An Introduction to Point-Set-Topology (Part II)
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Lecture 13
Paracompactness Continued

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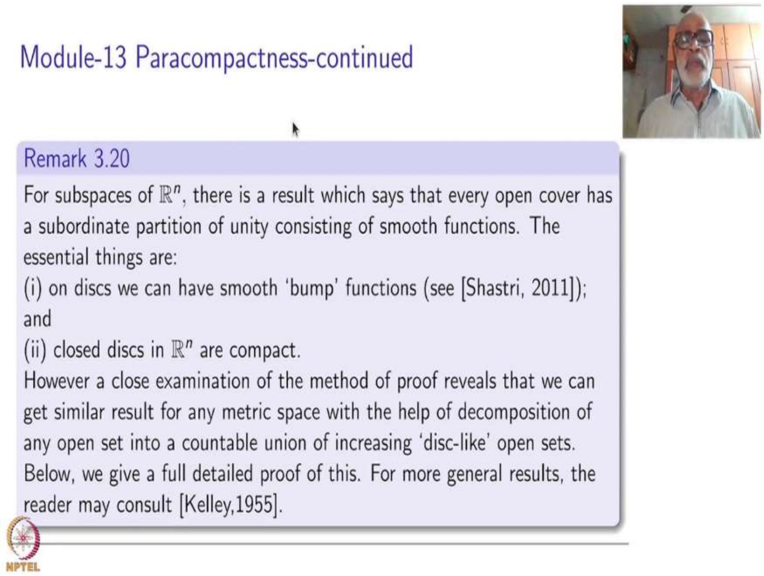

Hello, welcome to NPTEL NOC course on Point Set Topology Part II Module 13. We shall continue our study of paracompactness. For subspaces of $\mathbb{R}^n$, there is a result which says that every open cover has a subordinate partition of unity consisting of smooth functions. The word smooth will not have any meaning when you are studying arbitrary topological spaces and so members of a partition of unity for them are merely continuous functions.

Inside $\mathbb{R}^n$ such a thing is possible. And what is the additional thing that we have to do? namely, on every disc you can have what is called a bump function which is a smooth function with some additional properties. For more, you can see this reference here. The second thing is the local compactness of $\mathbb{R}^n$, in a very special way, namely, the closed discs themselves are compact. So, if you carefully study the proof that we have given earlier, then we can get a similar result for any metric space with the help of decomposition of any open set into a countable union of increasing disc-like open sets. What I mean to say is that the local compactness is not all that necessary here. So, something else will come to help namely, the metric property and that is what we are going to do. For more general results, you may consult again Kelley's book. So, I will do the bare minimum here to expose you to the ideas behind some results. This course is not meant to be concise or comprehensive.

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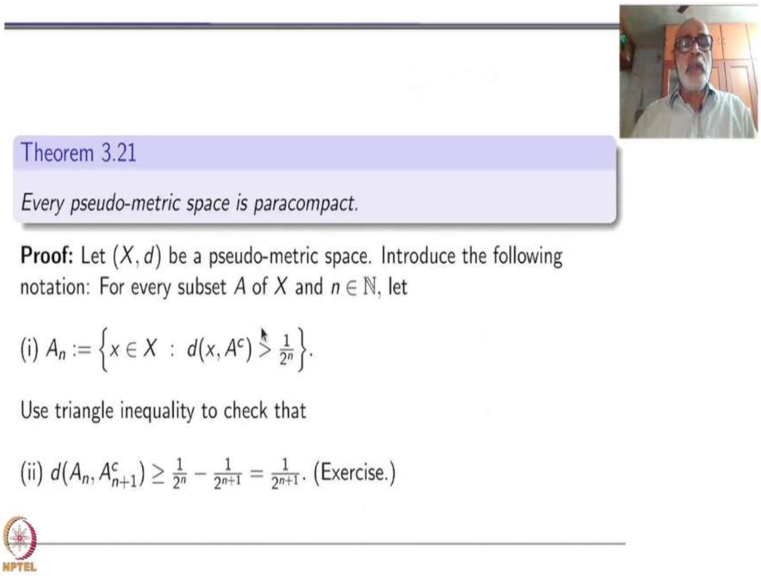

Every pseudo metric space is paracompact. So, this is a big theorem now. You have to have some patience. We will develop the proof of this result slowly, and the development itself is quite educative. So, first of all, let us have some notation here.

For every non empty subset A of a pseudo metric space (X, d), for each positive integer n, let us have a notation: A_n is the set of all $x \in X$ such that distance of x from A^c is bigger than $1/2^n$.

A may be any non empty subset of X. You take the complement and from there, the distance must be at least $1/2^n$, automatically these are subsets of A, may be empty. So, one thing very easy to see is that using triangle inequality, distance between A_n and A^c_{n+1} will be bigger than equal to $1/2^n - 1/2^{n+1}$ which is equal to $1/2^{n+1}$ (with the convention that the distance is infinite if one of the two sets is empty).

So, I again leave verifying this elementary thing as an exercise to you.

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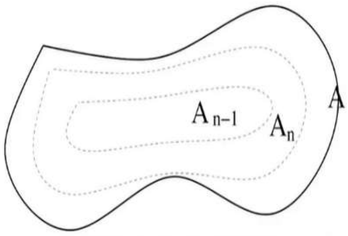
Note that for each A and each n, we have

(iii) $A_n \subset \bar{A}_n \subset A_{n+1} \subset A$ and each A_n is open. Moreover, if A is open then

$$A = \cup_n A_n.$$

(This is what we meant by countable union of 'disc-like' subsets in the remark above.)



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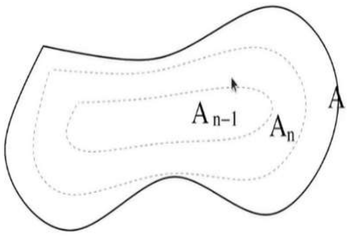

Figure 4: Nested, disc-like subsets

Theorem 3.21

Every pseudo-metric space is paracompact.

Proof: Let (X, d) be a pseudo-metric space. Introduce the following notation: For every subset A of X and $n \in \mathbb{N}$, let

(i) $A_n := \left\{x \in X \ : \ d(x, A^c) > \frac{1}{2^n}\right\}$.

Use triangle inequality to check that

(ii) $d(A_n, A^c_{n+1}) \geq \frac{1}{2^n} - \frac{1}{2^{n+1}} = \frac{1}{2^{n+1}}$. (Exercise.)

Note that A_n is contained in $\overline{A_n}$ (that is obviously true always, but we also have) $\overline{A_n}$ is contained inside A_{n+1}. So, already we have these is nested subsets. Only thing is that these A_i's may not be compact or even non empty. However, we have these increasing phenomena A_n contained inside $\overline{A_n}$ contained inside A_{n+1} contained inside A.

And each A_n is open, so that is because in this condition we have strict inequality, bigger than $1/2^n$. So, closure of A_n will be given by `bigger than or equal to'. That is all. So, that is the reason why you have these are open subsets.

Moreover, if A itself is open then and then only A will be the union of all the A_n.

What we have proved earlier was that when you have locally compact, Lindelof space then every open subset can be written like this with each $\overline{A_n}$ compact and so on. That is not exactly what we have here, but something of that which you have saved here, namely, with the use of the pseudo-metric, we were able to write every open set as a countable union of open sets, A_n with this property $\overline{A_n}$ contained A_{n+1}.

So, that is what I meant by writing every open set as a union of disc-like open sets. They are nested very strongly in the sense that the closure itself is contained inside A_{n+1}. So, these are just notations. Now, you have to remember this one for the rest of the proof.

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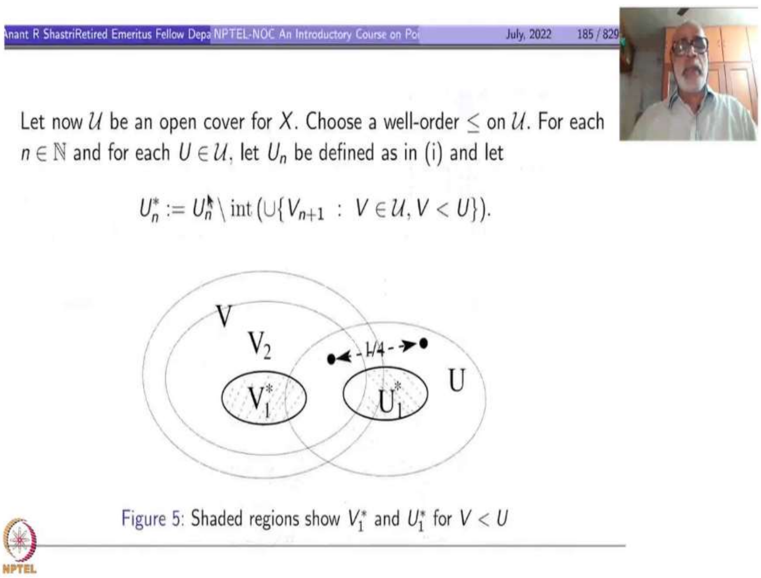

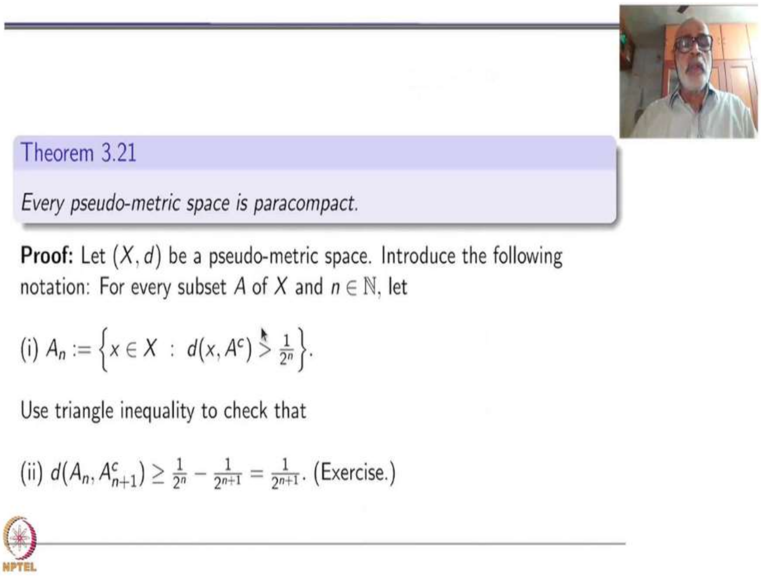

Let now $\mathcal{U}$ be an open cover for X. We want to extract a locally finite open refinement that is our purpose finally. Choose a well order on $\mathcal{U}$. (So, here again, we have use axiom of choice equivalently, the well-ordering principle. Every set can be well ordered.) For each $n \in \mathbb{N}$ and for each U inside $\mathcal{U}$, let us introduce another notation.

Let U_n be defined as in (i), there is no change, U_n is got by replacing A by U in (i). But I am going to give you, namely, U_n^* is the subset of U_n wherein I have thrown away something, this is a set theoretic complement, set theoretic minus. That some thing is the interior of union of all V_{n+1}''s, where V occurs before U, in the well ordering of $\mathcal{U}$. We have put a well order on $\mathcal{U}$. So take all the initial elements V to U and take the corresponding V_{n+1} portion, (not the whole of V) then take their union (n is fixed, V is varying), take the interior of this union and throw it away from U_n to get U_n^*.

So, here is a picture of U_1 and V_1. U is the given one this V occurring before U. Suppose V was the first one and U was the second one in $\mathcal{U}$, with respect to the well order. So, if you look at V_1^*, there is nothing before that V and hence nothing is thrown out of V_1. So, V_1 start will be full V_1. But for U_1^*, what will happen? I will have to throw away V_2, V is before U and so V_2 has to be thrown away from U_1. So that is U_1^*. Only this part.

Because of the definition of U_n and V_n, if you take any element of $U \setminus V$, and any element of V_2, the distance between them will be at least one-fourth. So, this is shown in this picture. So, why we are subtracting some portions of earlier elements is illustrated, in the simplest case, by this picture I have shown.

So, in general, you have to subtract V_{n+1} for all V which occur before U. The well order could be any arbitrary one, does not matter, but fixed once for all.

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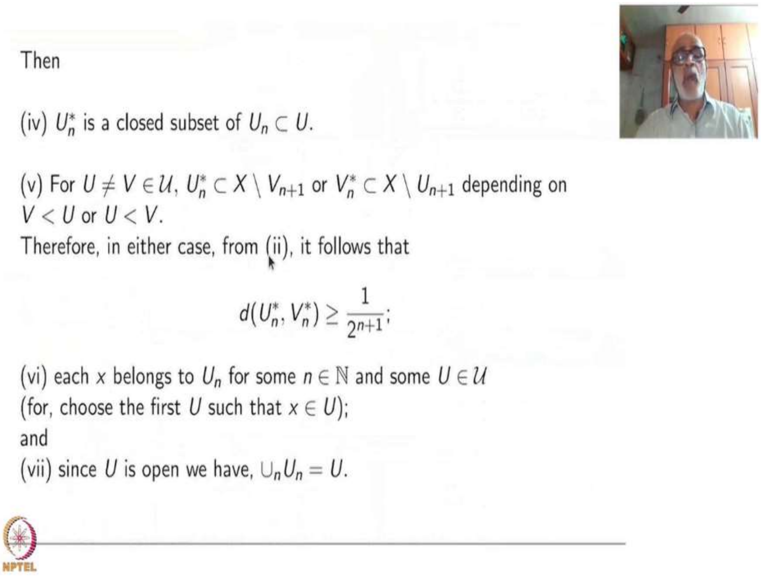

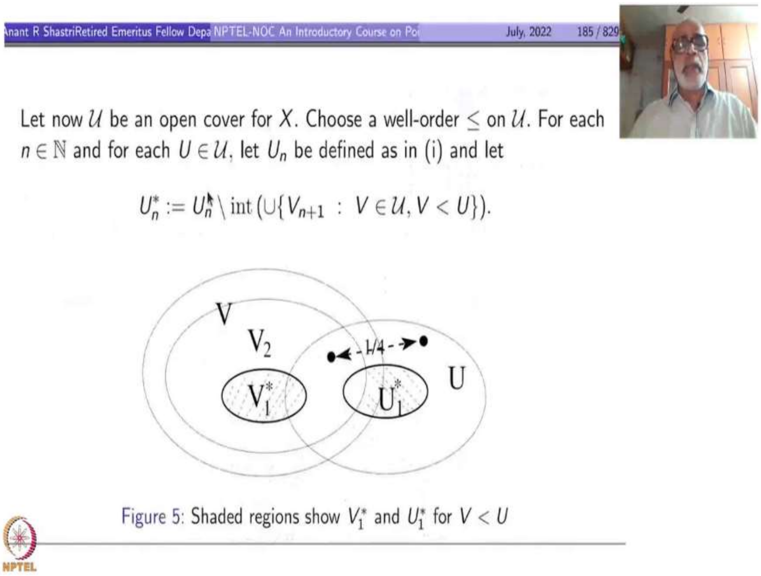

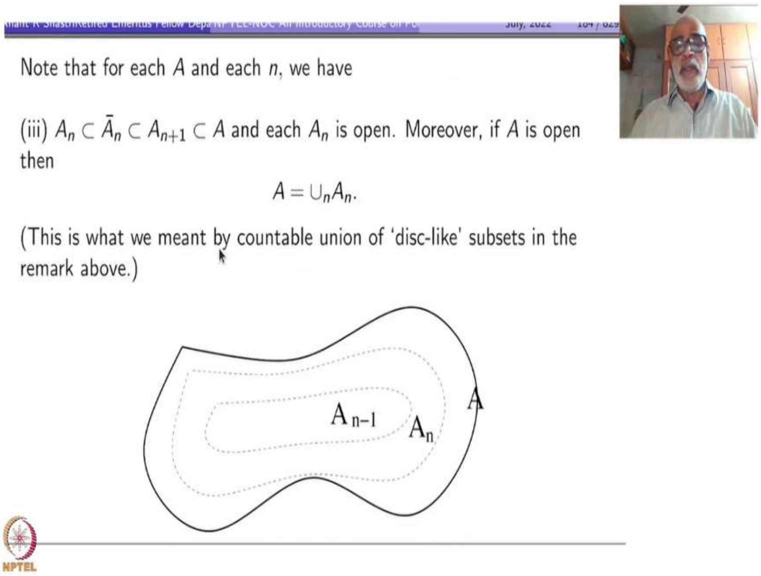

Then U_n^* is the closed subset of U_n contained inside U. So, why it is a subset? What I have said, deleted some open subset that is all. You see interior of something whatever it is, if you delete an open subset it will be a closed subset of the original set. So, that is what it is and it is a subset U_n and is contained inside U.

If U and V are not equal, if they are different distinct elements of $\mathcal{U}$, then U_n^* is inside $X \setminus V_{n+1}$ or V_n^* is in $X \setminus U_{n+1}$, depending on whether V is before U or U is before V. Because if V is before U, V_{n+1} will get subtracted from U_n or the other way round. So, that is from the definition. Therefore, in either case what happens is from this general remark (ii) here, that the distance between A_n and A_{n+1}^c is bigger than $1/2^{n+1}$, which I have shown you in this picture, what happens is that the distance between U_n^* and V_n^* is always bigger than or equal to $1/2^{n+1}$.

So, in this picture n was an equal to 1, so it is one-fourth. So, you do not have to do any pictures at all if you follow the logic here. Step by step, for each step get a very small picture in your mind. After that, you have to just use whatever you have proved before. So, if you use property (ii) this will be obvious.

The next thing is that each x belongs to U_n for some n and some U in $\mathcal{U}$.

First of all, x belongs to some U because $\mathcal{U}$ is a cover for X. But once it belongs to some U and from the complement of U, its distance will be positive. So, there will be n such that $1/2^n$ will smaller than this distance and hence x will be in U_n. So, first you choose U such that x is inside U and then since U is open and we have union of all U_n is equal to U, x must be inside one of the U_n's.

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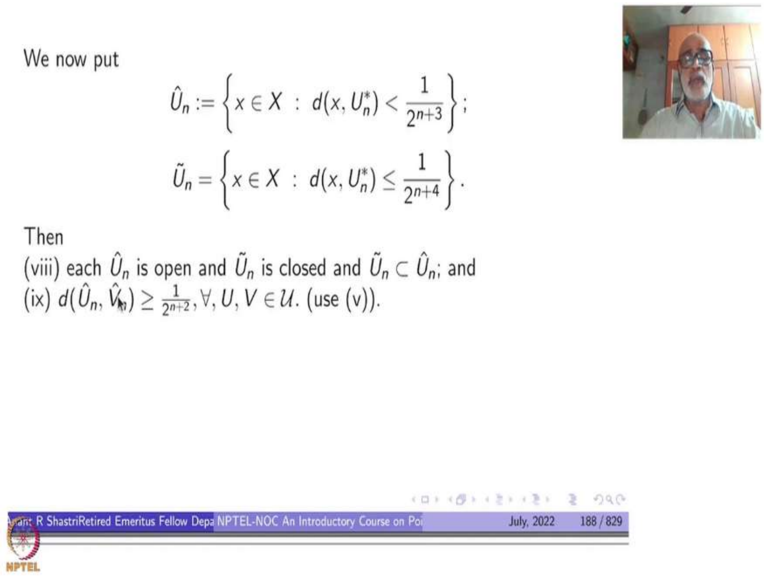

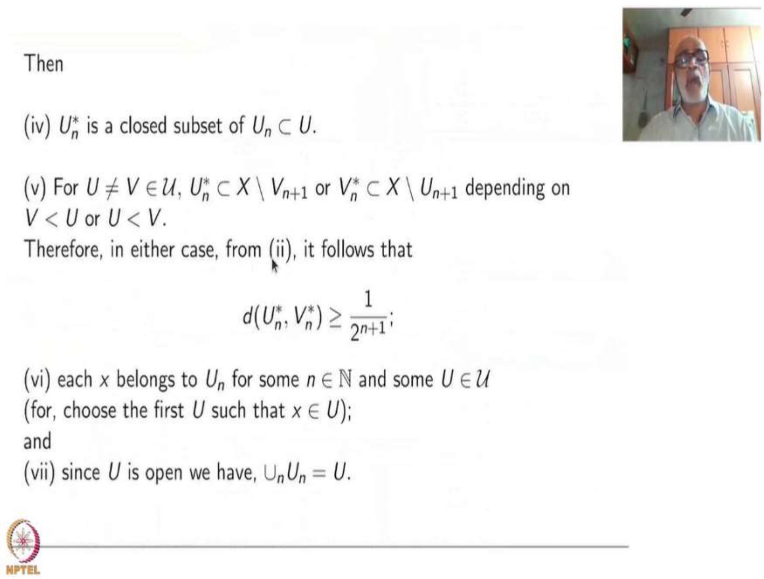

We now introduce two more notation, $\widehat{U_n}$ ... (You may be already bored!) So first you had U_n and then U_n^* here. Now, $\widehat{U_n}$ is the set of all x in X such that distance between x and U_n^* is less than $1/2^{n+3}$; and $\widetilde{U_n}$ is the set of all x in X, such that distance between x and U_n^* is less than or equal to $1/2^{n+4}$. So, both are enlargements of U_n^*. pay attention to the inequalities, the first one is strict one. So, Each $\widehat{U_n}$ is open and $\widetilde{U_n}$ is closed.

And $\widetilde{U_n}$ is contained inside $\widehat{U_n}$. Once again, the same property (ii) will tell you that the distance between $\widehat{U_n}$ and $\widehat{V_n}$ is bigger than or equal to $1/2^{n+2}$ for every distinct U, V inside $\mathcal{U}$.

Indeed, this time, you can directly use the (v) property here, that distance between U_n^* and V^{n*} is bigger than equal to $1/2^{n+1}$.

Similarly, you can talk about $\widetilde{U_n}$, this $\widetilde{U_n}$ also. Though that will not be needed in the final proof, but will play some auxiliary roll, so, I have kept it. So, the distance between $\widetilde{U_n}$ and $\widetilde{V_n}$ is bigger than equal to $1/2^{n+2}$. This hold no matter U occurs first or V occurs first.

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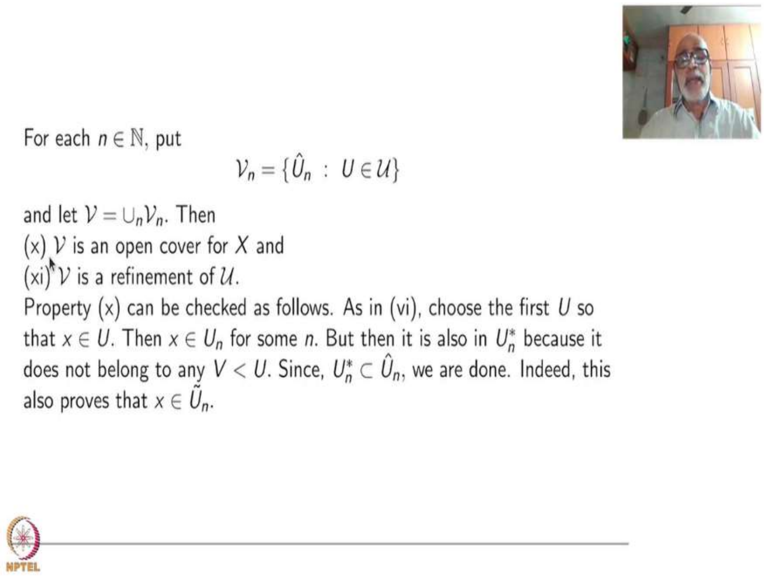

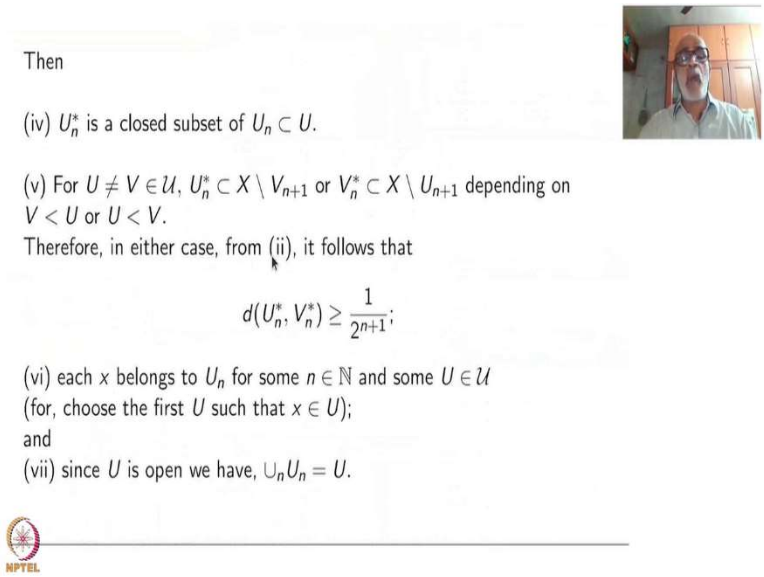

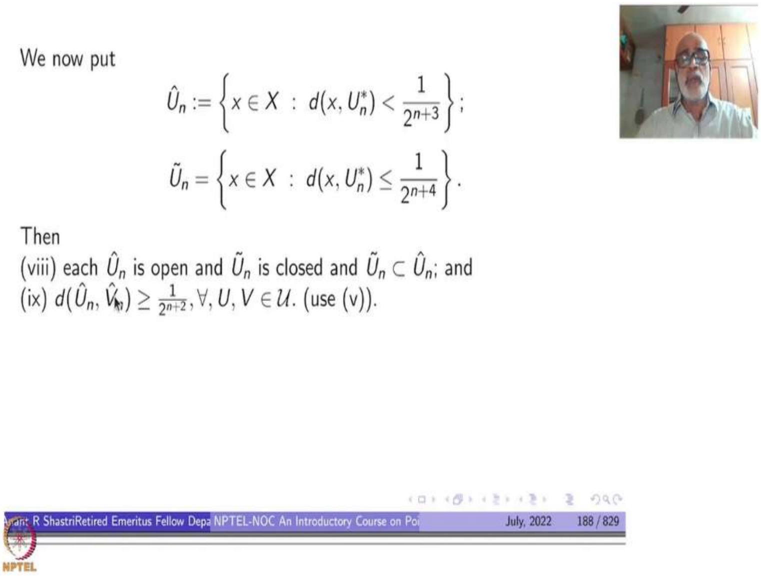

Next, for each natural number n, put $\mathcal{V}_n$ equal to this collection of all $\widehat{U_n}$ where U ranges over all of $\mathcal{U}$. So, each member of this family is an open subset. Take $\mathcal{V}$ to be the union of all $\mathcal{V}_n$'s. So, we have written $\mathcal{V}$ as a countable union of these sub families. What are the property of $\mathcal{V}$?

(x) It is an open cover for X; not the individual $\mathcal{V}_n$'s, but when you take all of them, namely, $\mathcal{V}$, that is an open cover for X and

(xi) $\mathcal{V}$ is a refinement of $\mathcal{U}$.

So, how to check this (x)? This can be checked as in the case (vi). What we have done there is that each x belongs to some U_n. Similarly, we want to say that each x belongs to some U_n^*. For that choose the first U so that x is inside U. Then x will be inside some U_n for some large n. But then it is also in U_n^* because it does not belong to any V, which comes before U, which you have subtracted from U_n to get U_n^*.

But why it is in $\widehat{U_n}$? Since U_n^* is contained inside $\widehat{U_n}$, we are done. Indeed, this also proves that x itself is in $\widetilde{U_n}$ also because U_n^* is contained inside $\widetilde{U_n}$ also. Remember that $\widehat{U_n}$ and $\widetilde{U_n}$ are actually fattening of U_n^*. (All those points which are at a distance smaller than a some positive number from a given set is called a fattening of that set.) Note that U_n and U_n^* are cutting downs from U whereas $\widehat{U_n}$ and $\widetilde{U_n}$ are fattening of U.

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Since $U_n^* \subset U_n$, we get

$$\hat{U}_n \subset \{x \in X \ : \ d(x, U_n) < \frac{1}{2^{n+3}}\} \subset U_{n+2} \subset U.$$

This proves (xi).

Finally, for each n and each $U \in \mathcal{U}$, put

$$U_n^{\#} := \hat{U}_n \setminus \cup\{\tilde{V}_k \ : \ V \in \mathcal{U}, k < n\}.$$

Note that $\{\tilde{V}_k \ : \ V \in \mathcal{U}, k < n\}$ is a locally finite family of closed sets. It follows that:

(xii) each $U_n^{\#}$ is open.

For each $n \in \mathbb{N}$, put

$$\mathcal{V}_n = \{\hat{U}_n \ : \ U \in \mathcal{U}\}$$

and let $\mathcal{V} = \cup_n \mathcal{V}_n$. Then

(x) $\mathcal{V}$ is an open cover for X and

(xi) $\mathcal{V}$ is a refinement of $\mathcal{U}$.

Property (x) can be checked as follows. As in (vi), choose the first U so that $x \in U$. Then $x \in U_n$ for some n. But then it is also in U_n^* because it does not belong to any $V < U$. Since, $U_n^* \subset \hat{U}_n$, we are done. Indeed, this also proves that $x \in \tilde{U}_n$.

We now put

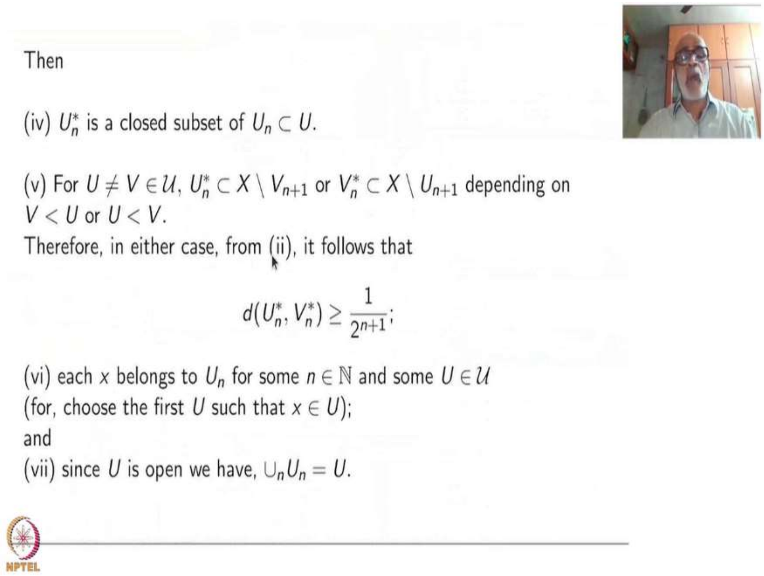
$$\hat{U}_n := \left\{x \in X \ : \ d(x, U_n^*) < \frac{1}{2^{n+3}}\right\};$$

$$\tilde{U}_n = \left\{x \in X \ : \ d(x, U_n^*) \leq \frac{1}{2^{n+4}}\right\}.$$

Then

(viii) each $\hat{U}_n$ is open and $\tilde{U}_n$ is closed and $\tilde{U}_n \subset \hat{U}_n$; and

(ix) $d(\hat{U}_n, \hat{V}_n) \geq \frac{1}{2^{n+2}}, \forall, U, V \in \mathcal{U}$. (use (v)).



(xi) And $\mathcal{V}$ is a refinement of $\mathcal{U}$. U_n^* is a subset of U_n, both are subsets of U. Therefore, $\widehat{U_n}$ which is by definition, the set of all those points which are at a distance less than $1/2^{n+3}$ from U_n^*, is contained inside the set of all those points which are at a distance less than$1/2^{n+3}$ from U_n also. This latter set is definitely contained in U_{n+2}, by triangle inequlatiy, so that $\widehat{U_n}$ is inside U.

So, finally, I will have one more notation here: Let $U_n^{\#}$ equal to $\widehat{U_n}$ minus union of all $\widetilde{V_k}$, V belong to $\mathcal{U}$ and $k < n$. So, this is where the twiddles are used. This is the only place where we need them.

What we are doing now? We do not want full $\widehat{U_n}$'s. So, I am chucking away some portion of them, namely these are all closed subsets now, take the union of all these $\widetilde{V_k}$'s where V is arbitrary in $\mathcal{U}$, but integer $k < n$ now. So, you throw away that part. Note that is the family $\{\widetilde{V_k} : V \in \mathcal{U}, k < n\}$ is a locally finite family of closed sets.

Therefore, it follows that each $U_n^{\#}$ is open. Remember this result about locally finite family of closed sets. When you take their union, it is still a closed set. So, this whole thing is closed set, the complement will be an open subset now. $\widehat{U_n}$'s are open subset. So, $U_n^{\#}$ are open subsets.

Student 1: I have question. Why this family is locally finite?

Professor Anant: Because we had taken union of only finitely many of them. For each fixed k, by property (v) the family $\{\widetilde{V_k} : V \in \mathcal{U}\}$ is locally finite. If you take any two different members, they are not only disjoint, the distance between them is bigger than or equal to a positive constant viz., $1/2^{k+2}$. Therefore, at any point x if take an open ball or radius less that $1/2^{k+3}$, then the ball may intersect at most one member of this family.

Student 1: Sir, I have one more question here. So, starting with the open cover, first we made U_n^*, a collection of which is also a cover for X. But that was not open.

Professor Anant: Yes.

Student 1: That is why you consider that $\widehat{U_n}$.

Professor Anant: Right, right, they are not they are closed subsets actually. Yeah?

Student 1: Yeah. And so, this $\widehat{U_n}$ collection was open refinement, but that may not be locally finite. That is why you are coming to $U_n^{\#}$, right?

Professor Anant: Yes, that is the precise reason. Subtracting these things, $\widehat{V_k}$ that will make $U_n^{\#}$ locally finite we will see that, okay? Yeah? So, first of all, $U_n^{\#}$ is open now, you see we did not even stop it $\widehat{U_n}$ also. So, first of all, these are open itself you have to look for that these things are locally finite, that is fine, $\widehat{V_k}$. But they are closed things. So now, these are open subsets, first thing.

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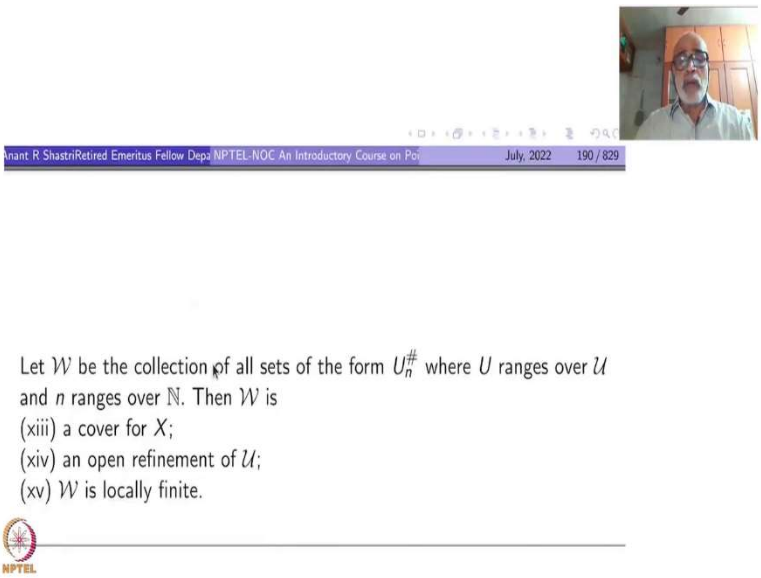

So, what we want to do is that take $\mathcal{W}$ to be the collection of all subsets of the form $U_n^{\#}$, U ranges over $\mathcal{U}$, n ranges over natural numbers, okay? That is like take all $U_n^{\#}$'s is first with n fixed and then taking the union over n, you can say a double union. That is

(xiii) a cover for X,

(xiv) it is an open refinement of $\mathcal{U}$ and

(xv) is locally finite itself. (There is no need to say sigma locally finiteness here.) This $\mathcal{W}$ is locally finite itself.

So, we have to prove this, (xiii), (xiv) and (xv).

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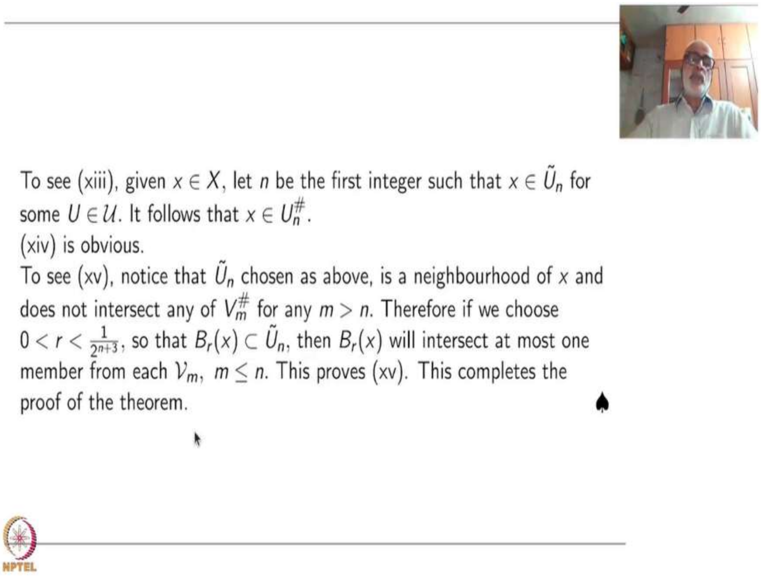

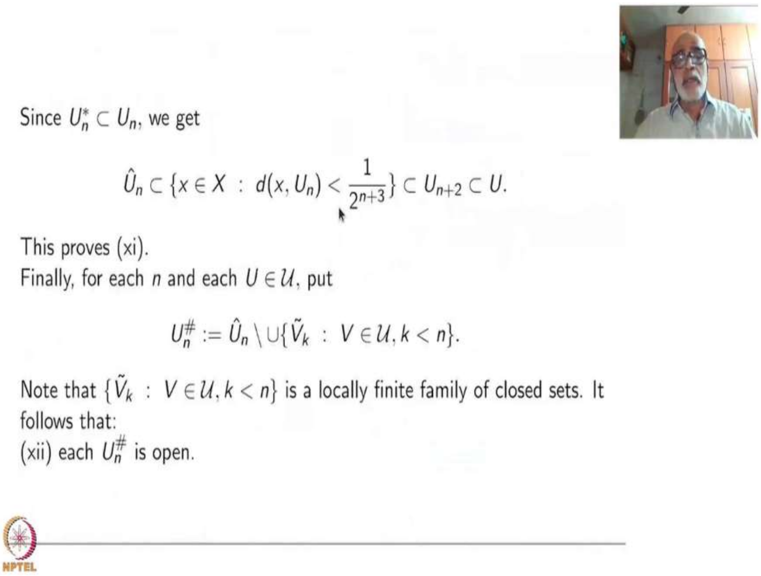

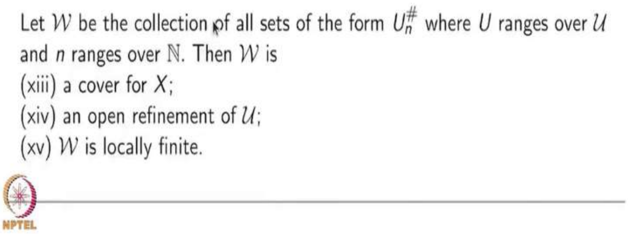

So, say how do you prove (xiii) that $\mathcal{W}$ covers the whole of X? Given any x in X, let n be the first integer such that x belongs to $\widetilde{U_n}$ for some U. Once it is in some U there will be some positive integer n for which it belongs to $\widetilde{U_n}$. So, let us take n to be the first such integer. It follows x is inside $U_n^{\#}$ itself, because all $\widetilde{V_n}$ which I have thrown out from $\widetilde{U_n}$ none of them contain the point x. So, it must be inside $U_n^{\#}$. So, that is the trick here. So, these things cover X.

(xiv) $\mathcal{W}$ it is an open refinement? We have already told that these members are open. Also they are all subsets of of some U that is clear.

(xv) Why it is locally finite? To see this, one notices that $\widetilde{U_n}$ chosen as above is a neighborhood of x and does not intersect any $V_m^{\#}$ for $m > n$, because these $\widetilde{U_n}$ would have been subtracted from $\widetilde{V_m}$ to get $V_m^{\#}$.

So, it does not intersect $\widetilde{V_m}$ for $\widetilde{V_m}$. Therefore, if we choose $0 < r < 1/2^{n+3}$, such that this ball $B_r(x)$ is contained $\widetilde{U_n}$. This possible because $\widetilde{U_n}$'s are open subsets. Then $B_r(x)$ may intersect only one of the $\widetilde{V_k}$ for each k such that $1 \leq k \leq n$. it happens that it will intersect only one of them. So, this completes the proof of a theorem.

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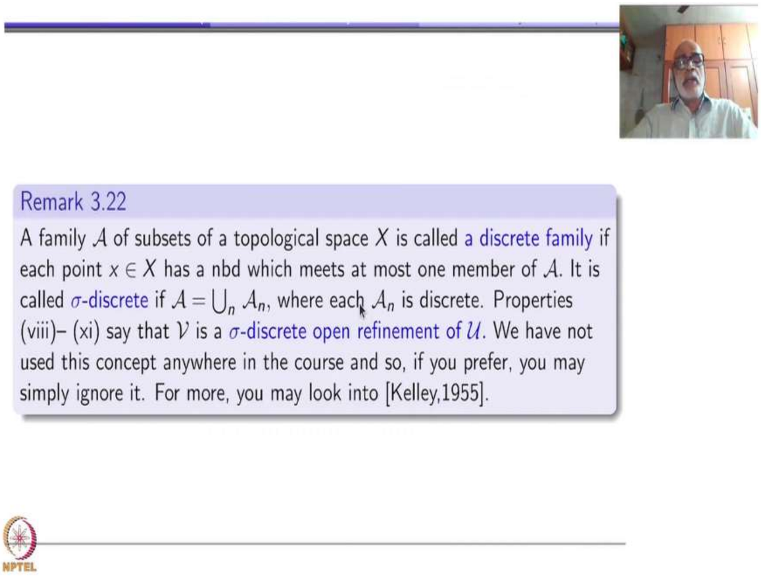

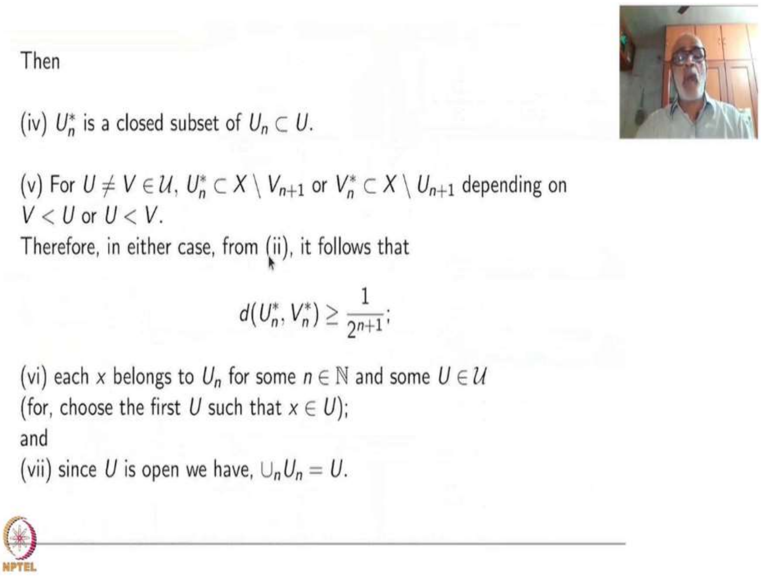

There is a remark here which is a bit deeper. So, I do not mind even if you do not understand it in the first reading. So, but I will make this remark. A family $\mathcal{A}$ of subsets of a topological space is called discrete family, if each $x \in X$ has a neighborhood which meets at most one member of $\mathcal{A}$. If $\mathcal{A}$ is the countable union of subfamilies $\mathcal{A}_n$'s where each $\mathcal{A}_n$ is discrete, then $\mathcal{A}$ is called a sigma-discrete family.

So, one of you asked this question. That is why this remark. It is even more relevant here now. So, why is it more relevant? What happens is, you see this condition right in the beginning distance between these two is bigger than $1/2^{n+1}$. So, this U_n^* and V_n^*, same n but U and V are different elements of the same cover.

What happens to these sets? They are disjoint, not only that, this is stronger than just being disjoint. Namely, I can take small open subsets around them for all of them simultaneously, such that all these neighborhood are disjoint, because of this metric property, we are able to do that one. Such a property can be made as an axiom in the general case. Then it is called sigma-discrete or that is what I am trying to say here.

A family is discrete family if each point x belongs to X has a neighborhood which meets at most one member of $\mathcal{A}$. You see, if the distance between as something positive each point x, I can take the ball of radius half of that distance whatever positive half that radius, then what I get is that that open ball cannot intersect both of them that is all.

So, that is what we have achieved here, it is called sigma discrete if it is union of, countable union of $\mathcal{A}_n$, where each n is discrete. So, that is why that n has come. When you fix n, it is a discrete family, you take the union it becomes a cover and so on, only countable union you have to take. Such a thing is called sigma discrete. This family $\mathcal{A}$ itself may not be discrete. That is one thing you have to understand, it is sigma discrete.

Properties (viii) and (ix) say that this $\mathcal{V}$ which we have constructed is sigma discrete open refinement of $\mathcal{U}$. So, in general, what one does without assuming metric property? You would like to prove this one out of some other properties by making this definition sigma-discreteness. We have not used this concept anywhere in the course. And so that is what I want to say, if you prefer, you can simply ignore it for the time being. If you are interested more, then you can look into Kelley's book.

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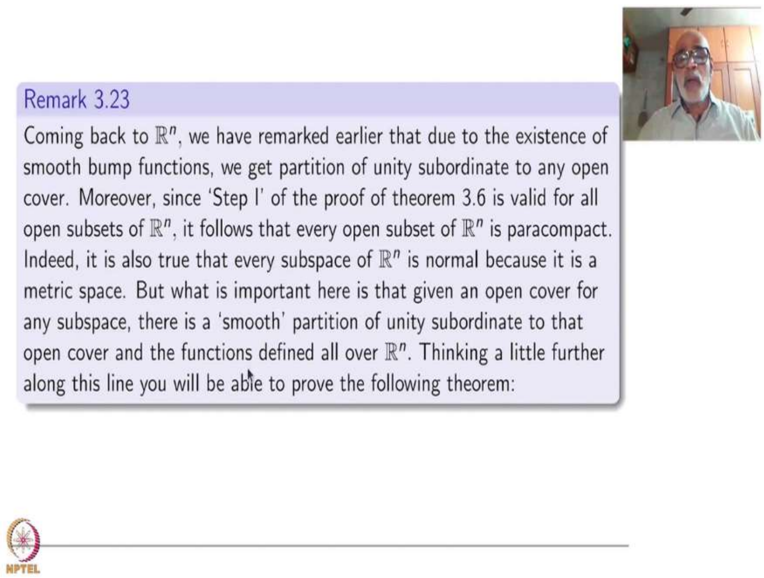

So, coming back to $\mathbb{R}^n$, we have remarked earlier that, due to the existence of certain smooth functions, we get smooth partitions of unity subordinate to an open cover. Moreover, since step one of the proof of 3.6 is valid for all open subsets of $\mathbb{R}^n$, it follows that every open subset of $\mathbb{R}^n$ is paracompact. Indeed, it is also true that every subspace of $\mathbb{R}^n$ is normal, because it is a metric space.

But what is important here is that given an open cover for any subspace there is a smooth partition of unity subordinate to that open cover, but functions are all defined on the entire of $\mathbb{R}^n$, not on just that open subset. Thinking a little further along this line, you will be able to prove the following theorem.

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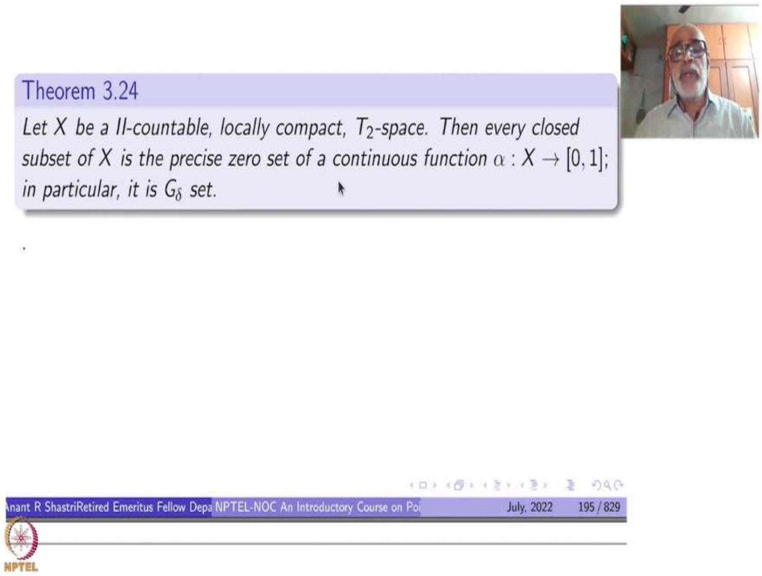

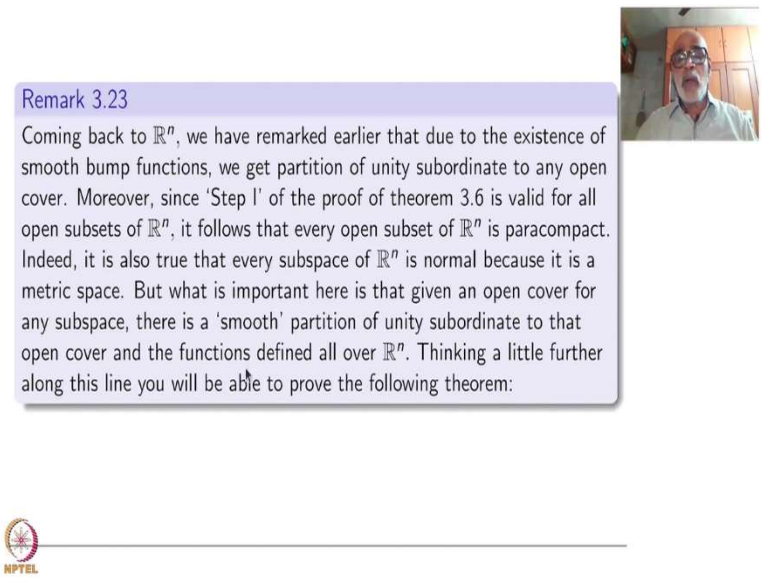

This I get it as a corollary to whatever we have done so far. All these remark is for getting such a motivation from $\mathbb{R}^n$. Let X be a second countable, locally compact, Hausdroff space.

Then every closed subset of X is the precise zero set of a continuous real valued function on X. You can choose the codomain of this function to be $[0,1]$, or any other closed interval.

Once you have this it follows that such a subset is a G_δ set, because the precise zero set of a continuous function is a G_δ set. You can just write as the intersection of $\alpha^{-1}[0,1/n)$ as n ranges over all positive integer. So, how do we prove this one? It is not difficult.

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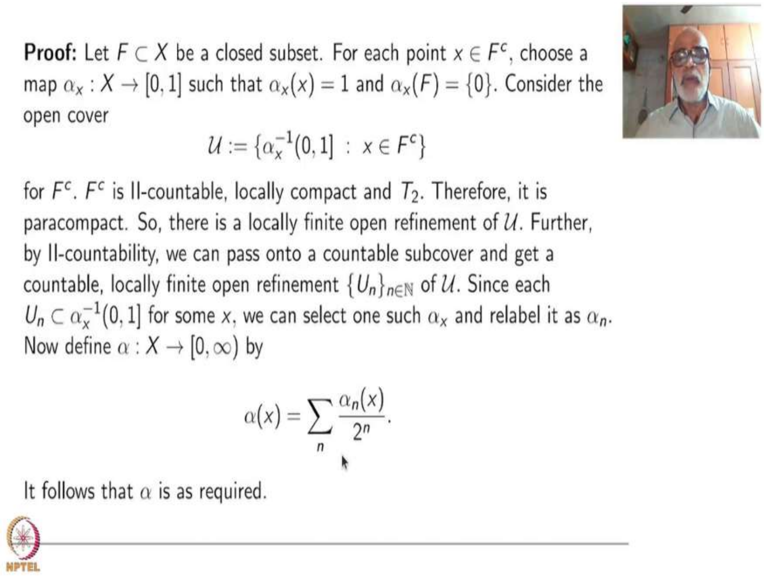

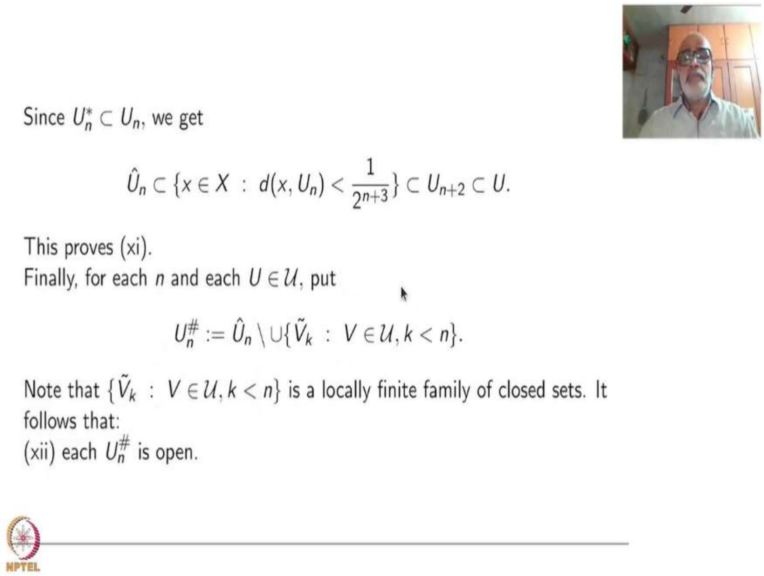

Start with any F contained inside X closed subset. For each x in the complement, choose a function α_x from X to $[0,1]$ such that $\alpha_x(x) = 1$ and $\alpha_x(F) = 0$. Consider the open cover $\mathcal{U} = \{\alpha_x^{-1}(0,1], x \in F^c\}$ for F^c. F^c being a closed subset of X is also second countable, locally compact and T_2.

Therefore, it is paracompact. So, there is a locally finite open refinement. Further by second countability, you can get a countable subcover, it will be again a locally finite refinement. So, we get a countable locally finite open refinement $\{U_n, n \in \mathbb{N}\}$ of $\mathcal{U}$. These U_n are not necessarily members of $\mathcal{U}$, but each of them is contained in some members of $\mathcal{U}$.

So for each n, we shall choose x_n such that U_n is contained in $\alpha_{x_n}^{-1}(0,1]$ and relabel the function by $\alpha_n := \alpha_{x_n}$. So, define α now, from X to $[0,\infty)$ (no index here,) equal to sum of all αn's, but divided by 2^n each of them. After dividing 2^n, you take the sum.

So, go through this proof carefully again and again. Maybe three times it does not matter, Each of these steps has a meaning there.

Next time we will do some general results which seemingly come from nowhere. But the motivation is here. If you know this one, you know where it is coming form. Thank you.