An Introduction to Point Set Topology (Part 2)
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Lecture No: 17
Arzela Ascoli's Theorem

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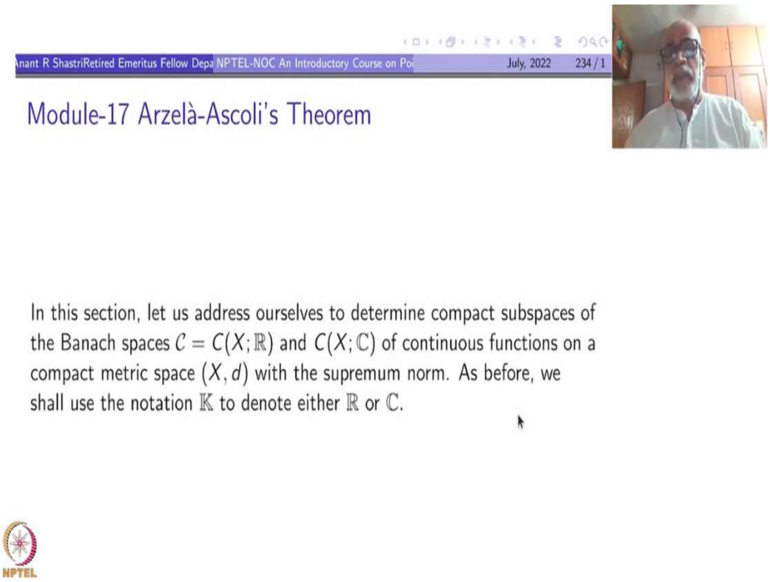

Hello, welcome to NPTEL NOC course on Point Set Topology Part II Module 17. As promised last time we shall do some function space study today, Arzela-Ascoli's Theorems. So, in this section let us address ourselves to determine compact subspaces of the Banach space that we have introduced, namely, space of all continuous functions from X to R and continuous function X to $\mathbb{C}$.

So, both of them we handle simultaneously, no separate proof or no separate techniques necessary. Here we start with X which is a compact metric space. And then the vector space $\mathcal{C}$ of all continuous functions from X to $\mathbb{R}$ (or $\mathbb{C}$) is given a norm viz., the supremum norm. That is how it becomes a Banach space. As before, we shall use the notation $\mathbb{K}$ to denote either $\mathbb{R}$ or $\mathbb{C}$.

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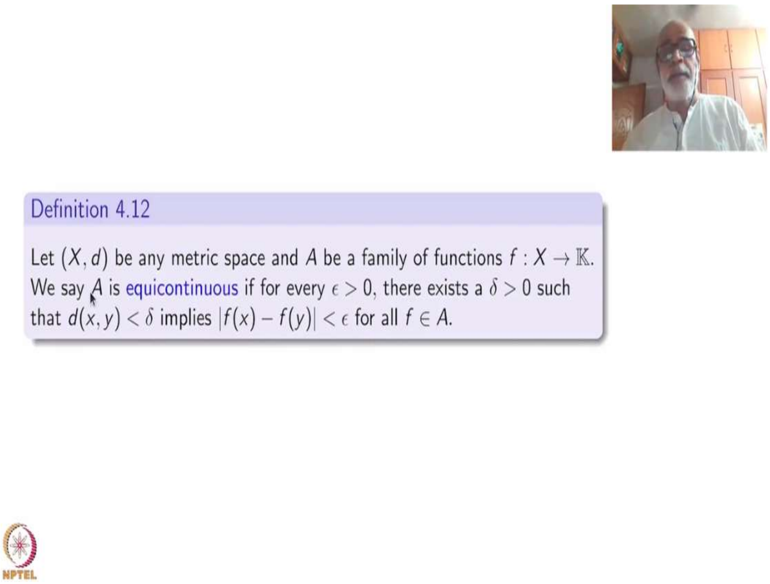
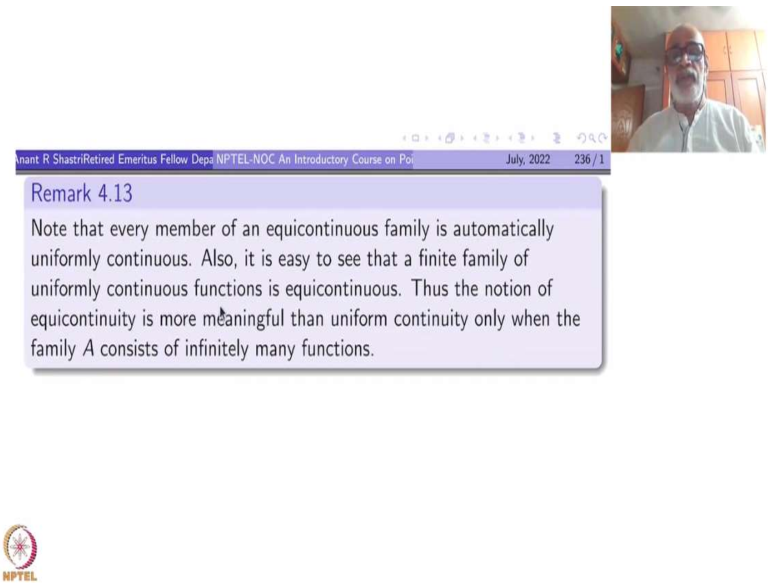


I will make the historical comments later. So, we need a new and important notion here, viz., equicontinuity. Let (X, d) be any metric space. Let A be a family of functions f from X to $\mathbb{K}$. This is the starting data for us. We say A is equicontinuous if for every $\epsilon > 0$, there exists a $\delta > 0$ such that the distance between x and y is less than δ implies distance between $f(x)$ and $f(y)$ is less than ϵ for all f in A and for all $x, y \in X$.

So, this is an $\epsilon - \delta$ definition. So, why it is called equicontinuous? Suppose, there is only one function $f \in A$. Then this is like uniform continuity of f, nothing else. But in the general case, $\epsilon - \delta$'s are the same for all $f \in A$. So that is why it is called equicontinuous family, so this is a new name for new concept. (Refer Slide Time: 02:58)



So, every member of an equicontinuous family is automatically uniformly continuous. If you just read this one a single f, this is uniform continuity. Also it is easy to see that any finite family of uniformly continuous functions is equicontinuous because for each f we have $\delta(f)$. Then you take δ as the minimum of all these $\delta(f)$'s; that will work for all f simultaneously.

So, finite families of uniform continuous functions is automatically equicontinuous. So, this kind of notion is important only when A is an infinite family.

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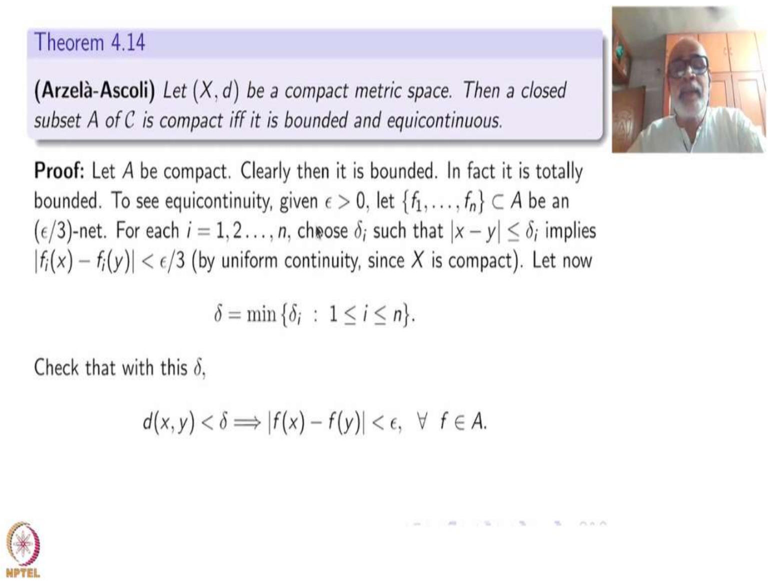

So, here is the famous theorem of Arzela-Ascoli. Let (X, d) be a compact metric space. Then a closed subset A of $\mathcal{C}$ (remember $\mathcal{C}$ denotes the space of all continuous functions from X to $\mathbb{K}$) is compact if and only if it is bounded and equicontinuous.

We are working in a Hausdorff space; you a Banach space is metric space and Hausdorff. So, if you want a subset to be compact, you have to assume A is closed. So, that is a standing assumption that A is a closed family of continuous functions from X to $\mathbb{K}$.

It will be compact in the supremum-norm topology if and only if it is bounded and equicontinuous. So, this is the statement. One part of the proof is very easy. The second part is a little involved. with a new technique which is quite illuminating. So that is what we have to do some work.

Start with a compact A. Then it is clearly bounded, because in any metric space compact subsets are bounded. In fact, we know that it is totally bounded.

To see equicontinuity, given $\epsilon > 0$ let $f_1, f_2, \ldots, f_n$ in A be such an $\epsilon/3$-net because I am now using the total boundedness of this family A. For each i equal to 1, 2, etc. upto n, these $f_1, f_2, \ldots, f_n$ are uniformly continuous and so we can choose δ_i such that $d(x, y)$ is less than δ_i implies $|f_i(x) - f_i(y)| < \epsilon/3$, by uniform continuity since X is compact.

Let now δ be the minimum of $\delta_1, \delta_2, \ldots, \delta_n$. You see I have already used the fact that there is only finitely many of them. So as indicated before in my remark, I am taking the minimum of this δ_i's. Check that this δ has the property that distance between x and y is less than δ implies $|f_i(x) - f_i(y)| < \epsilon/3$ for all i.

Now you have to use the fact that these $\{f_1, f_2, ..., f_n\}$ is an $\epsilon/3$-net. So, you have to do little more triangle inequality business here. So, that much I am leaving it to you as an exercise. for all $f \in A$ this will be true. So, from infinite set to the role of A the link is that this finite set is an $\epsilon/3$-net. So, that comes from total boundedness.

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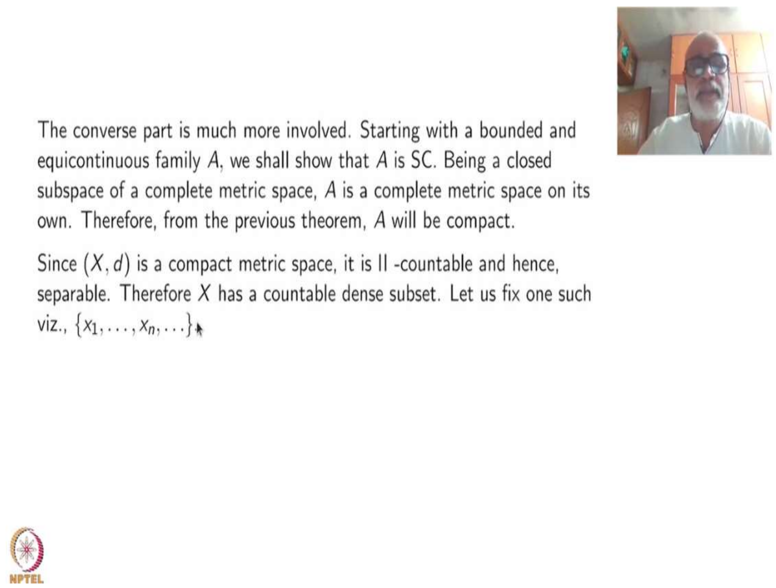

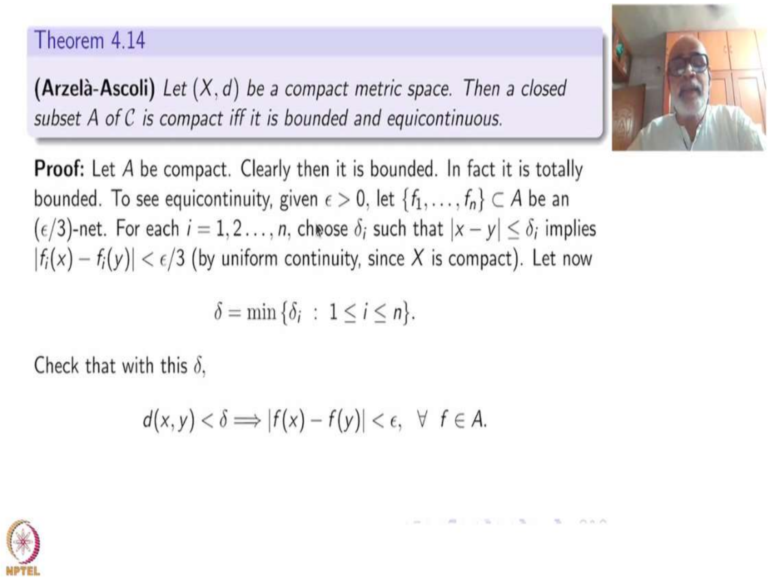

The converse part is much more involved. If these tow conditions are satisfied by A, then you have to show that this A compact. Starting with a bounded and equicontinuous family A, we shall show that A is sequentially compact. Then by our earlier theorem compactness of A will follow.

Since (X, d) is a compact metric space it is second countable also and hence separable. Therefore, X has a countable dense subset. Let us fix one such. Any countable set which is dense will do, no need to be anything special here. After all (X, d) is an arbitrary space and we do not know anything more than that. Fix one such countable dense subset $\{x_1, x_2, \dots, x_n, \dots\}$ of X.

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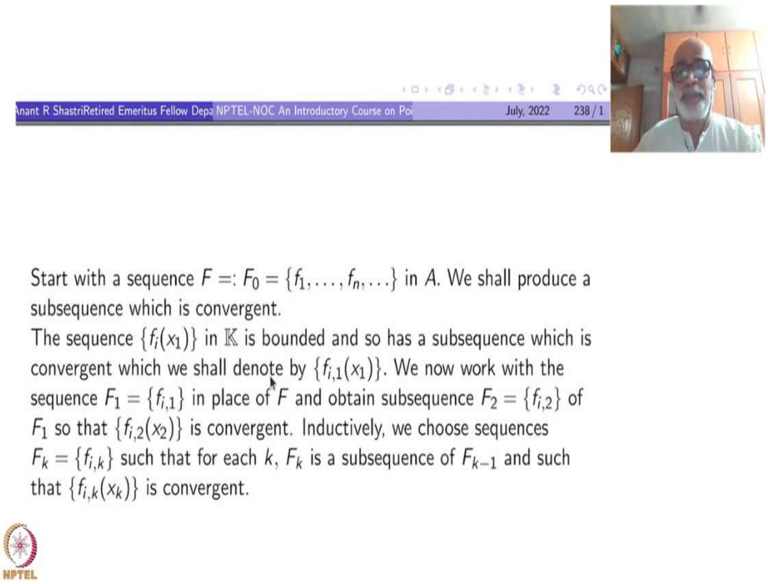

Start with a sequence say $\{f_1, f_2, \ldots, f_n, \ldots\}$ in A. So I am calling it F_0. It is our aim to show that this F_0 has a subsequence which is convergent. So, what we are going to do is we are going to produce a sequence of subsequences almost like improving F with each step which will produce a subsequence which is convergent, finally.

So, the first subsequence is chosen as follows: look at values of f_i's at the single point x_1, viz., $f_1(x_1), f_2(x_1), \ldots$ and so on, the entire sequence evaluated at x_1, at one single point. So, now what we have got is a sequence inside $\mathbb{K}$ which is bounded because the entire A. A bounded sequence inside $\mathbb{R}$ or $\mathbb{C}$ has a subsequence which is convergent.

So, this property of the codomain is essential here. Convergent subsequence is coming pointwise, in the codomain, so we are using that here. So, it has convergent subsequence. We shall denote it by $\{f_{i,1}(x_1)\}$ and the corresponding subsequence $\{f_{i,1}\}$ of F_0 by F_1.

We shall now work with the sequence F_1, and apply the same procedure to this F_1, but with the point x_2, viz., we now evaluate F_1 at x_2, get a convergent subsequence of it and denote the corresponding subsequence of F_1 by F_2 and so on.

Inductively, choose a subsequence F_k of F_{k-1} such that the sequence got by evaluating it at the point x_k is convergent.

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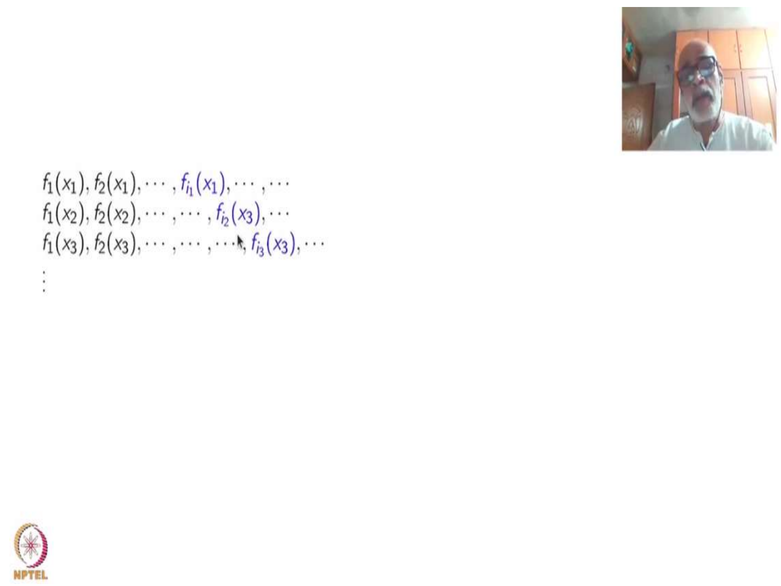

So, here is a picture. We start with $f_1(x_1), f_2(x_1), \ldots$ and so on. I have picked up $f_{i_1}(x_1)$, in the first row, f_{i_1} being the first element of the subsequence F_1. There are many more terms here, I am not bothered about that. I am looking at a first term here. Similarly, from the second row, I am picking up the second term of its subsequence F_2 and so on.

It is important that I am going forward viz, i_1, i_2 etc is strictly increasing so that $f_{i_1}, f_{i_2}, \ldots$ is a subsequence of F_0. In simpler way let me simply put $s_j = f_{j,j}$, the j^{th} term of the subsequence F_j.

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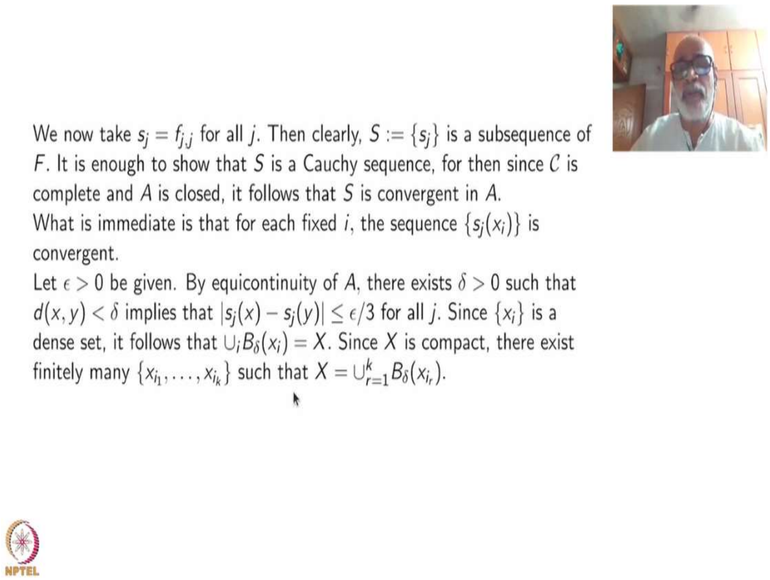

Then clearly $S = \{s_j\}$ is subsequence of F_0. What is the property of this subsequence that is what we want to know. So, in order to show that this sequence is convergent, it is enough to show that it is a Cauchy sequence in $\mathcal{C}$. For each point if it is Cauchy what happens inside $\mathbb{K}$, ($\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$) it will be convergent.

But here what we are showing is that it is a Cauchy sequence with respect to the metric of $\mathcal{C}$ coming from the supreme norm. So, what you get is uniform Cauchy this Cauchy sequence just means that in each point if you take those sequences in uniform Cauchy therefore you will get uniform convergence automatically.

So, let us not bother about this at all. Directly, I want to show that in the Banach space $\mathcal{C}$, S is a Cauchy sequence. Since $\mathcal{C}$ is already complete, (that we have already proved in part I) S will converge in $\mathcal{C}$. But A is closed in $\mathcal{C}$, what happens? the limit, the limit will be inside A. So, we have found a subsequence of F_0 which is convergent inside A.

So, how to prove S is a Cauchy Sequence?

So, what is immediate is the fact that for each fixed i, the sequence $\{s_j(x_i)\}$ is convergent, because after certain stage it becomes a subsequence of $f_i(x_i)$. All that I have to do is $j > i$. So, once it is subsequence of that is a sequence is convergent subsequence will be convergent. So, what we have achieved is a sequence which is convergent at this countable subset $\{x_i\}$, which is not arbitrary but a dense subset. From the density we want to conclude that this s_j is convergent at all the points. So, what we will do is, we will just show that it is Cauchy sequence that is all. (All this explanation does not mount to any proof!)

Given ϵ is positive, by equicontinuity of A, we get a δ positive such that $d(x, y) < \delta$ implies $|s_j(x) - s_j(y)| < \epsilon/3$, for all j. You can choose anything here given ϵ choose $\epsilon/3$ so we will do that one. Accordingly, there is some δ.

Next, since $\{x_i\}$ is a dense set, it follows that if you take the union of all open balls of any positive radius say, $B_\delta(x_i)$ where i range from 1 to infinity, will cover the whole of X. Every point must belong to one of the open balls. Since X is compact you will get a finite subset $x_{i_1}, x_{i_2}, \ldots, x_{i_k}$ such that X is the union of j ranges from 1 to k of the open balls $B_\delta(x_{i_j})$. So, we have got a δ-net for X itself now. So ϵ-net was coming implicitly for the family A here now it is coming for X itself. That is the role of equicontinuity here. So, what is the net result?

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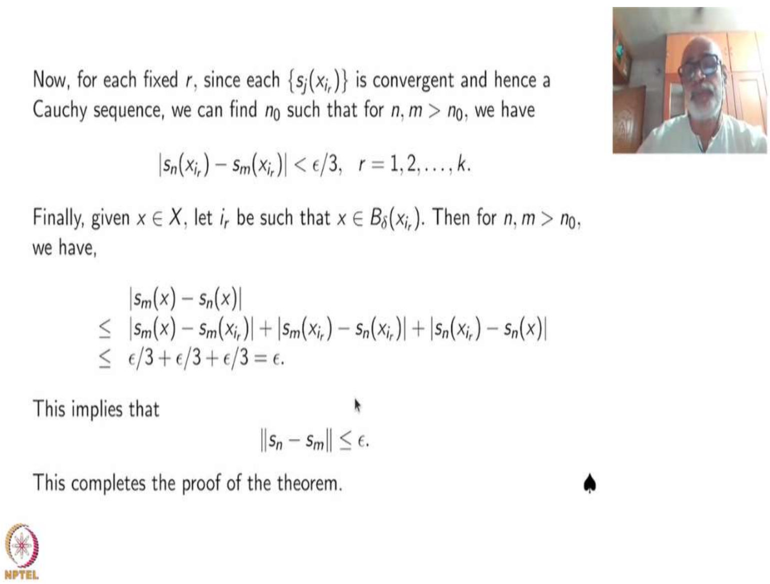

Now for each fixed r look at the sequences $\{s_j(x_{i_r})\}, r = 1, 2, \ldots, k$, only k of them. All these are convergent and hence Cauchy sequences. Therefore you can find a uniform n_0 such that if n and m are bigger than n_0, then$|s_n(x_{i_r}) - s_m(x_{i_r})| < \epsilon/3$. Each $r = 1, 2$ up to k, will get $n_1, n_2, \ldots, n_k$, you take their maximum as n_0.

Then for n_0, the inequality will be true for all $r = 1, 2$ up to k. Finally, given x belonging to X, now let i_r be such that x is one of the balls as above. Then x will be at a distance less than δ from x_{i_r}. Therefore, for $n, m > n_0$, I have this inequalities. So, what I do? I break it into three parts by adding and subtracting $s_m(x_{i_r})$ and then $s_n(x_{i_r})$. So, $|s_m(x) - s_m(x_{i_r})|$ is less than or equal to $\epsilon/3$ by equicontinuity. So, this one comes from inequality, since x is inside in this ball. Similar conclusion for the third term. The middle term is less than $\epsilon/3$ due to Cauchyness. Therefore $|s_m(x) - s_n(x)| < \epsilon$.

This is true for all x. Therefore, $\|s_n - s_m\| < \epsilon$. So, this means that the sequence $\{s_n\}$ is uniformly Cauchy, (same as saying that it is Cauchy in $\mathcal{C}$, Cauchy with respect to the norm that we have been using. This completes the proof.

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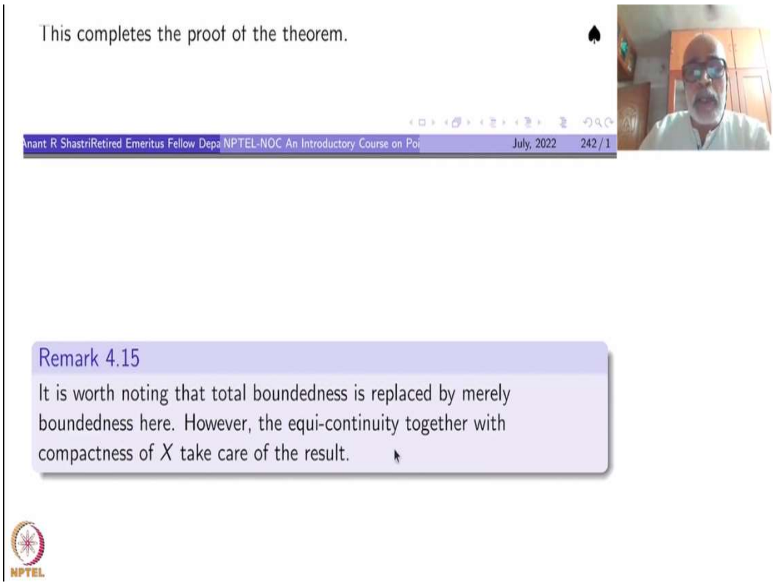

So, here are a few remarks. The total boundedness is replaced by merely boundedness here, but what we have here is already under the assumption that X is a compact metric space. That is how this is important. The equicontinuity together with compactness is taking care of that. Remember that in the general theorem, in arbitrary metric spaces we want total boundedness and completeness to get compactness. But we are not putting total boundedness on X. It is in $\mathcal{C}$ we are working, a function space. For that the equicontinuity helps us.

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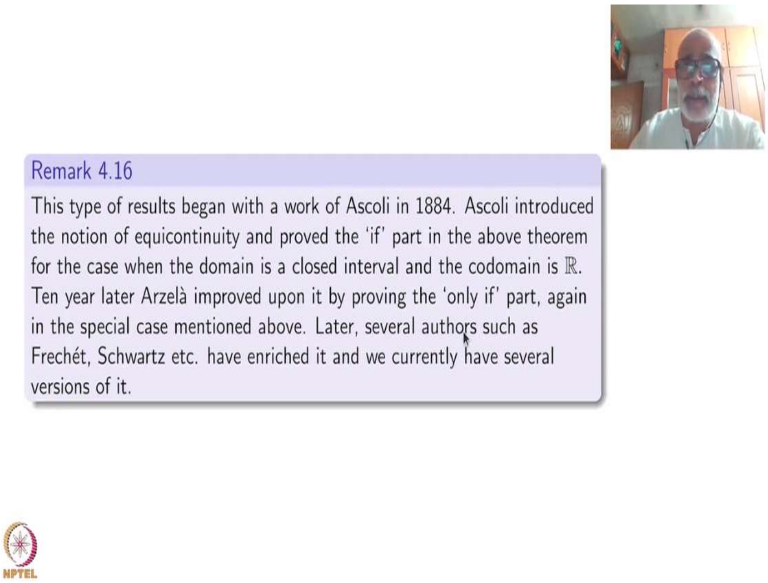

This type of results began with some work of Ascoli in 1884 or 1883. Ascoli and Arzela are both Italian mathematicians, almost contemporary. Ascoli introduced a version of the notion of equicontinuity and proved the `if' part which was more complicated tah what we have here. However, 10 years later, Arzela improved upon it by proving the `only if' part. I say `improving upon it' because he also brought much clarity to it. His paper is much more readable also in that sense. So, notion of equicontinuity belongs to Ascoli, but the present day has seen contributions from many other authors like Frechet, Schwartz etc. They have enriched it with many other versions which are quite often more general and so on. The original version of Ascoli is for only functions on a closed intervals inside $\mathbb{R}$ to $\mathbb{R}$.

We immediately can generalize by replacing the codomain $\mathbb{R}$ by $\mathbb{C}$. There is no problem. but there are many more other versions wherein the codomain can be replaced by what are called uniform spaces and so on.

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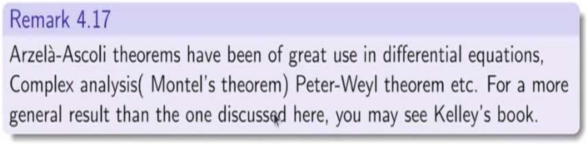

So, Ascoli-Arzela theorems have been of great use in differential equations, complex analysis especially in Montel's theorem, Peter-Weyl theorem etc. For a more general result than the one discussed here you may see Kelly's book which we have been referring to all the time.

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So, here are some easy exercises for you. Show that in a metric space second countability, Lindelofness and separability they are all equivalent. The second exercise: show that totally bounded metric space is separable and hence second countable.

So, you see how different concepts are related. Of course only when we are working with metric spaces. So, that is for today. Let us meet next time. Thank you.