viz, $(\eta_1, X_1)$ is bigger than or equal to $(\eta, X^*)$, that is all.

(Refer Slide Time: 36:02)

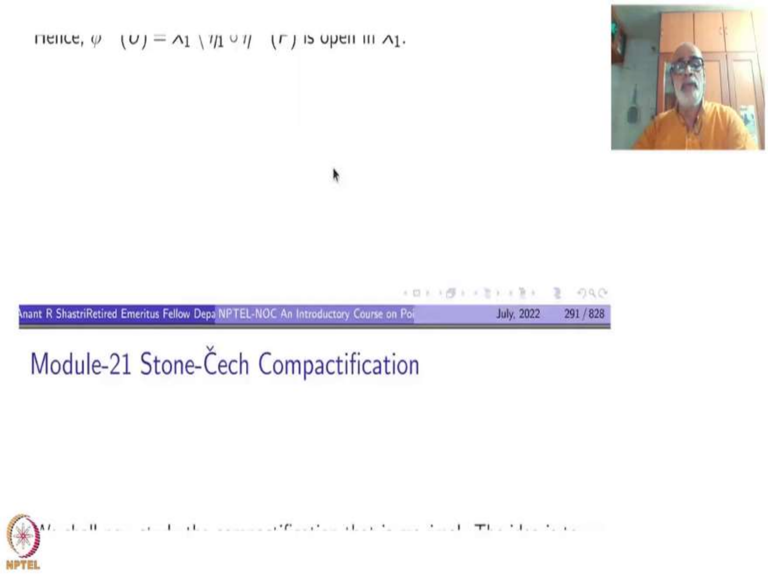



So, let us see why this map $\phi$ is continuous. If $U$ is an open neighborhood of infinity in $X^*$, then by the definition, $F = X^* \setminus U$ must be a closed and compact subset of $\eta(X)$, that is the definition of the Alexandroff's compactification. Hence, $\eta_1 \circ \eta^{-1}(F)$, say equal to $K$, is a compact subset of $\eta_1(X)$, because $\eta_1 \circ \eta^{-1}$ this is a homeomorphism.

Since, $X_1$ is Hausdorff, $K$ will be closed in $X_1$ also. Any compact subset of a Hausdorff space is closed. ($K$ is closed in $\eta_1(X)$ but why should it be closed in $X_1$? Observe that without Hausdorffness of $X_1$, we would not be able to conclude this.) Hence, the $\phi^{-1}(U)$, ($\phi$ is remember it extends the identity map on the entire of $X^*$ by pushing all the extra elements to single point infinity), so $\phi^{-1}(U)$ is nothing but $X_1 \setminus K$. That is an open subset because $K$ is a closed subset. Thus, we have proved that the map phi is continuous at all the points away from $\eta_1(X)$, since they are all going to the point at infinity in $X^*$. So, this completes the proof.

So, one of the fall outs of this study is the concept of universally closed functions and proper maps. There are many other applications of one point compactifications themselves. But let us go to other things now. So, today we stop here.

(Refer Slide Time: 38:26)




Next time we will study another compactification, Stone-Cech compactification. Thank you.