An Introduction to Point-Set-Topology (Part II)
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Lecture 14
Paracompactness Continued

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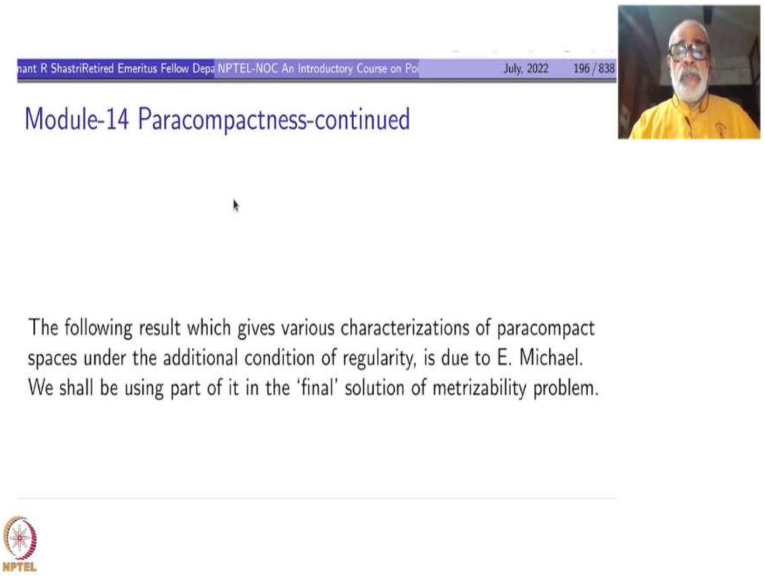

Hello, welcome to NPTEL NOC an introductory course on Point Set Topology Part II, Module 14. We continue our study of paracompactness. The following result which gives various characterization of paracompact spaces under additional condition of regularity is due to Michael. We warn you about this condition of regularity. For some of the implications, this may not be necessary, but there are other implication for which it will be necessary and there are counter examples otherwise. We are not going that much deeper into it.

Basically, the presentation that I have taken is close to what you can see in Willard's book. If you want more elaborate description of this paracompactness with more characterization then you can have a look at Kelley's book. Whatever we are doing today is part of it. We will be using a part of this in the final solution of metrizability problem.

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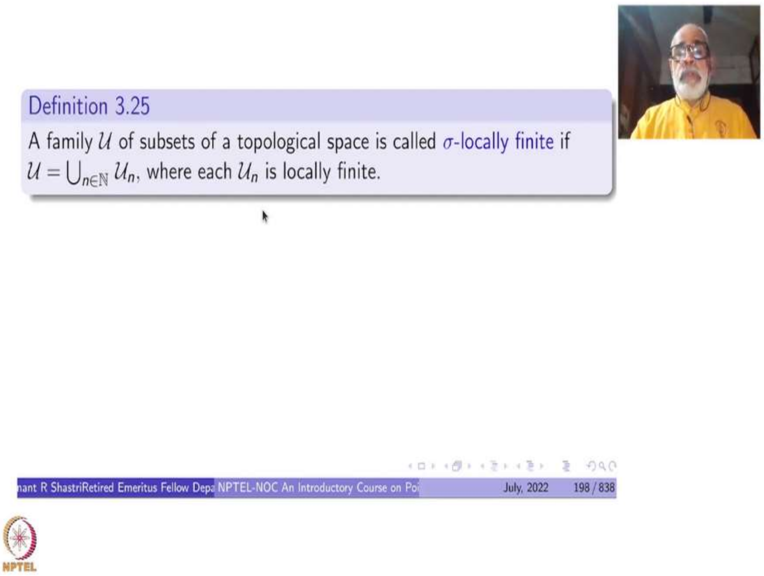

So, we begin with a definition. A family of subsets of a topological space is called sigma-locally finite if the family can be expressed as a countable union of sub families $\mathcal{U}_n$ each of them is locally finite. Obviously, if something is locally finite already, then it is sigma-locally finite.

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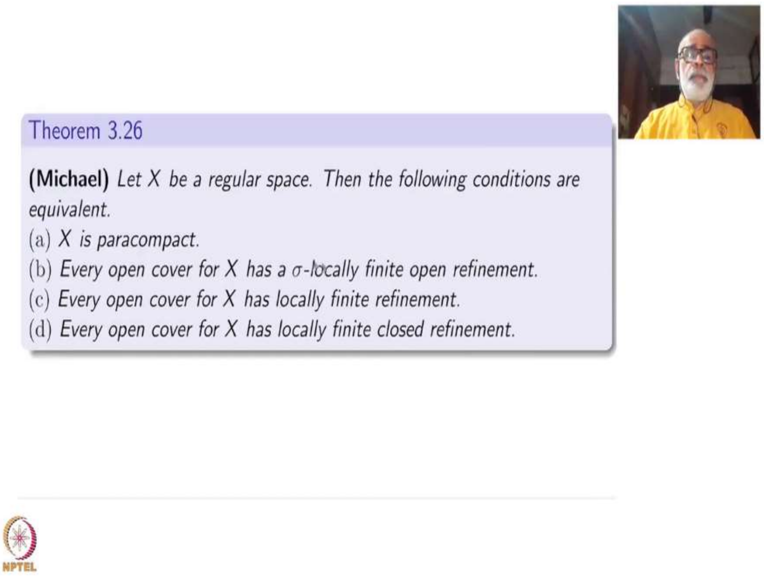

So, here is the theorem that we want to go through today. Let X be a regular space. Then the following conditions are equivalent.

(a) X is paracompact.

(b) Every open cover for X has a sigma-locally finite open refinement.

(c) Every open cover of X has locally finite refinement.

(d) The last statement here is: every open cover for X has locally finite closed refinement.

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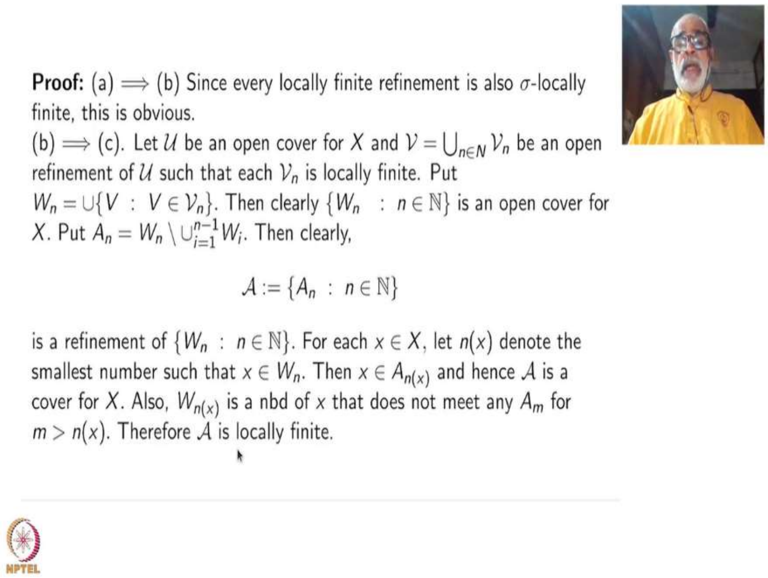

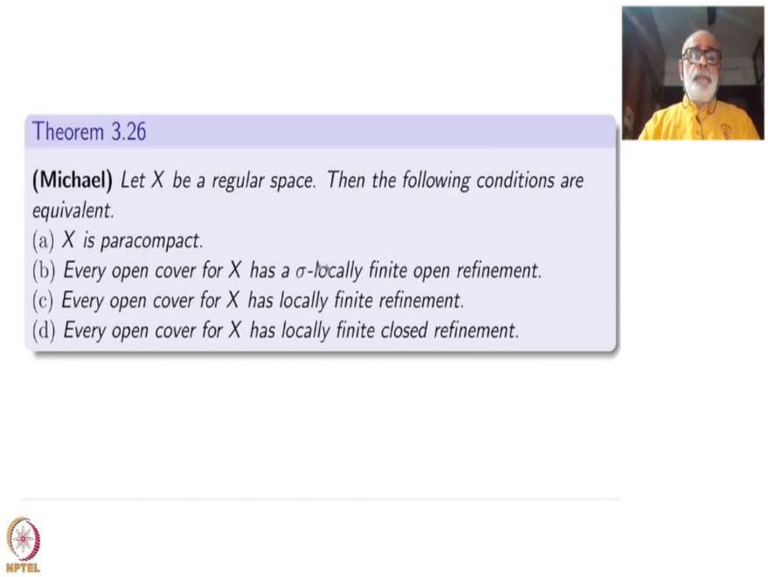

So, let us go through these proofs quickly. (a) implies (b) is very obvious because X is paracompact implies every open cover has a locally finite open refinement, locally finite already implies sigma-locally finite as we have pointed out earlier. So, (a) implies (b) is obvious.

(b) implies (c): We start with the condition that every open cover of X has a sigma locally finite open refinement. And then we want to produce a locally finite refinement.

So, this is an improvement. Remember that we have not put openness here. So, we will not be achieving openness in this step. So, we are not actually proving paracompactness, but something seemingly weaker here. So, let us see how to prove that. Start with an open cover $\mathcal{U}$ for X, and let $\mathcal{V}$ be an open refinement of it such that it is countable union of $\mathcal{V}_n$'s where each $\mathcal{V}_n$ is locally finite.

So, sigma locally finite open refinement is there for you. So, start with an open cover which admits a sigma locally finite open refinement. Put W_n equal to the union of all the members of $\mathcal{V}_n$. Then, as n ranges over the natural numbers the collection of all W_n's will form an open cover for X because this $\mathcal{V}$ is an open cover.

Now, we are defining A_n equal to W_n minus the union of all the previous W_i's, i ranges from 1 to $n-1$. This is an open subset, I am subtracting an open subset from an open subset so that will not give an open subset; A_n will not be an open subset in general, unless something additional condition is given, such as W_i are also closed and so on.

Clearly what is true is that this $\mathcal{A}$ which is a collection of all A_n's is a refinement of $\{W_n\}$ because each A_n is contained inside W_n.

And refinement is fine, but when you have a cover, W_n's for a cover for X, so I have to prove that $\mathcal{A}$ is also a cover for X. For each x inside X, let n_x denote the smallest number or you can say the first number n such that x is inside W_n. So, x is inside W_n. Then x will be inside An also because it is not in any of these W_i's. for $i < n_x$. So, x will not get deleted. So, it is inside W_{n_x} and it is not getting deleted. Hence $\mathcal{A}$ is a cover.

So, now what we have got is $\mathcal{A}$ is a cover and it is refinement of $\{W_n\}$.

Also, now you look at W_{n_x} which is a neighborhood of x. It will not meet any A_m for $m > n_x$ because A_{n_x} gets deleted from W_m. So, A_m will not intersect W_{n_x} for $m > n_x$. This means that this family is locally finite. So, we have got a locally finite cover but this is not a refinement of U. I want to get a refinement of U, these are somewhat larger.

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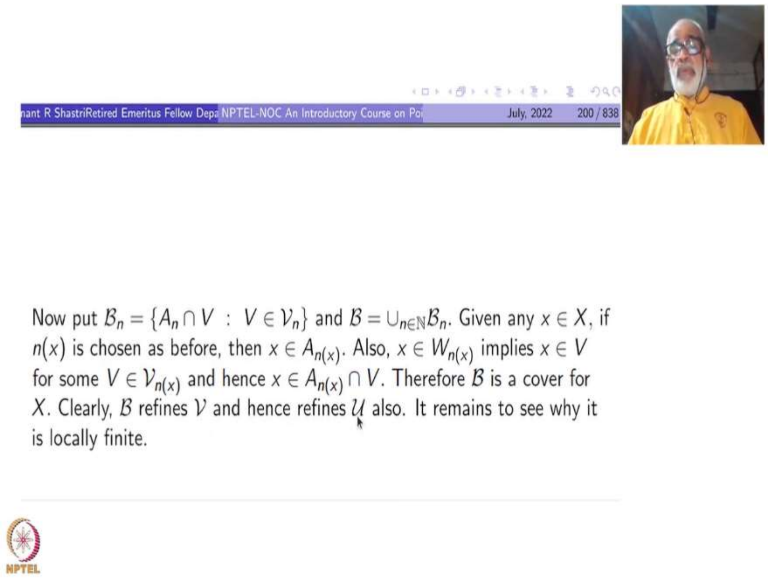

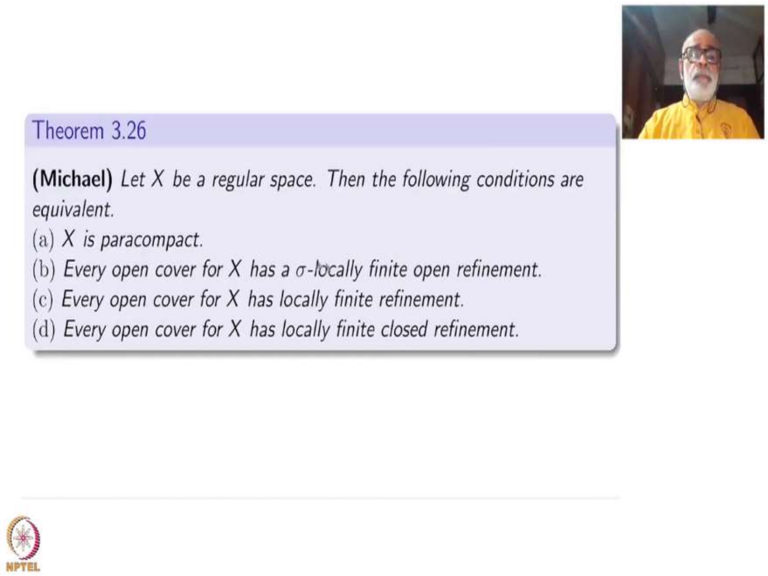

So, here is what you have to do. Put $\mathcal{B}_n$ equal the set of all $A_n \cap V$, where V belongs to $\mathcal{V}_n$. So, everything is now happening inside $\mathcal{V}_n$. So, you take An intersect with each member of $\mathcal{V}_n$, put them in $\mathcal{B}_n$. Then you take $\mathcal{B}$ to be the union of all these $\mathcal{B}_n$'s, n ranging over the natural numbers.

Given any x in X, if n_x is chosen as before, namely the first n for which x belongs to W_n remember that, then we have shown that x is inside A_{n_x}.

Also, this implies x is inside W_{n_x} which in turn implies x is inside some V, for V in $\mathcal{V}_{n_x}$. It is already in A_{n_x}, it must be in some V here. So, it will be the intersection, and that intersection is in $\mathcal{B}_{n_x}$. Therefore, $\mathcal{B}$ is a cover for X.

So, this $\mathcal{B}$ is now written as a countable union of this $\mathcal{B}_n$'s and this is a cover. Still there is no openness here. So, $\mathcal{B}$ is refinement of $\mathcal{V}$. because each element of $\mathcal{B}$ is looks like $A_n \cap V$ and hence contained in V. $\mathcal{B}$ is the refinement of $\mathcal{V}$ and $\mathcal{V}$ is the refinement of $\mathcal{U}$, so $\mathcal{B}$ is a refinement of $\mathcal{U}$. All that remains to see now is that this $\mathcal{B}$ is sigma-locally finite.

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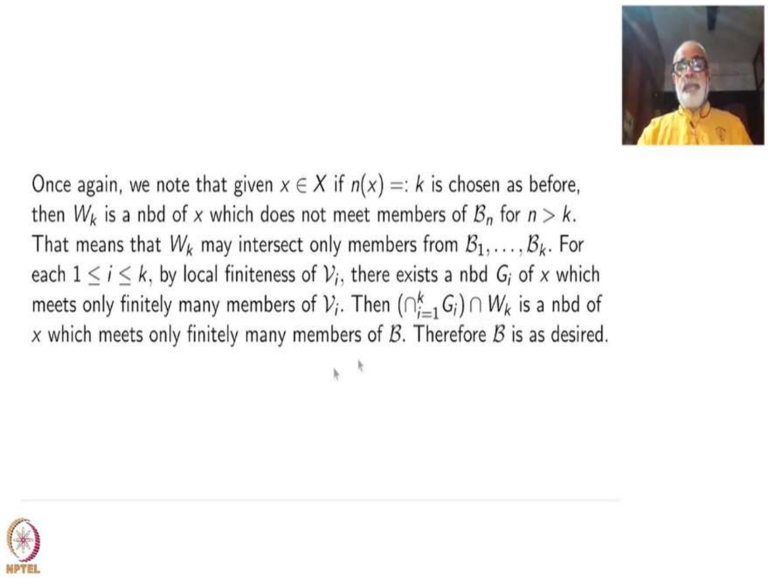

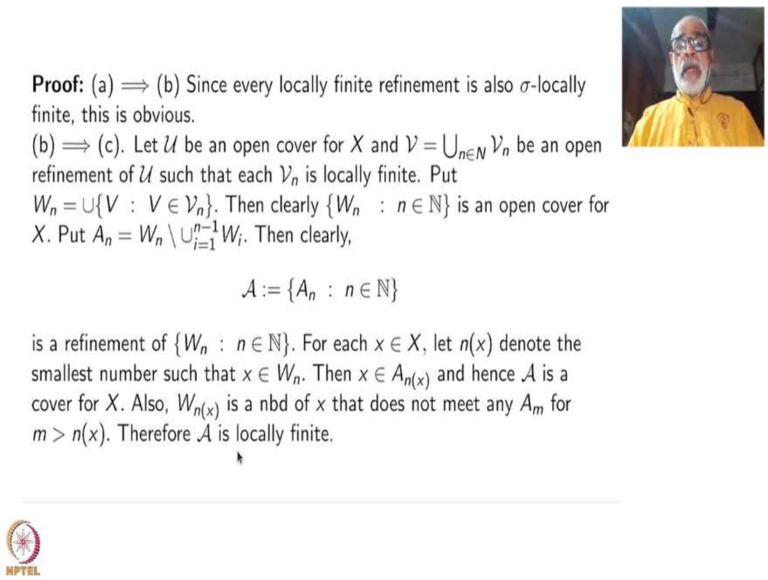

Once again, we note that given x inside X, if $n_x := k$ is chosen as before, once again I repeat what is n_x, n_x is the smallest number n such that x is inside W_n. So, I am now writing n_x as k. This W_k is a neighborhood of x because x belongs W_k and W_k is open. W_k does not meet members of $\mathcal{B}_n$, for $n > k$. W_k may intersect only members of $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_k$. If they are in $\mathcal{B}_{k+1}$ and so on, it will not intersect them.

Now, for each 1, 2, up to k what happens? Each $\mathcal{B}_i$ is locally finite because they are sub families of $\mathcal{V}_i$. Since each V_i is locally finite at least. So, for each fixed i, there exists neighborhoods G_i of x which meets only finitely many members of $\mathcal{V}_i$.

Now, you look at the intersection of all these G_i as i ranges 1 to k and W_k also. W_k will cut off things from $k+1$ onwards and this intersection will have the property that it will meet only finitely many members from each of $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_k$ and hence, totally, it will be only finitely many members of the $\mathcal{B}$. So, therefore, this $\mathcal{B}$ is locally finite.

So, some proof here is needed from sigma local finiteness to local finiteness, but the prize we have paid is that, we are not bothered to get members of the family to be open. Now, we have to improve on that. So, in the next step, we will do that.

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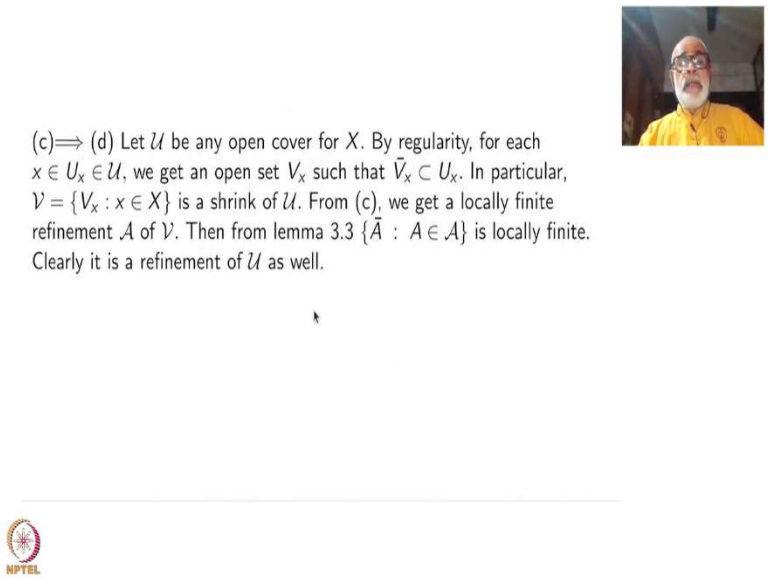

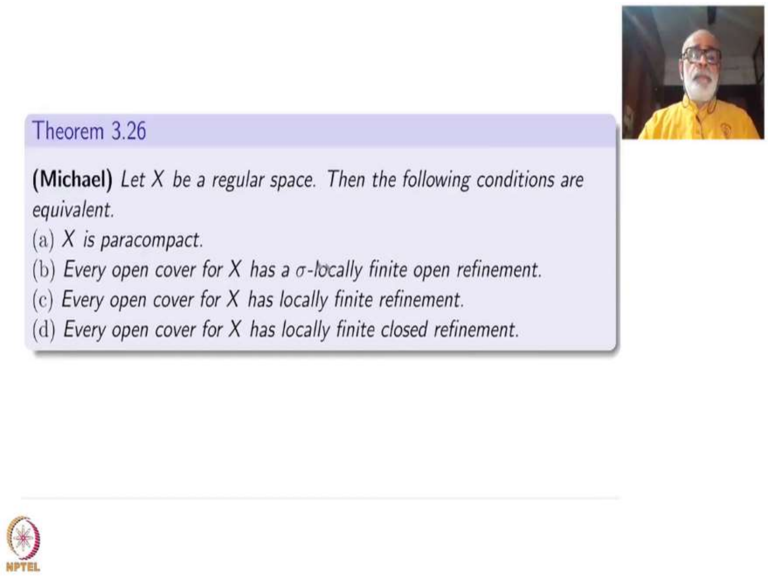

Now take up the proof of (c) implies (d). Starting with locally finite refinement, we want to get locally finite closed refinement. And from closed refinement finally, ((d) implies (a)) we will get open refinements, that is paracompactness. So, that is the idea. So, now, let us prove (c) implies (d). So, let $\mathcal{U}$ be any open cover. Regularity comes here now. By regularity, for each x belongs to X we choose U_x in $\mathcal{U}$ and then an open set V_x such that x is in V_x and $\overline{V_x}$ is contained in U_x.

In particular, if you take the family $\mathcal{V}$ of all V_x's as x varies over X, this is a shrink of $\mathcal{U}$, From hypothesis (c), we get a locally finite refinement $\mathcal{A}$ of $\mathcal{V}$. Then from lemma 3, I really want this lemma 3, the family of all closure of A as A belongs to $\mathcal{A}$ is locally finite. If you have a locally finite family of subsets, then their closure is also locally finite, this was the lemma. In fact that lemma says more things.

In fact, union of arbitrary families of closed subsets here, closures of here will be again closed that is the second part here. So, this we have proven, you have used it elsewhere also. So, these closures are locally finite. So, clearly it is a refinement of $\mathcal{U}$ because each $\overline{V_x}$ are contained inside U_x for all x. So, and $\bar{A}$ will be contained inside some $\overline{V_x}$ which in turn is contained in U_x. So, it is a refinement of $\mathcal{U}$ as well. So, that completes the proof of (c) implies (d).

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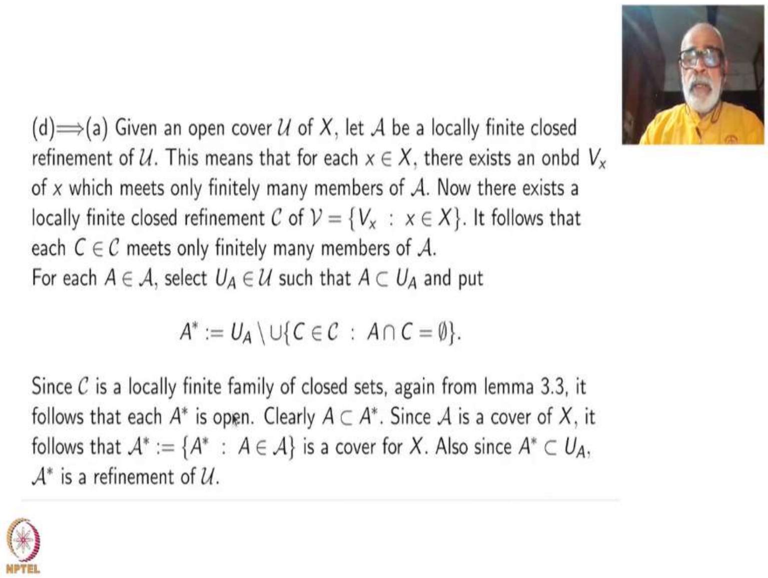

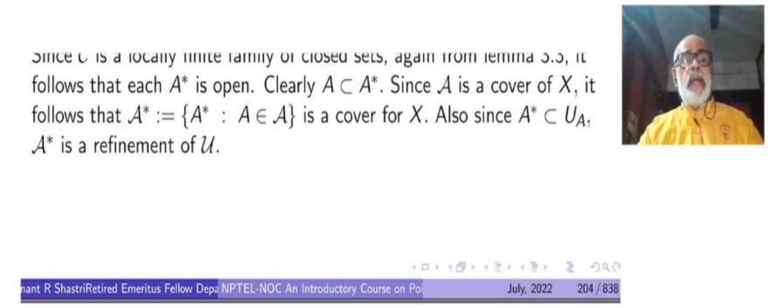

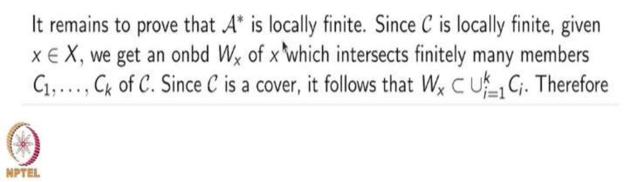

So, let us finally take the proof of (d) implies (a). Given any open cover, from existence of closed locally finite refinements, we are now proving that there is a locally finite open refinement. So, that will be paracompactness. So, start with any open cover $\mathcal{U}$ and a locally finite closed refinement $\mathcal{A}$ of $\mathcal{U}$. We are going to produce an open refinement of $\mathcal{A}$ now, so you can as well assume to begin with that $\mathcal{U}$ itself is a locally finite closed covering. But we are doing that now).

This means that for each $x \in X$, there exists a neighborhood V_x of x which meets only finitely many members of $\mathcal{A}$. Now, there exists locally finite closed refinement $\mathcal{C}$ of this cover namely $\{V_x : x \in X\}$.

I repeat. So, starting with an arbitrary open cover $\mathcal{U}$, we pass on to locally finite closed refinement $\mathcal{A}$. The local finiteness of this cover $\mathcal{A}$, gives an open cover $\mathcal{V}$ of X each of whose members meet only finitely many members of $\mathcal{A}$. And once again we pass onto a closed locally finite refinement $\mathcal{C}$ of $\mathcal{V}$.

It follows that each C belonging to $\mathcal{C}$ meets only finitely many members of $\mathcal{A}$. For each A inside $\mathcal{A}$, select U_A inside $\mathcal{U}$ such that A is contained inside U_A.

Remember this $\mathcal{U}$ was an open cover with which we started and $\mathcal{A}$ is a refinement. Put A^* (this is just a notation) equal to U_A (this is an open subset) setminus union of all the members of $\mathcal{C}$ which do not intersect A. So, what I am doing here is that I am taking this A and I am fattening it inside U_A.

See member of $\mathcal{A}$ are closed, it is a locally finite family it follows from the lemma 3.3. that A^* is an open subset.

Starting with $\mathcal{U}$, you got a shrink $\mathcal{A}$, which is a closed refinement. Now, you are going to expand them, using yet another, very special closed cover. How do I expand, I expand it inside U_A by throwing away a closed set from U_A.

What is the closed set? It is union of all those closed sets C inside $\mathcal{C}$, which do not meet A. Because you do not want to throw away points of A, that is all. Whatever you have thrown away, they does not intersect A, so the part of A inside U_A is kept as it is. Since, this is locally finite family and they are closed, arbitrary union of such closed set is closed this again use this lemma 3.3, used here.

So, U_A minus this set is an open subset. So, each A^* is a fattening of A to an open subset. Moreover, they are inside U_A. That means they are refinements of $\mathcal{U}$. This family $\mathcal{A}^* = \{A^* : A \in \mathcal{A}\}$ is a cover for X because $\mathcal{A}$ is a cover for X. It is a refinement of $\mathcal{U}$. So, there is lots of hope with this family. We will show that $\mathcal{A}^*$ is locally finite also and that will complete the proof that (d) implies (a).

Again use the fact that $\mathcal{C}$ is locally finite. Given any $x \in X$, we can find a neighborhood W_x of x which intersects finitely many members, say, $C_1, C_2, \ldots, C_k$ of $\mathcal{C}$. Since $\mathcal{C}$ is a cover, it follows that this neighborhood W_x must be contained in the union of those members of $\mathcal{C}$ which intersect it, namely. W_x is contained in the union of i range to 1 to k of C_i. Other members do not intersect, so they are not needed to cover any element of W_x. But all of them together has to cover W_x. Therefore, W_x is contained inside union of i range to 1 to k of C_i.

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It remains to prove that $\mathcal{A}^*$ is locally finite. Since $\mathcal{C}$ is locally finite, given $x \in X$, we get an onbd W_x of x which intersects finitely many members $C_1, \ldots, C_k$ of $\mathcal{C}$. Since $\mathcal{C}$ is a cover, it follows that $W_x \subset \cup_{i=1}^{k} C_i$. Therefore

$$W_x \cap A^* \neq \emptyset \Longrightarrow C_i \cap A^* \neq \emptyset \text{ for some } 1 \leq i \leq k.$$

But

$$C_i \cap A^* \neq \emptyset \Longrightarrow C_i \cap A \neq \emptyset.$$

Since C_i meets only finitely many members of $\mathcal{A}$, it follows that $W_x \cap A^* = \emptyset$ for all but finitely many members of $\mathcal{A}^*$. Therefore $\mathcal{A}^*$ is locally finite. ♠

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(d)$\Longrightarrow$(a) Given an open cover $\mathcal{U}$ of X, let $\mathcal{A}$ be a locally finite closed refinement of $\mathcal{U}$. This means that for each $x \in X$, there exists an onbd V_x of x which meets only finitely many members of $\mathcal{A}$. Now there exists a locally finite closed refinement $\mathcal{C}$ of $\mathcal{V} = \{V_x : x \in X\}$. It follows that each $C \in \mathcal{C}$ meets only finitely many members of $\mathcal{A}$.
For each $A \in \mathcal{A}$, select $U_A \in \mathcal{U}$ such that $A \subset U_A$ and put

$$A^* := U_A \setminus \cup\{C \in \mathcal{C} : A \cap C = \emptyset\}.$$

Since $\mathcal{C}$ is a locally finite family of closed sets, again from lemma 3.3, it follows that each A^* is open. Clearly $A \subset A^*$. Since $\mathcal{A}$ is a cover of X, it follows that $\mathcal{A}^* := \{A^* : A \in \mathcal{A}\}$ is a cover for X. Also since $A^* \subset U_A$, $\mathcal{A}^*$ is a refinement of $\mathcal{U}$.

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You want to finally prove that $\mathcal{A}^*$ is locally finite.

Take a member of $\mathcal{A}^*$, say, A^*. Suppose W_x intersects A^* is non-empty. But W_x contained inside this union, therefore, one of the $C_i \cap A^*$ must be non-empty. For all i, $C_i \cap A^*$ cannot be empty. So for several of them it may be non empty, that does not matter. For at least one i, we have $C_i \cap A^*$ is non-empty.

Look at this definition of A^*, $C_i \cap A^*$ is non-empty means C_i is not here, not here means what, $A \cap C_i$ itself is non-empty. So, $C_i \cap A^*$ is non-empty implies that $A \cap C_i$ itself is non-empty. So, C_i meets only finitely many members of $\mathcal{A}$, that is how we have constructed $\mathcal{C}$, right from the beginning, using local finiteness of $\mathcal{A}$.

It follows that $W_x \cap A^*$ is empty for all but finitely many members of $\mathcal{A}^*$. Given $x \in X$, once we have W_x, we get finitely many members C_i of $\mathcal{C}$, each of these C_i will give you finite members of $\mathcal{A}$, only these members can meet W_x. Therefore, there will be only finitely many members of $\mathcal{A}$, and only the corresponding members of $\mathcal{A}^*$ can meet W_x. Therefore $\mathcal{A}^*$ is locally finite. So, this completes the proof of theorem of Michael.

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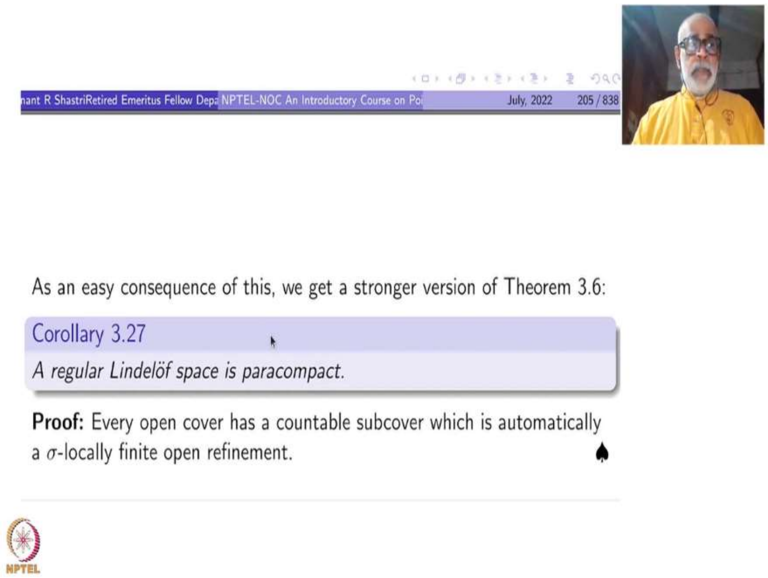

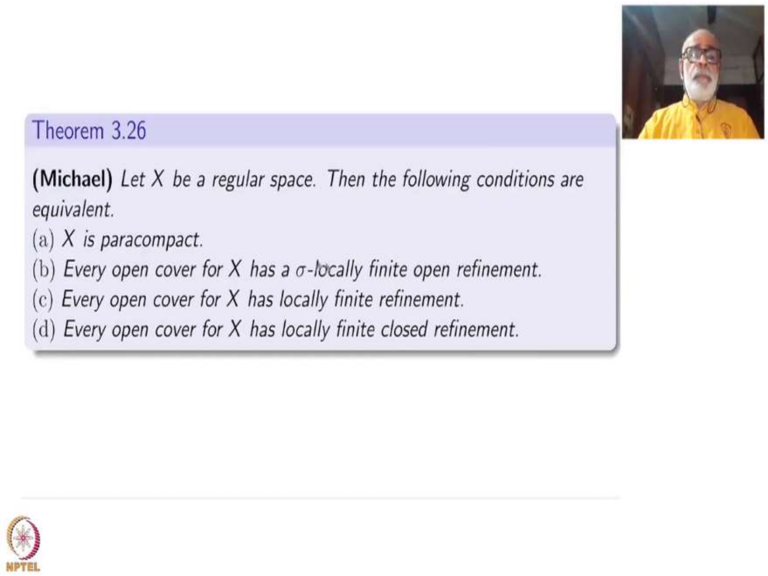

As an immediate consequence, we get a stronger version of this theorem 3.6 that we approved. Namely, if you have a locally compact of Hausdroff space, which is Lindelof, then it is paracompact. Locally compactness automatically implies regularity. Therefore, regular

Lindelof space is paracompact. It is a stronger theorem now. How do you get this one? Every open cover has a countable sub cover which is automatically sigma-locally finite.

A countable family you can write it as countable union of singleton open sets, singleton families are automatically locally finite. So, using this criterion, once you have regularity sigma-locally finite refinement or the first thing here, a and b here, every locally finite open refinement that will give you X is paracompact. So, easy proof of this one, you can directly prove this one also, things will not be all that easy. I mean, much easier than what we have done already.

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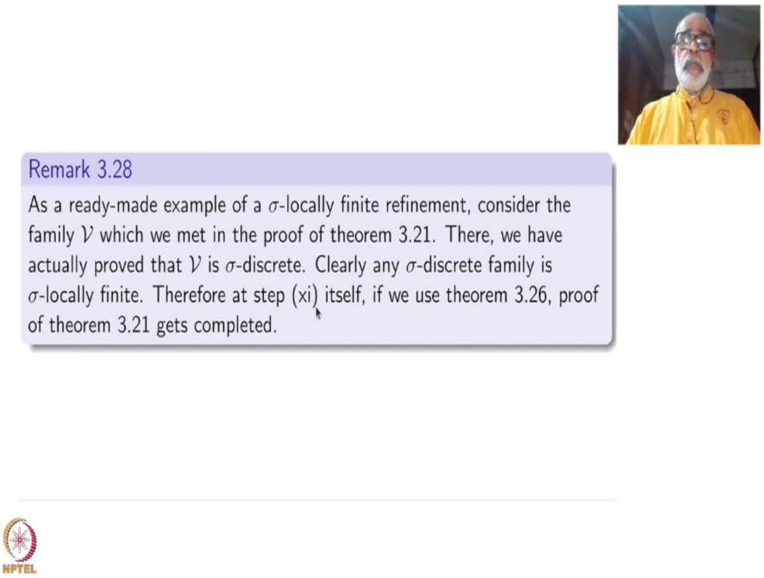

Here is a ready-made example of a sigma locally finite refinement. So far we have defined but we have not given an example. So, I want to give an example. Consider the family $\mathcal{V}$ which we met in the proof of theorem 3.21. Namely, while proving that a pseudo metric space is paracompact. There we have actually proved that the family $\mathcal{V}$ is sigma-discrete.

Remember sigma-discrete means what, it is a countable union of sub families, each sub family is discrete, i.e., for each point there will be a neighborhood which will intersect only one of the members. So that is discreteness, sigma discreteness means that it is a countable union of such discrete families, that is what we have done. So, sigma discrete is automatically sigma locally finite. So, we have such readymade examples there.

So, sigma locally finiteness is slightly more general than sigma discreetness, sigma discreetness is a very strong condition. Therefore, when you come up to several steps, at step (xi), if we use theorem 3.26 of Michael, you do not have to go further at all, because immediately you can conclude that it is paracompact. Whereas, there we have to work harder. Because we wanted to have an independent proof of paracompactness for pseudo metric spaces, that is all.

Finally, the results of Michael are based on experience which we one gets while studying metrics spaces. So, if you dig deeper into them, you get better and better theorems, that is all. So, let us stop here. And we shall meet next time with other notions of compactness, a new chapter. Thank you.