An Introduction to Point-Set-Topology Part (II)
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Lecture No 15
Various Notions of Compactness

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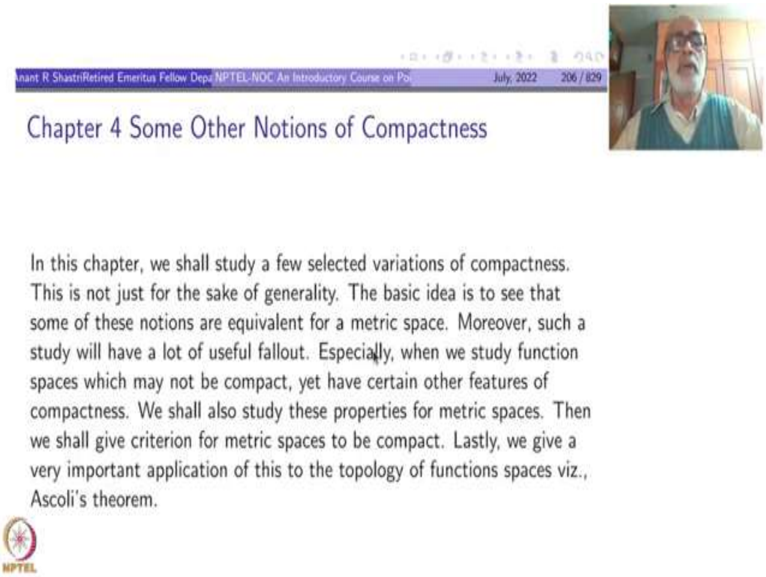

Hello welcome to NPTEL NOC course on Points Set Topology Part (II). We will begin Chapter 4 with certain other notions of compactness, `other' in the following sense. We have studied compactness itself. Closely related was Lindelof property. Then we have local compactness and paracompactness.

So, now we should study some other notions here, these notions have been selected maybe be can say, just my personal taste or rather one's belief that these are the more useful ones in analysis and topology, central analysis and topology. When we study function spaces which may not be compact, yet have certain other features of compactness.

So, that is what we would like to study. Like paracompactness was one of our obsession. You should also study these properties for metric spaces wherein many of them come together. Then we shall do a criterion for metric spaces to be compact. Lastly, we give a very important application of this to topology of function spaces, namely, what are known as Ascoli’s theorems. Actually we will do only one of them. So, that is the general idea for chapter 4.

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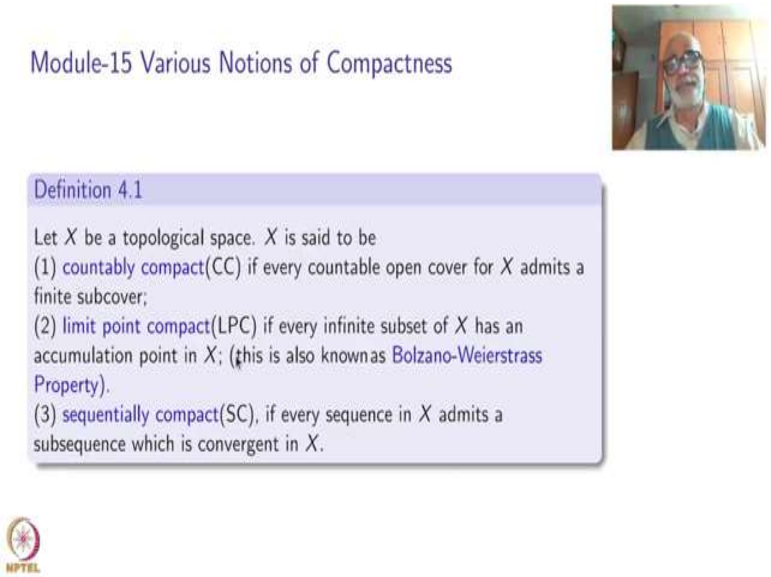

So, let us now begin with this module 15. Start with any topological space. Then I am going to introduce three different notions of compactness here. The first one is countable compactness. So, I may just denoted it (CC) in the following section all the time. So, countable compactness: If every countable open cover for X admits a finite sub cover.

This is similar to Lindelof property, but in contrast Lindelof property gives you a countable cover out of any cover, whereas countable compactness gives you a finite cover cover out of a countable cover. So, you can see that Lindelof plus countable compactness implies compactness. That is one easy consequence of this definition.

The second one is limit point conpactness, which has something to do already with our experience with metric spaces, namely, dealing with sequences, dealing with limit points. So, if every infinite subset of X has an accumulation point or what is called a limit point, that is called limit point compactness (LPC). This is also called Bolzano-Weierstrass property. Bolzano-Weierstrass property is also reflected in the next one, but we will call it a sequential compactness and denoted by (SC): if every sequence in X admits this subsequence which is convergent.

In fact, in standard real analysis, whenever you come across this property `every bounded sequence has a subsequence which is convergent' and so on, either you prove it or you state it, you refer to it as a Bolzano-Weierstrass theorem. But this property is not Bolzano-Weierstrass property. So, you have to make a slight distinction here. Bolzano-Weierstrass property if you want to refer to is is the limit point compactness defined above.

That is why I call it limit point compact not to confuse with the standard Bolzano-Weierstrass theorem as such.

So, countable compactness, somewhat similar to Lindelof, but it is somewhat closure to limit point compactness and sequential compactness.

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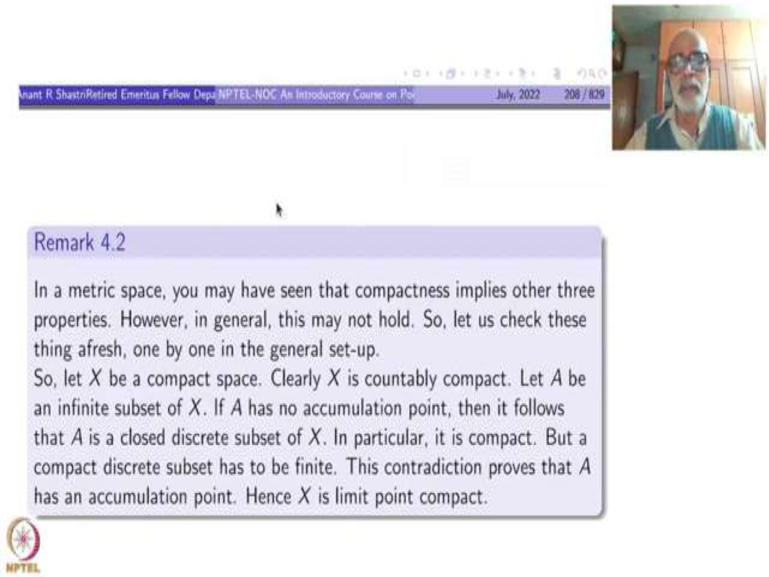

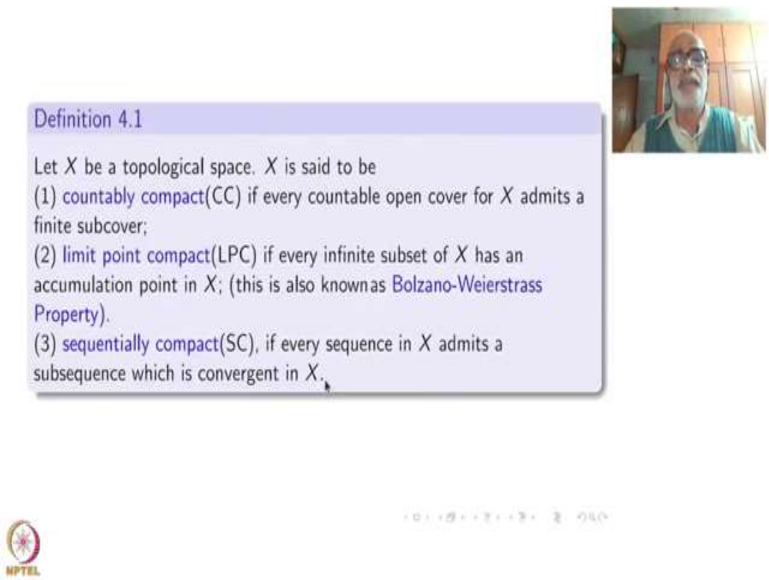

In a metric space, you may have seen that compactness implies these three properties. These are the standard properties of compact metric spaces. So, we extracted these properties and made then into definitions. In general, this may not be true that compactness implies any one of these things. We will have to be careful here. Something maybe true some of them may not be so.

So, let us go through this carefully. So, let us check these things a fresh one by one, in the general setup. Start with compact space X. It is clearly countably compact. After all every open cover will admit a finite sub cover, so countable covers also admit a finite sub cover. So, compactness implies countable compactness.

Now, take the second one. Let A be an infinite subset of X. If A has no accumulation point then it follows that A is a closed discrete subset of X. In any closed set there are two types of points, those which are in the set itself and those which are not in the set. Points of the second type are accumulation points of the set. Therefore, if A has no accumulation points it is closed. Further it follows that each singleton $\{x\}$ is an open set in A. Therefore, if A has no accumulation points then it follows that A must be closed discrete subset. In particular being closed subset of a compact space X, it is a compact space. But in a compact discreet subset, there are only finitely many points. So, this is a contradiction, we started with infinite set. Therefore, A must have a limit point. So, properties (1) and (2) are fine. May be compactness implies (3) also? And then you are lucky, but you have to be careful.

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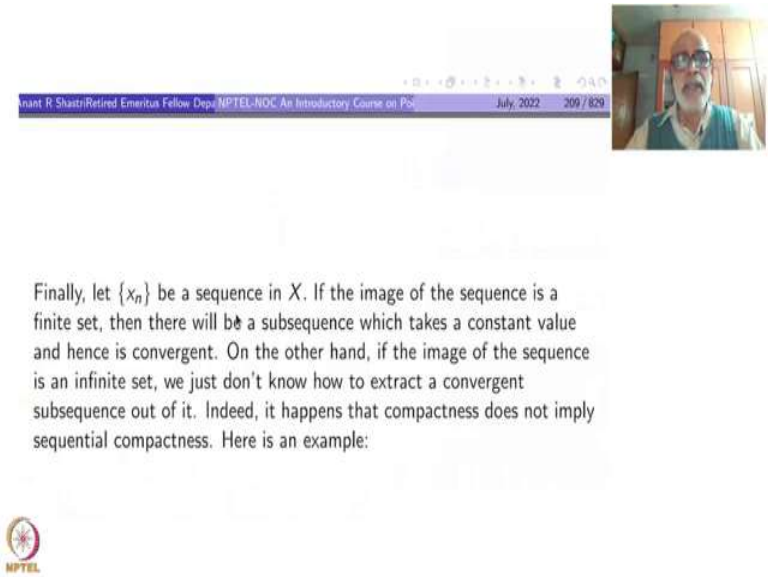

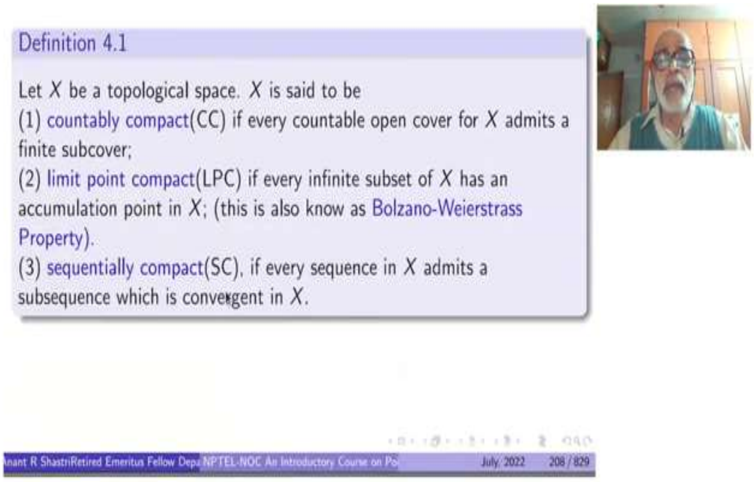

Finally, let $\{x_n\}$ be any sequence in X, where X is a compact space. If the image of the sequence is a finite set, then there will be a subsequence which takes a constant value and such a subsequence is convergent. On the other hand, if the image of the sequence is an infinite set, we just do not know how to extract a convergent subsequence out of it. Indeed, it happens that compactness does not imply sequential compactness. This may be a small surprise for you.

So you have to be careful with this property. So, let us first understand an example which makes sure that compactness in general need not imply sequential compactness. So, do not confuse it with Bolzano-Weierstrass properties all the time, so that is the whole idea.

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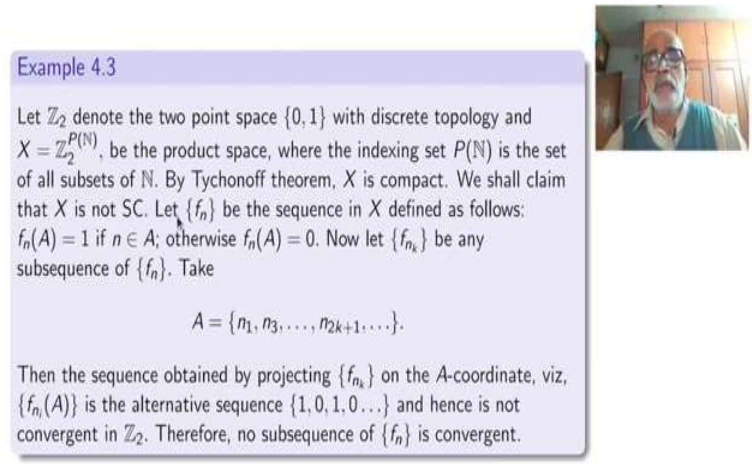

So, here is an example. Take $\mathbb{Z}_2$. That is a standard notation for the subspace of $\mathbb{R}$ consisting of -1 and 1. It is just a discrete space with two points, that is all. We do not care about the group structure here, though I have used the notation which is common for the group with two elements.

Take X equal to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \dots$ product of how many copies of $\mathbb{Z}_2$? As many as the power set $\mathcal{P}(\mathbb{N})$, where $\mathbb{N}$ is the set of natural numbers. The cardinality of the power set of $\mathbb{N}$ is nothing but the cardinality c of $\mathbb{R}$. One could have just written $(\mathbb{Z}_2)^c$ here. So, but I want to use this specific description of this power set. So, this is the power set of $\mathbb{N}$ all subsets of the natural numbers.

Take the Cartesian product of so many copies of the discrete space $\mathbb{Z}_2$. So, indexing set is the power set of $\mathbb{N}$. So by Tychonoff theorem, we know that X is compact. We shall claim that this space is not sequentially compact.

To show that X is not sequentially compact, I have to produce one sequence which has no subsequence which is convergent. So, I will give you an example, namely, take the sequence $\{f_n\}$ defined as follows: what are f_n's? They are functions from $\mathcal{P}(\mathbb{N})$ into $\mathbb{Z}_2$. So, f_n of a subset A of $\mathbb{N}$ is equal to 1 if this n belongs to A; otherwise defined $f_n(A) = 0$. For example $f_1(\{2, 3, 4, 5\}) = 0$; $f_1(\{1, 2, 3, 4\}) = 1$ and so on.

Now, let $\{f_{n_k}\}$ be any subsequence. Take A be the set $\{n_1, n_3, n_5, \dots\}$ i.e., just take the set of all $\{n_k\}$, where k is odd. To see that the sequence $\{f_{n_k}\}$ is not convergent, it is enough to check that the sequence $\{f_{n_k}(A)\}$ in $\mathbb{Z}_2$ is not convergent. But this latter sequence is nothing but the alternating sequence $1, 0, 1, 0, \dots$

Here we have used the fact that the coordinate projections from a product space into any factor are continuous and applies it to the A-th projection from $(\mathbb{Z}_2)^{\mathcal{P}(\mathbb{N})}$ to $\mathbb{Z}_2$. So, we have proved that this space X is not sequentially compact.

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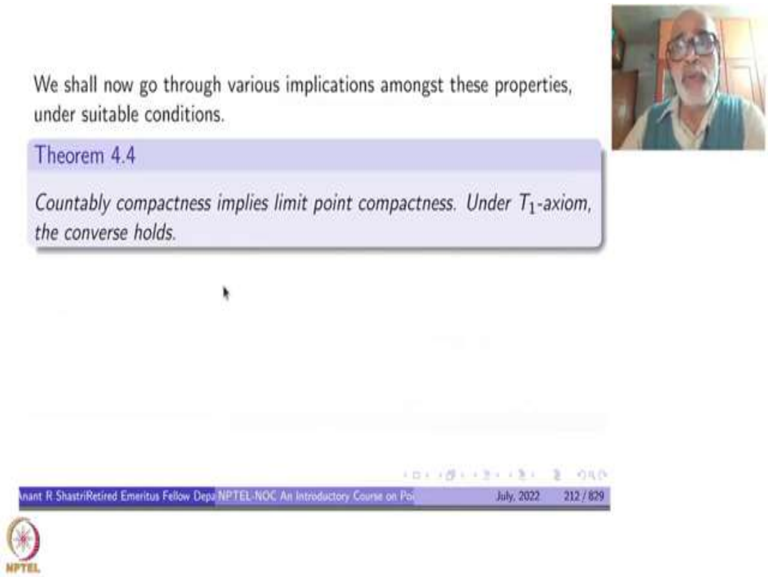

Now, let us try to do some other implications on the positive side.

The first thing is countable compactness implies limit point compactness. Under T_1 axiom the converse always true. So, you can just say (1) implies (3) and under T_1 axiom they are equivalent. You can see that slowly we are bringing them nearer to the metric spaces. T_1 spaces are slightly closer to metric spaces. Metric spaces are very strong being T_5 spaces, completely normal and T_1. So, let us see how one proves it, without using any distance function. You have to do everything purely topologically.

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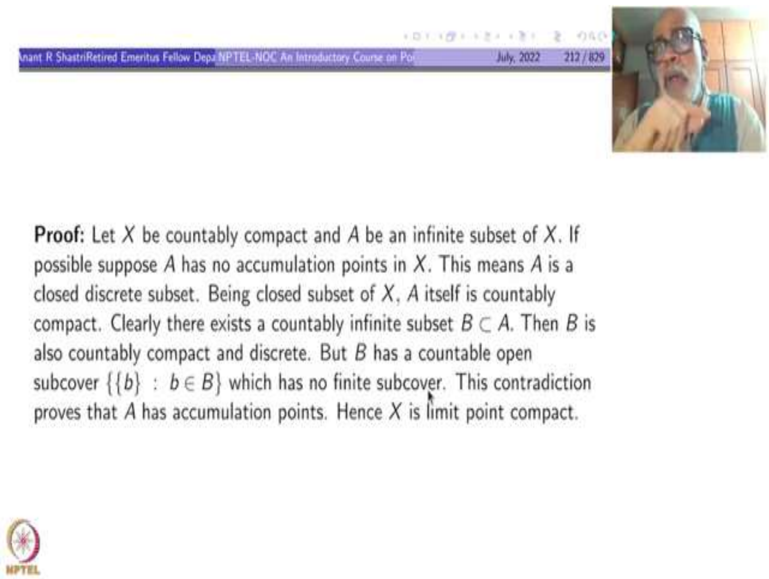

So, let X be countably compact and A be an infinite subset of X. If possible suppose A has no accumulation point. This means that A is a closed discrete subset (as seen before). Also being closed subset of X, A itself is countably compact.

It is an easy thing to see that closed subsets of countably compact space are countably compact, just like closed subset of a compact space are compact, exactly same proof will work. Clearly there exist a countably infinite subset B of A because we started with an infinite subset A of X. Then B is also countably compact and discrete, countably compact because the set itself is countable and discrete because it is a subspace of a discrete space. But B has a countable open cover viz., $\{\{b\} : b \in B\}$ which has no finite subcover. So, that is a contradiction.

So, any infinite set must have a limit point if it is countably compact.

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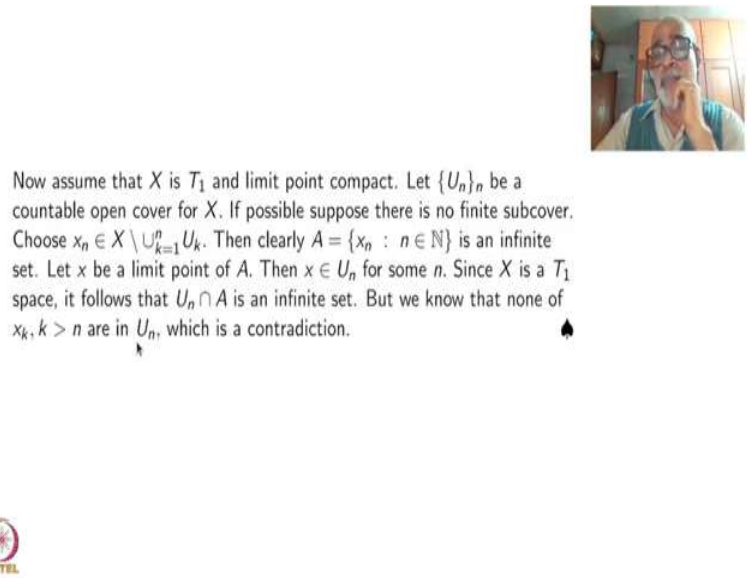

Let us prove the converse now, under the axiom T_1 for X. So, X be a T_1 space and limit point compact. Now, you take any countable open cover $\{U_n\}$ for X. If possible, suppose there is no finite subcover. What does that mean? U_1 does not cover X. So, there will be x_1 which is in the complement of U_1, next you can find an x_2 and $n_2 > n_1$ such that x_2 is different for x_1 and is not in the union of U_i for $i \leq n_2$. And so on, you find an increasing sequence $\{n_i\}$ of integers such that x_i is not in the union of all U_k, $1 \leq k \leq n_i$ and different from all the x_j chosen earlier.

Here we have used the assumption that $\{U_n\}$ has no finite subcover. So, you get an infinite subset $A = \{x_n\}$.

Now, use the limit point compactness property of X. Let x be a limit point of this infinite set A. Let x be in U_n for some n. Since X is a T_1 space and x is a limit point of A, it follows that $U_n \cap A$ itself is an infinite set.

So, this is where we are using the T_1 property. If a limit point of A belongs to an open set U, then $U \cap A$ must be an infinite set. But we know that none of $x_k, k > n$ are inside U_n. That is a contradiction.

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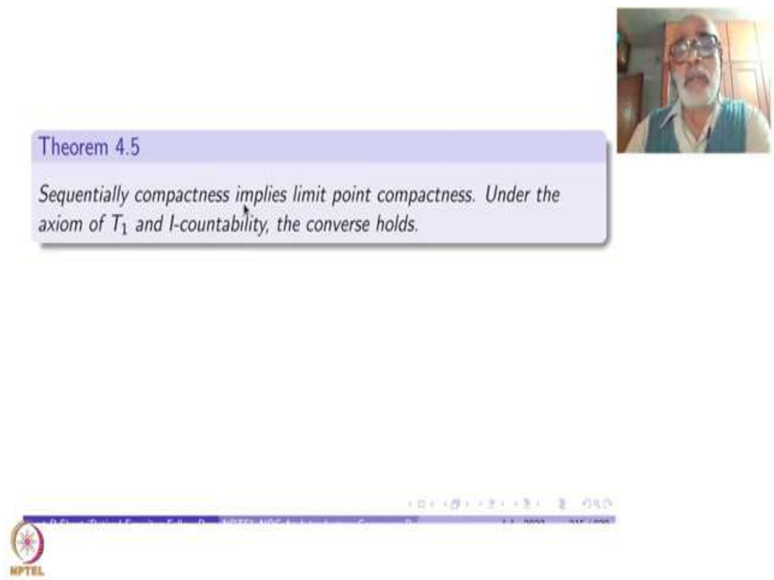

So what we have proved? We have proved that under T_1 axiom, (1) and (2), viz., countable compactness and limit point compactness are the same.

Next, a similar result: sequential compactness implies limit point compactness;

(3) implies (2) now. And under the axiom T_1 and first countability, the converse holds.

(1) implies (2) always, and (3) implies (2) always. So 2 is the weakest one, but under T_1

(1) and (2) are same and under T_1 and first countability (2) and (3) are equivalent.

That is the theorem.

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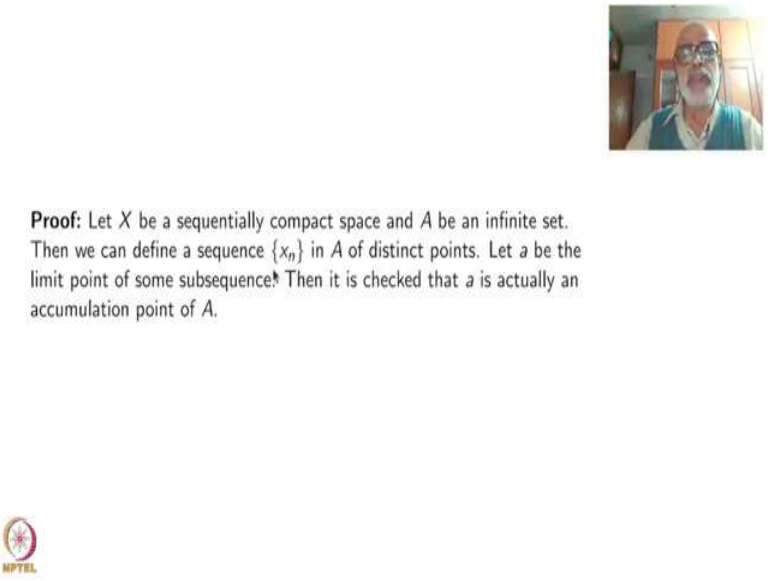

So, let us prove this theorem also. This is also equally easy. Set X be a sequentially compact space and A be an infinite subset. Then we can define a sequence $\{x_n\}$ in A of distinct points. Let a be a limit point of some subsequence. Then it is easily check that a is actually an accumulation point of A. So, having an infinite set of distinct points is important here. So, very straightforward there is nothing hidden here.

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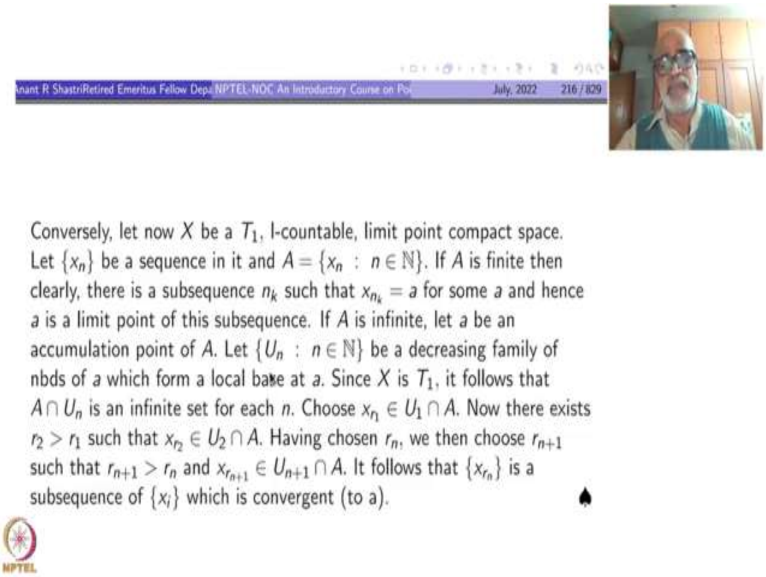

Now, conversely, let now X be a T_1 and I-countable space and a limit-point-compact space. Now start with a sequence $\{x_n\}$ in it and let A equal to the image set of this sequence. If A finite then clearly there is a subsequence which is a constant and hence convergent.

So, if A is finite is no problem. Suppose now A is infinite. Then let a be an accumulation point of A. Now, let $\{U_n\}$ be a decreasing family of neighbourhoods of a which forms a local base at a. This is where I-countability is used. I-countabilty gives a countable local base $\{V_n\}$. Then you can take $U_n = V_1 \cap V_2 \cdots \cap V_n$.

So, you can make a decreasing sequence of neighbourhoods that will form a local base at a. Since X is T_1, it follows that $A \cap U_n$ is an infinite set for all n (this idea, we have used earlier also) because a is an accumulation point of A. So, $A \cap U_n$ is an infinite set for each n. Therefore, we can choose a subsequence $\{x_{r_i}\}$ such that x_{r_i} belong to U_i for each i. It follows that this subsequence is convergent to a.

So, first countability as well as T_1 is used here to extract a subsequence out of a sequence. In the case when the sequences infinite. If sequences consists of finite number of elements that there is no problem that will always have a subsequence which is convergent.
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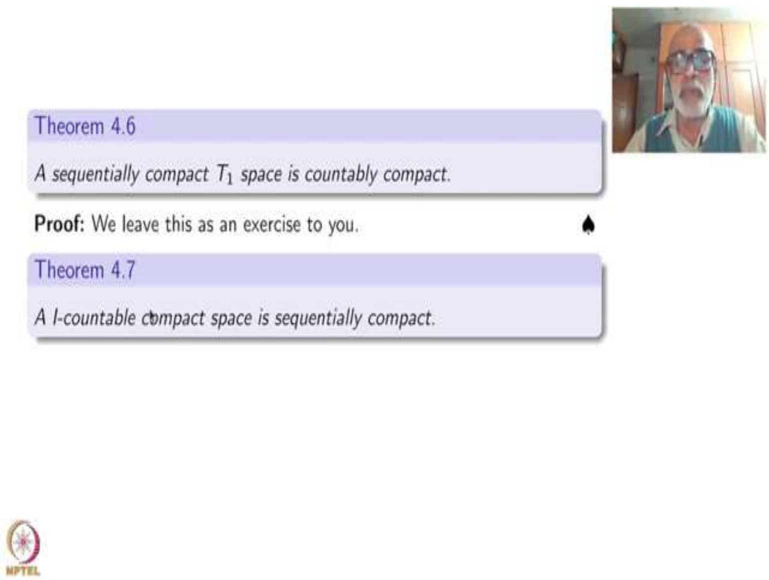

The next theorem is: A sequentially compact T_1 space is countably compact. Now, I am going from (3) to (1) sequentially compact T_1 space is countable compact. The proof of this does not need any new ideas, after seeing the proofs of the previous two theorems. So, I feel that you should be able to extract a proof on your own. So, try your hand okay? This will be left as an exercise, as an assignment to you.

Next theorem is that if X is first countable compact space then it is sequentially compact. Right in the beginning, we saw that compactness does not imply sequential compactness. However, under I-countability compactness implies sequential compactness.

So, that may be the reason why compact metric spaces are sequentially compact. You see, because metric spaces are always I-countable. So, this way the essence of all these properties comes out, rather than just if you study the non-metric spaces.

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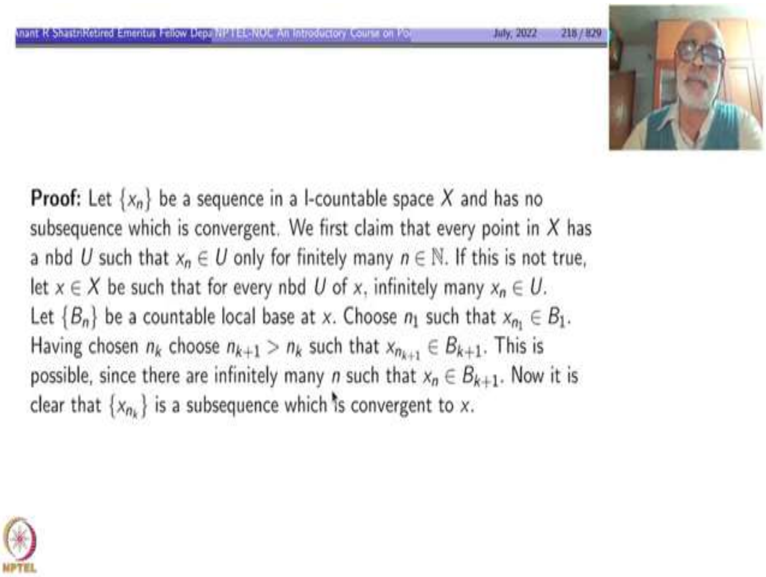

So, how do we proved it? Start with a sequence $\{x_n\}$ in a first countable space X and suppose this has no subsequence which is convergent. We first claim that every point in X has a neighbourhood U_x such that only for finitely many $n \in \mathbb{N}$, x_n will be inside U_x. There may not be any point that is fine, if at there will be a finite number of them.

Suppose this is not true, what is the meaning of this is not true? I am claiming that for every point something is happening, that is not true means that that is some point, at least one point $x \in X$ such that every neighbourhood U of x has infinitely many x_n in it. No matter how small you choose that U to be. So, this is the denial (negation) of our claim.

Now, let be $\{B_k\}$ be countable local base at x which is decreasing. So, the I-countability enters into the discussion. Choose n_1 such that x_{n_1} is inside B_1. Having chosen n_k such that x_{n_k} is in U_k, choose $n_{k+1} > n_k$ such that x_{k+1} is in U_{k+1}. This is possible because there are infinitely many k for which x_k belongs to U_{k+1}.

Now, it is clear that this $\{x_{n_k}\}$ is a subsequence and this sub sequence will converge to x because they are inside smaller and smaller members of this local base. This contradiction proves that for every point $x \in X$, there is an open set U_x for which there are only finitely many n such that x_n belongs to U_x.

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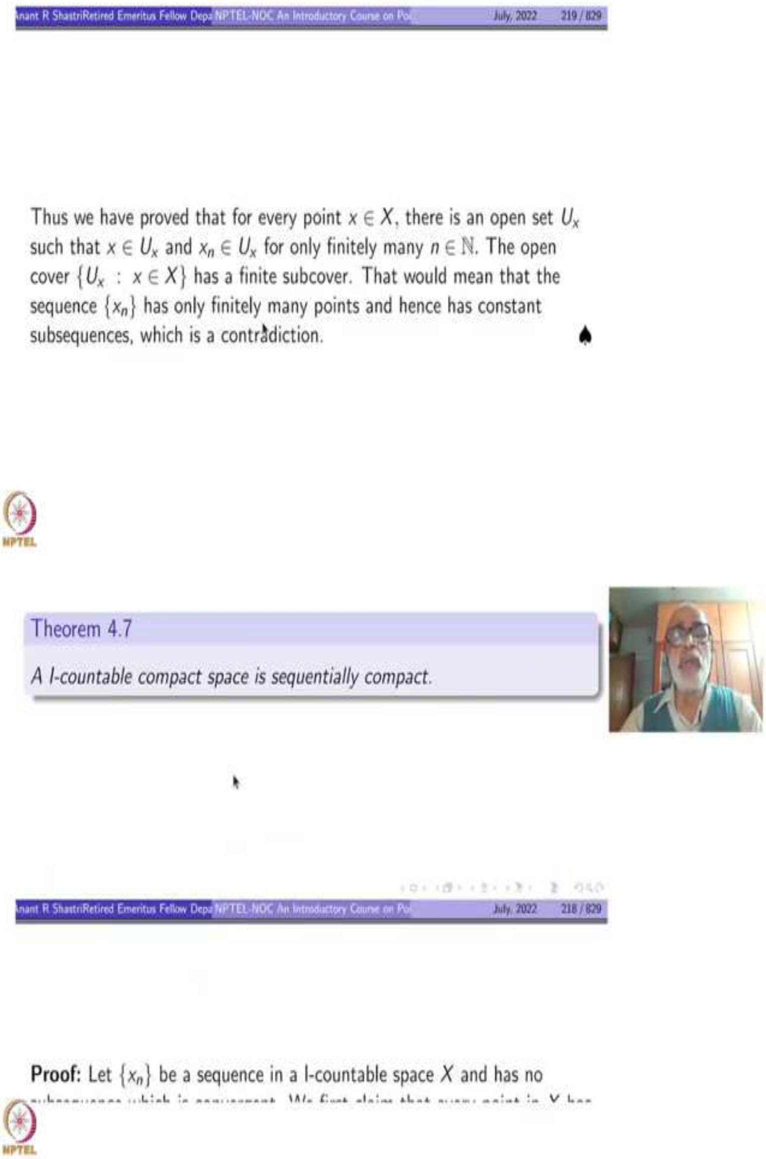

Now look at the open cover $\{U_x, x \in X\}$. This has a finite subcover, because X is compact. See compactness enters only now. So, that would mean that the sequence $\{x_n\}$ has only finitely many points. Where are all the points x_n's? They have to be in one of these finitely many $U_{x_1}, U_{x_2}, \ldots, U_{x_k}$. But of each of these U_i's contains only finitely many. Therefore there are only finitely many points. But then it has a subsequence which is a constant and hence convergent. A contradiction.

So, you must be satisfied now that whatever you have learned in metric spaces is correct, even if we have not proved them at that time. Direct proofs in the case of metric spaces is not much easier either; you need to use T_1-ness and I-countability without specifically mentioning them. That is all.

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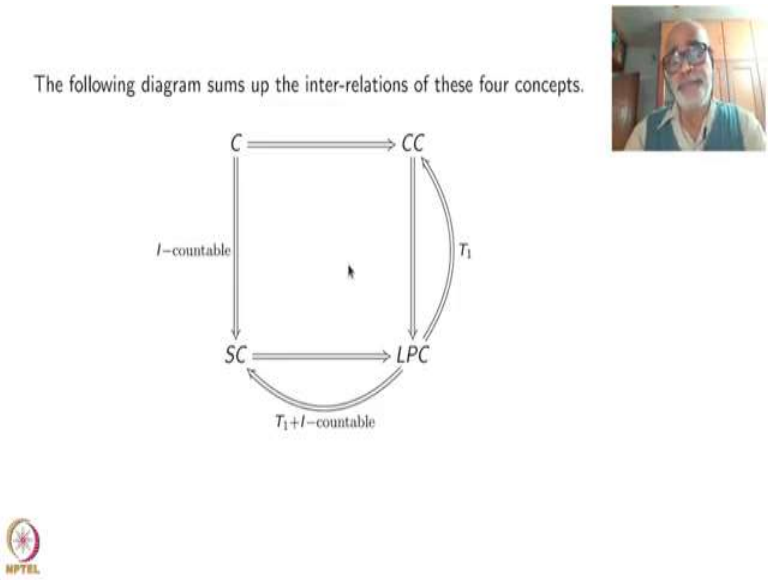

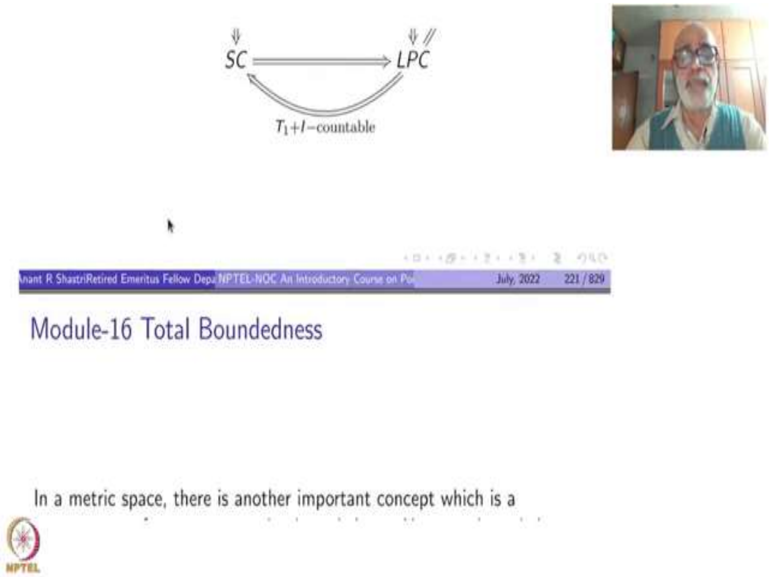

Here is a picture which will help you to remember what are all the implications compactness, countable compactness, limit point compactness, start with compactness add first countability you can go to sequential compactness. Start with sequential compactness, you can always go to limit point compactness; put T_1 and first countability you can come back; start from here you can always come to limit point compact, but under T_1-ness you can go back also.

So, this is all what we have proved so far. So, far we have never gone back here to compactness itself. So, that is our next step what will imply this one that is the kind of thing we would like to do. I think that is enough for today. So, next time we will go into deeper into the, this one namely, we will bring now metric spaces and some condition, criteria, which will finally give you will allow you, to go from somewhere here to here and so on. Thank you.