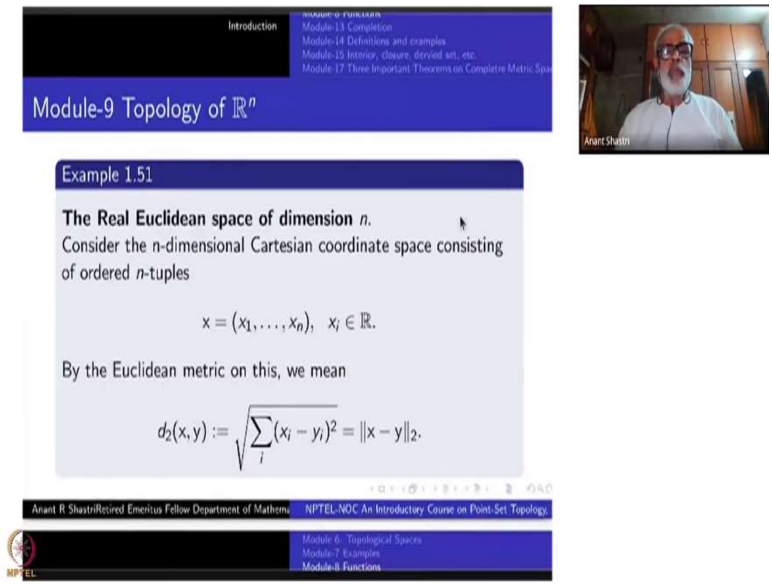
Introduction to Point Set Topology, (Part I)
Prof. Anant R. Shastri
Department of Mathematics
Indian Institute of Technology, Bombay

Module - 09
Lecture - 09
Topology of $\mathbb{R}^n$

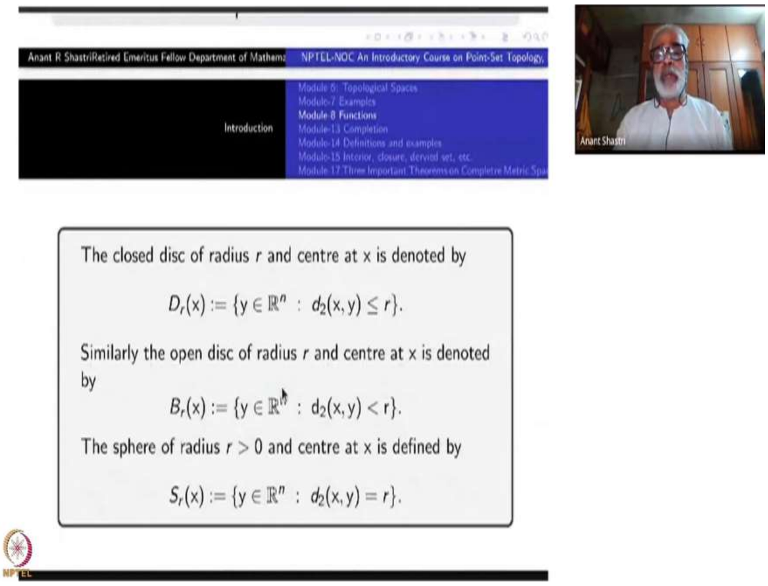
(Refer Slide Time: 00:16)



Welcome to module 9, today we will concentrate on Topology of $\mathbb{R}^n$ of course, we cannot cover the entire topic as such, but few glimpses of what is happening inside $\mathbb{R}^n$ including $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3$ etc all together one single thing. I am not making any separate cases here ok.

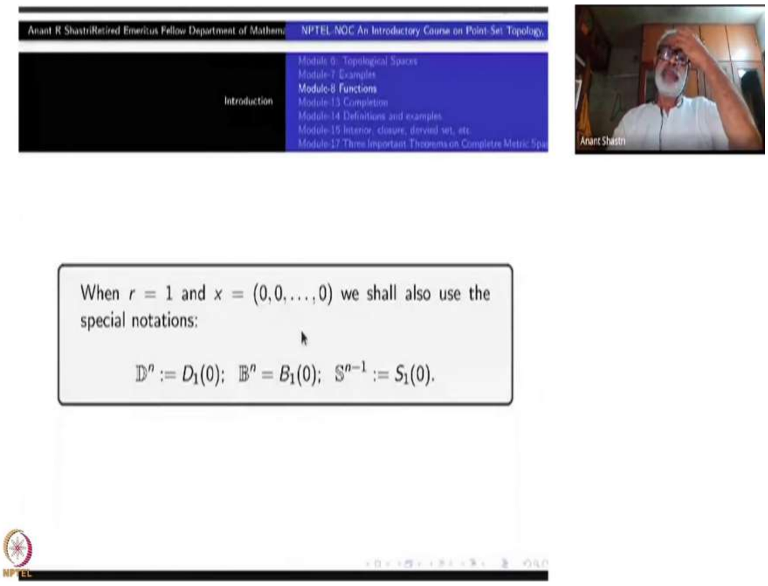
So, let us recall that a n dimensional Cartesian coordinate space consists of n tuples of real numbers $(x_1, x_2, \ldots, x_n), x_i$ is are inside $\mathbb{R}$. The Euclidean metric you have denoted by $d_2(x, y)$ is nothing, but the norm 2 of $x - y$, and the 2 norm which is the square root of the sum of the squares of the differences $x_i - y_i$.

(Refer Slide Time: 01:20)



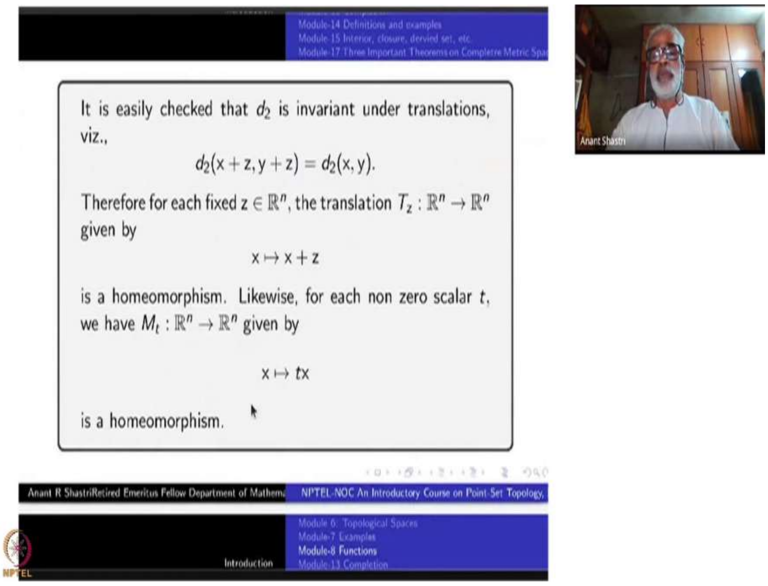
Recall that a closed disc was defined as the set of all points which are at a distance less than or equal to r from a given point and is denoted by $D_r(x)$. So, all points y such that, now I am using this distance, I can drop out this one because for now onwards unless I mention otherwise, it is always the Euclidean norm. So, I could have done just $d(x,y)$, but here I am elaborately writing $d_2(x,y)$ less than equal to r ok. Similarly, the open disc is defined $d_2(x,y)$ less than r only ok? And when you just take the equality that will be the boundary sphere. It is called as sphere of radius r inside. everything inside $\mathbb{R}^n$ ok? So, it should be $n-1$ dimensional sphere. Set of all y belonging to $\mathbb{R}^n$ such that $d_2(x,y) = r$. We do not take r equal to 0. We do not take a sphere of radius 0. So, r is positive, that is important here ok.

(Refer Slide Time: 02:29)



The special case when the radius is equal to 1, we will have a separate notation, because we have opportunity to use it again and again. So, I will just denote it by $\mathbb{D}^n$ instead of $\mathbb{D}_1(0)$ it is a decorated $\mathbb{D}^n$, Euler font ok. This is a closed disc of radius 1 centered at 0, just to indicate that I am working inside $\mathbb{R}^n$, I am putting this n here at the top.

(Refer Slide Time: 03:14)

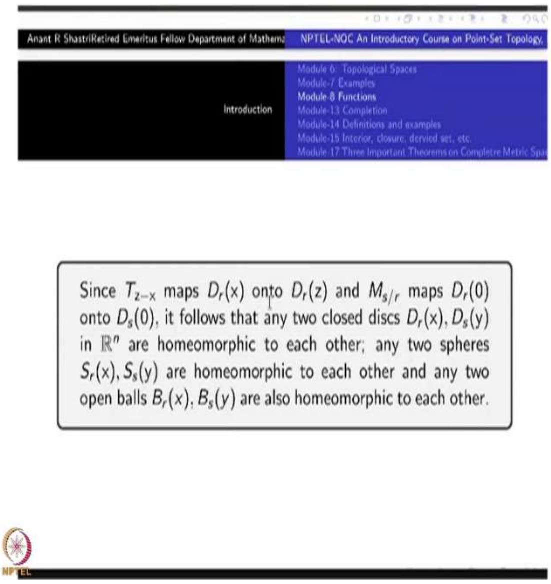


Similarly, $\mathbb{B}^n$ is the open disc and $\mathbb{S}^{n-1}$ is the sphere of radius 1 ok. So, this is the short notation. So, the first thing is that the d_2 metric is the Euclidean metric which is coming from the norm, has this obvious property, namely, if you add a vector z, to both of them namely x as well as y, $d_2(x+z, y+z)$ is the same thing as $d_2(x, y)$.

So, that is called the invariance of d_2 under translation. Adding z, a constant vector is called a translation by z. This I denote it by T_z, T_z is a map from $\mathbb{R}^n$ into $\mathbb{R}^n$ given by x going to $x+z$. Its inverse will be x going to $x-z$. And; obviously, this is a continuous map. Obviously, T_z and T_{-z} are inverses of each other. So, it is a homeomorphism.

Likewise, for each non zero scalar you can talk about M_t, the scalar multiplication by t, x going to tx. This time you have to have t non zero if you want M_t to be a homeomorphism. If t is 0, this will take entire thing to 0, a constant function 0 right? It would not be a homeomorphism that is all. So, what is the inverse, inverse is x going to $t^{-1}x$.

(Refer Slide Time: 04:56)



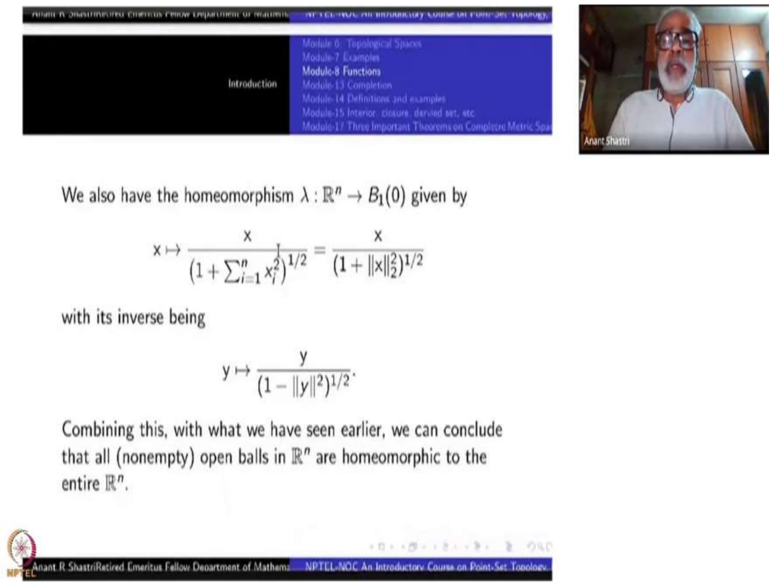
So, if you look at T_{z-x} maps $D_r(x)$ to what? $D_r(x)$ has centre x, subtract x and add z, the center goes to z, so, the disc will go to $D_r(z)$, the radius does not change that is precisely the point here. Namely, $d_2(x+z, y+z)$ is $d_2(x, y)$. So, this closed disc of radius r mapped onto a closed disc of radius r, but centres have changed ok. So, all closed disc with radius r, they are homeomorphic to each other under translations.

Similarly, now you can the centre at the origin 0. Use multiplication by arbitrary positive scalars, $M_{s/r}$. Here then it will be dividing by r and multiplying by s. So, the radius r will become s. So, $D_r(0)$ is mapped to $D_s(0)$. And vice versa if you multiply by r by s. That will be the inverse of $M_{s/r}$. Thus $D_r(x)$ and $D_s(y)$, they are all homeomorphic to each other because you can compose the two maps, first translate, multiply and then translate again and so on ok?

So, the conclusion is that any two discs any two open discs are homeomorphic, any two closed discs are homeomorphic, any two spheres are homeomorphic. So, first you understand what is going on here if you put equality everywhere you will get spheres ok, If you put strictly less than strictly less than s you will get everywhere open discs, less than equal to you will get closed discs.

So, argument is the same. Same maps T_{z-x} and $M_{s/r}$ will work for all of them ok. So, they are all homeomorphic to each other ok. By the way all these are defined on the entire of $\mathbb{R}^n$ this these functions these homeomorphisms are defined on the entire of $\mathbb{R}^n$, just to see that this is homeomorphic this one we may not need it there may be many other homeomorphisms, but this is easy way of seeing that they are homeomorphic to each other ok?

(Refer Slide Time: 07:53)



Now, I come to another important one namely from the whole of $\mathbb{R}^n$ to the open ball of radius 1 centered at 0. I have this homeomorphism what is it is, $x/(1+\|x\|_2)^{1/2}$, this is again l_2 norm I have taken 1 plus that, then I have taken the square root here, raised to half ok?

You can directly check that this map here y going to $y/(1-\|y\|_2)^{1/2}$, this is the inverse of that. This map is defined only on the open ball strictly on the open ball because as soon as the $\|y\|_2^2$ is 1 this is not defined you have $1-\|y\|_2^2$ would be 0 ok. So, that indicates that as y approaches the boundary, this value goes to infinity ok?

So, the boundary points are not there in the domain. So, this is the inverse of that. We can check that directly. So, that will tell you that $\mathbb{R}^n$ is homeomorphic to $B_1(0)$. You combine it with whatever we have seen here all open balls are homeomorphic to the whole of $\mathbb{R}^n$. It is very important here that you take the open ball and not the closed ball ok? $B_1(0)$ not $D_1(0)$ ok? So, $B_1(0)$ I have notation here, but we have no we did not have the notation over there ok.

(Refer Slide Time: 09:54)

Now, I come to different subsets. Instead of round spheres and discs and so on, now I will take squares, rectangles etc. ok? All these things I call, in general in $\mathbb{R}^n$, as boxes. So, in this terminology an open interval is a box, open interval cross open interval is another box, open interval cross open interval cross open intervals is the usual box in $\mathbb{R}^3$, but I am using the same word box for all n ok? So, i ranges from 1 to n, (a_i, b_i) open these are all open ball open boxes ok?

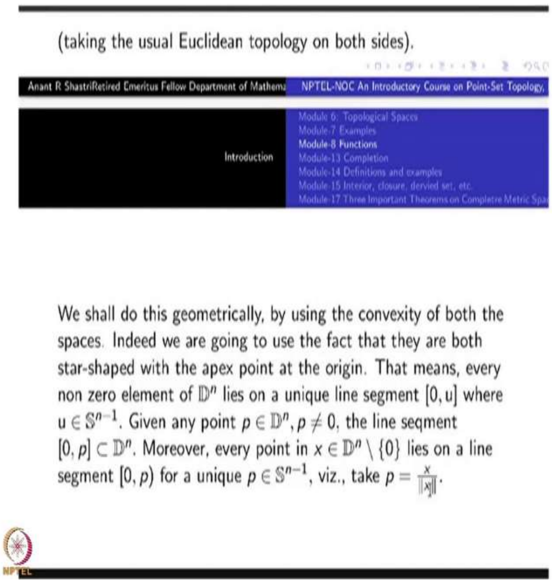
They happen to be open subsets of $\mathbb{R}^n$ also. That is very easy to see. Similar to what we have seen for any open ball. Take any point inside that. At that point you can give an open ball completely contained inside the original ball, centered this time at the point you have chosen. Similarly, for this open boxes also you can do the same thing. Therefore, these are all open subsets first of all ok?

So, what I want to do is that any box like this any two of them, they are homeomorphic to each other. So, first of all you should check that we have already seen this when n is equal to 1. Any two open intervals are homeomorphic to each other. Remember that ok? So, you can use that put several of this (a_i, b_i) to say (c_i, d_i) here ok, then take the product of this $f_i's$ which are homeomorphisms from (a_i, b_i) to (c_i, d_i) that will give you homeomorphism. Ok?

So, instead of that I am going to write just one case. Namely, this is just the way we have done earlier also namely from $(-1,1)$ to (a_i, b_i) you write down this one f_i ok then take product of $f_i's$. So, if all of them are homeomorphic to this $\mathbb{J}^n$ this is another standard notation I am going to use namely interval $(-1,1)$ taken n times product taken n times ok. So, if that is homeomorphic to any open interval any open box like this any two open boxes will themselves be homeomorphic to each other ok.

So, finally, I want to say that open boxes and open balls closed boxes and closed balls they will all be homeomorphic to each other ok. So, I am going to give you a homeomorphism from $\mathbb{J}^n$ to $B_n(0)$. If you put $\mathbb{D}^n$ it will be only into the interior because $\mathbb{J}^n$, I am taking this as open here ok. Of course, I am taking usual topology on both sides the Euclidean topology on both sides ok I would not define a homeomorphism here, but this should be the $B_1(0)$.

(Refer Slide Time: 14:00)



So, we shall do this geometrically, illustrated diagramatically. You can write down a formula also there is no problem by using convexity of both the spaces. So, this method will be used later on for different things also ok? What is the meaning of convexity? Inside any vector space, a subset is called convex if given any two vectors inside that subset t times this vector plus $(1-t)$ times that vector, which is the line segment joining any of the vectors, the entire line segment must be inside the subset. That is the meaning of convexity ok? I am just recalling it. what we are going to do is use the fact that they are both star shaped with the apex point at the origin. Star shaped means little less than convexity.

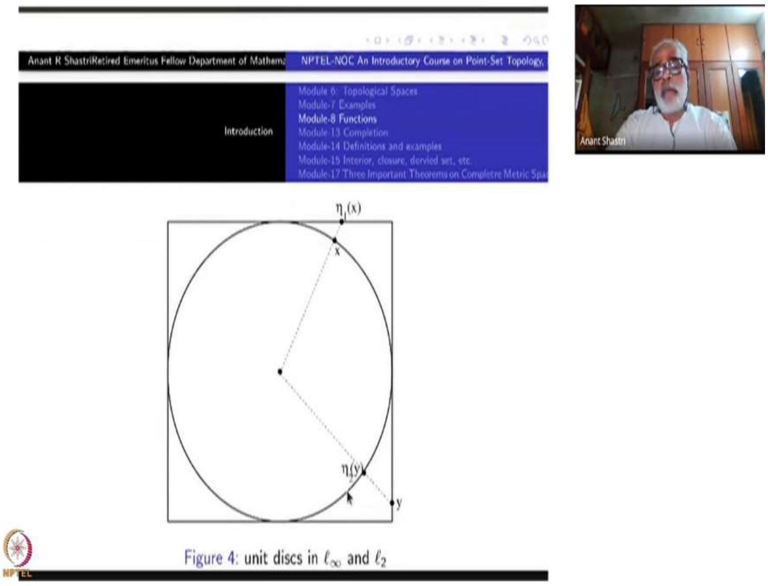
Here one of the point is fixed that is called the star. From there all the line segments which are emerging from that point to any of the point of the given set that is the entire line segment, must be there inside the set. Like, if you take the union of $(-1,1)$ open interval along with $(-1,1)$ on the $y-$axis OR the union of the entire $x-$axis and $y-$axis, these are not convex sets, but they are star-shaped at the origin ok? So, that is star shaped. Definition of star shaped with an apex point at the origin.

That means, I am writing it down here, every non zero element of $\mathbb{D}^n$ lies on a unique line segment $[0,u]$, where this u is a unit vector ok? So, this is the property I am going to use. this property can be termed as star shaped, if you do not understand this term, the property will tell you what I am going to use. So, that is enough.

Given any point p in $\mathbb{D}^n$, p not equal to 0, there is a unique line segment, the line segment $[0,p]$ will do, ok, that is contained inside $\mathbb{D}^n$ because for all these points the norm will be less than 1 the distance between 0 and that point will be less than 1. So, that is the criteria here, ok.

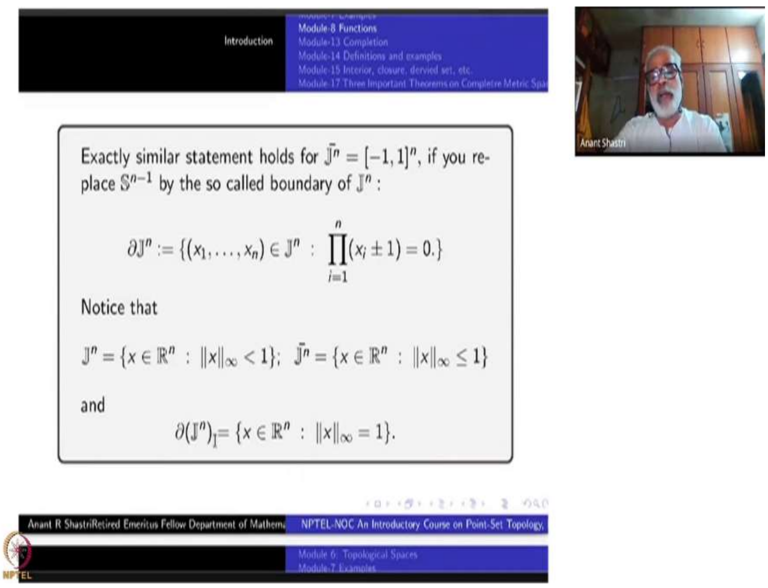
Indeed, every point x not equal to zero, lies on a line segment like this ok, for a unique p belonging to $\mathbb{S}^{n-1}$. Namely, take p equal to $x/\|x\|_2$ ok? If you take $x/\|x\|_2$, what is the norm of this, the ℓ_2 norm? It will be equal to 1. So, that is a point of S^{n-1} ok? Then the segment $[0,p]$ will contain this point x because to come back to x what you have to do you have to multiply by its norm. x is equal to $\|x\|_2 p$.

(Refer Slide Time: 17:56)



So, here is a picture ok? I am going to use this later on. So, I will come back to this picture a little later on ok.

(Refer Slide Time: 18:08)



So, what I want to do is now the same statement which I did for the sphere I can do it for the rectangle the closed rectangle ok. Take a point on the boundary of the rectangle join it to the center, every point inside this rectangle other than 0 will actually lie a unique line segment like that.

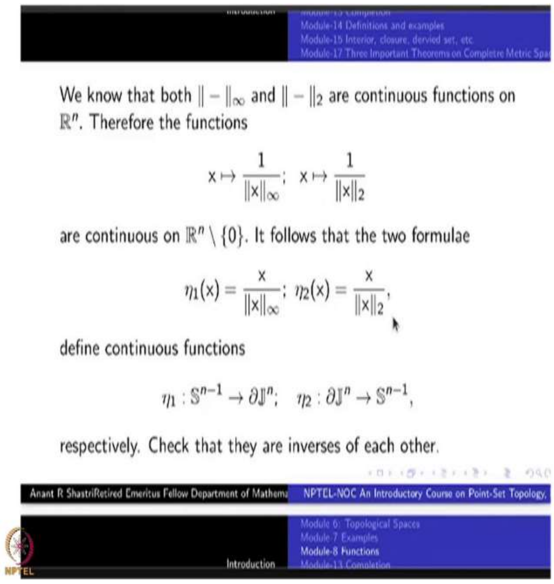
So, that is clear ok? Exactly same way you can do that, for the $\bar{\mathbb{J}}^n$. this $\bar{\mathbb{J}}^n$ is $[-1,1]$ raised to n ok that is the closure this I am taking closed the closed intervals here. If you replace $\mathbb{S}^{n-1}$ by the so called boundary of $\mathbb{J}^n$, what is the boundary of $\mathbb{J}^n$? See in $\mathbb{R}^2$, the boundary consists of this line, this line, this line and that line right? the 4 sides.

So, how do you define that, here the x_2 coordinate is -1 right, here x_2 coordinate is 1, here the x_1 coordinate is 1, here x_1 coordinate is -1. So, that is what you have to do, one of the coordinates is 1 or -1. So, that is what you have to do.

So, x_i is 1 or x_i is -1, product of $x_i \pm 1$ is 0 take the product that is 0 which is same thing as one of them at least must be 0, which is same thing as x_i is 1 or x_i is -1. ok that is the boundary of this $\mathbb{J}^n$ this is the boundary of $\bar{\mathbb{J}}^n$ also ok. Even if you do not know the word boundary in general does not matter this is the definition here for the boundary of $\mathbb{J}^n$.

Notice that $\mathbb{J}^n$ can be defined in a different way all points x belonging to $\mathbb{R}^n$ such that the maximum norm is less than 1 ok? This rectangle here in $\mathbb{R}^2$ can be defined as $[-1,1] \times [-1.1]$ or we can just defined it as all (x, y) such that the maximum of $|x|$ and $|y|$ is less than 1. So, that is precisely the maximum norm here ok and $\bar{\mathbb{J}}^n$ is all those points, which have their maximum norm less than or equal to 1. The boundary is all those in which maximum norm is exactly equal to 1, ok. At least one coordinate must be exactly equal to 1, I mean modulus of that coordinate. So, it could be 1 or -1 that is precisely what I told here how to how to get this $\mathbb{J}^n$ which I have got it here ok. So, these are some elementary descriptions of some interesting subspaces of $\mathbb{R}^n$, ok.

(Refer Slide Time: 21:48)



So, one thing which you can observe is the l_∞ norm taking the maximum of modulus of $x_1, x_2, \ldots, x_n$ or taking the square root of the sum of the squares etc both of these are continuous functions in $\mathbb{R}^n$. What is the meaning of $\mathbb{R}^n$ here with the usual topology and these are mapping into $(0, \infty)$ inside $\mathbb{R}$. So, there also, I am taking the standard topology with that these are continuous functions ok?

Because each modulus of x_i is continuous taking the maximum of continuous functions. That is what it is. And here, it is the difference is continuous, taking the square is continuous, taking the sum is continuous, again taking the square root is also continuous because these are non-negative ok? Wherever you are taking the square root, make sure that the inside thing is a non-negative quantity. So, this part is also continuous ok?

Now, look at these fraction here: $1/\|x\|_\infty$, $1/\|x\|_2$, ok? These things are continuous wherever the denominator is not 0, you take a function f which is continuous $1/f$ will be continuous wherever f is not 0. That is a fact I am using here from elementary real analysis ok. These are real valued functions x going to $1/\|x\|_\infty$ or $1/\|x\|_2$, they are continuous on $\mathbb{R}^n \setminus \{0\}$ because at 0 they are 0, everywhere else they are positive.

So, it follows that the following two formula $\eta_1(x)$ which is $x/\|x\|_\infty$, now I am multiplying this by x, x is continuous. So, multiplication of these two that will be continuous, $\eta_2(x)$ similarly is $x/\|x\|_2$ that is also continuous. So, they define continuous functions now where are they, I start from the unit sphere then I take η_1, unit sphere means norm of x is 1 the $\|x\|_2$ is 1, but it is 2 norm and it is $\| - \|_\infty$ may be something different I have divided into ball.

That means what if I take the l_∞ norm of this one now it will be equal to 1 therefore, it will be in the boundary of $\mathbb{J}^n$ similarly here start with anything on the boundary ok it is ℓ_2 norm could be different. So, divide by that now we should take the ℓ_2 norm of this whole thing it will be equal to 1. So, you have landed inside $\mathbb{S}^{n-1}$ here.

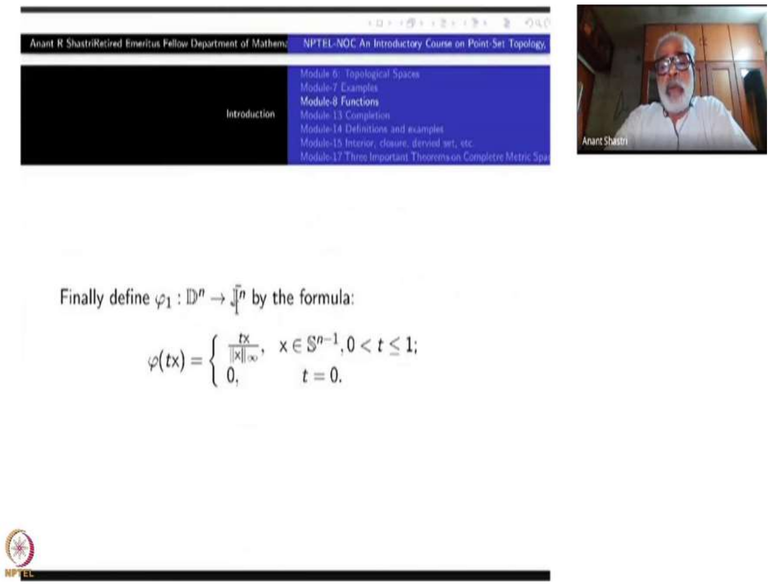
So, these two maps are continuous ok what I want to say is it is elementary geometrically to see that they are inverses of each other. In fact, take η_1 of this ok, now what is η_2 of this one, I have take whatever here and divide it by it is ℓ_2 norm ok, this is some real number. So, ℓ_2 norm of this is ℓ_2 norm of the numerator divided by this number ok, but what is the ℓ_2 norm of this one.

See, I want to say that it take η_1 of whatever from starting $\mathbb{S}^{n-1}$ then apply η_2 on that one you would comeback to $\mathbb{S}^{n-1}$ ok. So, we want to say x goes to x ok when you divide anything by ℓ_2 norm it will be come back to x because norm of x is already equal to norm of ℓ_2 norm of x was already equal to the 1 here, ok.

So, here is the picture that is what I wanted to say picture demonstration of this one. Take a point on the sphere unit sphere ok extend this line segment so that it hits the boundary of $\mathbb{J}^n$. How do you get that, on this line we have to take only one element there is only one element in the same line segment in the same ray the positive ray which has its l_∞ norm equal to 1 and that is this point ok because there is a unique point which will intersect this one.

So, x goes to $\eta_1(x)$, now start with this one if you divide it by it is ℓ_2 norm we got back this one because that is the only point with ℓ_2 norm equal to 1 this is the only point. So, that is the inverse of this. So, η_2 of this point would have been this one. So, this is this is why η_2 of that one is this one ok. So, geometrically it is very easy to see, but we can also do this through formula ok. So, from the boundary to the boundary I have a homeomorphism here because they are inverses of each other ok.

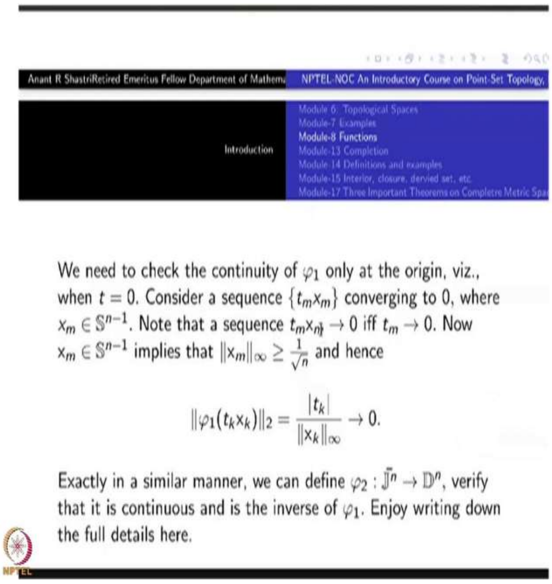
(Refer Slide Time: 27:51)



Now, what I do, I look at ϕ_1 from the interior the entire thing $\mathbb{D}^n$ to $\bar{\mathbb{J}}^n$. So, I define this by a formula $\phi(tx)$ remember every point here looks like tx, where x is a point on the boundary right, except when the scalar $t = 0$, this x will be an indeterminate, you could have taken any point x. Otherwise it is a unique point. And every element can be written as tx, for $x \in \mathbb{S}^{n-1}$. I do not want the 0 element here. So, $0 < t \leq 1$, ok, 0 strictly less than t less than equal to 1. So, this will give you all points other than 0, those points you take $tx/\|x\|_\infty$. At point 0, we have to define it in a different way. We define it as 0.

So, this is nothing but, remember it is t times $\eta_1(x)$. So, I have defined this one by extending this η_1, I am getting ϕ_1, extending how? By putting 0 going to 0 ok. The problem is that whenever you define a function by two different formula you have to verify the continuity. Here, this is only at one point I have to worry about. By taking the limit of this as t tends to 0 its limit is the same value here. Therefore the function is continuous ok?

(Refer Slide Time: 29:56)



So, that is what I am using here, elementary calculus. So, we need to check the continuity of ϕ_1 only at the origin ok, elsewhere it is given by this formula which is continuous already we have we have seen. So, at the origin what you want to do, consider a sequence $t_m x_m$ converging to 0. See x_m's are inside $\mathbb{S}^{n-1}$ they are not fixed, they may be different points ok? but t_m's are real numbers when does this sequence converge to 0? It converges to 0 if and only if t_m converges to 0, ok?

Because norm of x_m is equal to 1 always, ok, It is a bounded sequence this will be 0 only if t_m goes to 0 ok. These are inside $\mathbb{S}^{n-1}$, they may not converge to any point they may be converging if at all inside $\mathbb{S}^{n-1}$ if and only if t_m is 0.

Therefore, since x_m is $\mathbb{S}^{n-1}$ this implies $\|x_m\|_\infty$, you see this infinite norm of x_m is bigger than $1/\sqrt{n}$, ok? There are n coordinates each of them is less than $1/\sqrt{n}$, the square of the sum total will be less 1, right.

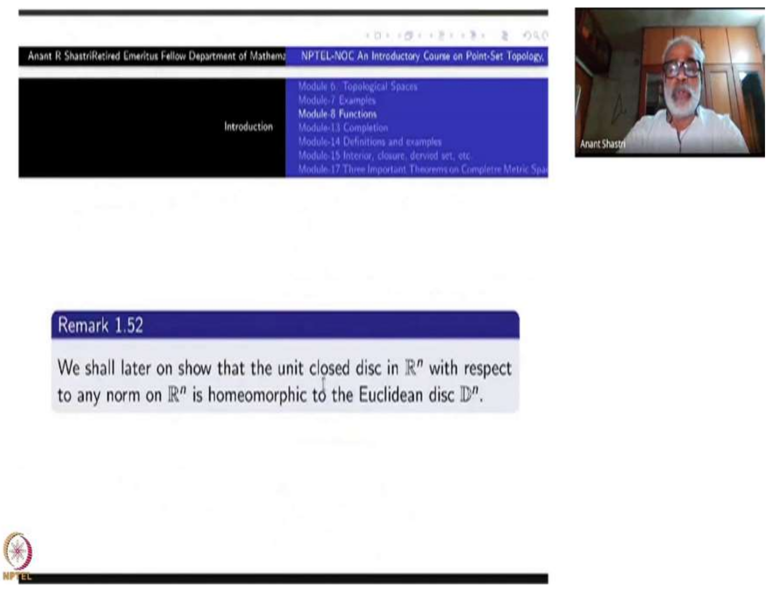
So, at least one of the coordinate must be bigger than or equal to $1/\sqrt{n}$ once one of the coordinate is bigger the infinite norm will be bigger. So, that is all I am using. So, the infinite norms are all bounded away from 0 this n is fixed here by the way, the m is the one which is changing here. The sequence is with respect to the variable m, ok?

And take $\phi_1(t_m x_m)$, take its ℓ_2 norm, by this definition, it is the ℓ_2 norm of $t_m x_m$ divided by l infinity norm of x_m, which is bounded below by $1/\sqrt{n}$. So, divided by that one will be bounded by less than or equal to $\sqrt{n}$.

So, it is bounded therefore, when t_k tends to 0 this goes to 0, ok? $|t_k|$ by something in the denominator which bigger than $1/\sqrt{n}$. Therefore, 1 divided by that is less than or equal to $\sqrt{n}$. Exactly similarly we can define ϕ_2 as an extension of η_2 here ok? And verify that it is continuous the same proof we will work there and it will be inverse of ϕ_1. That is all. Over. OK? Enjoy writing down the full details here alright?

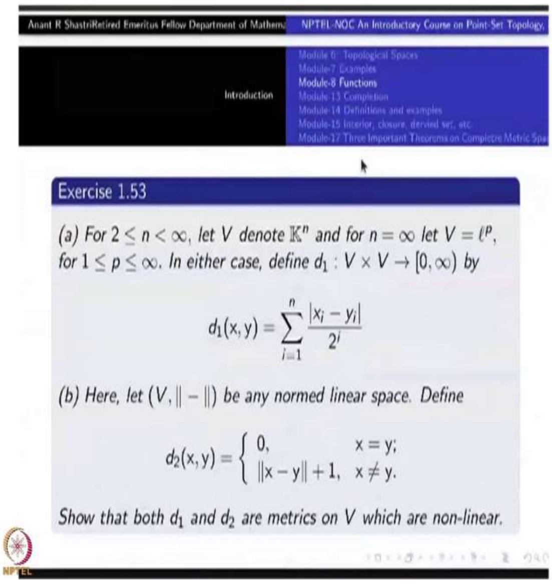
So, this will give you a homeomorphism from square to the circle ok, the entire square including this one into the disc and then generalise this one, this is the picture for n equal to 2, but generalise this one. I have never used that this n equal to 2 only in the illustration, in the diagram it is there ok? So, all thing these things are varied for all $\mathbb{R}^n$ alright.

(Refer Slide Time: 33:56)

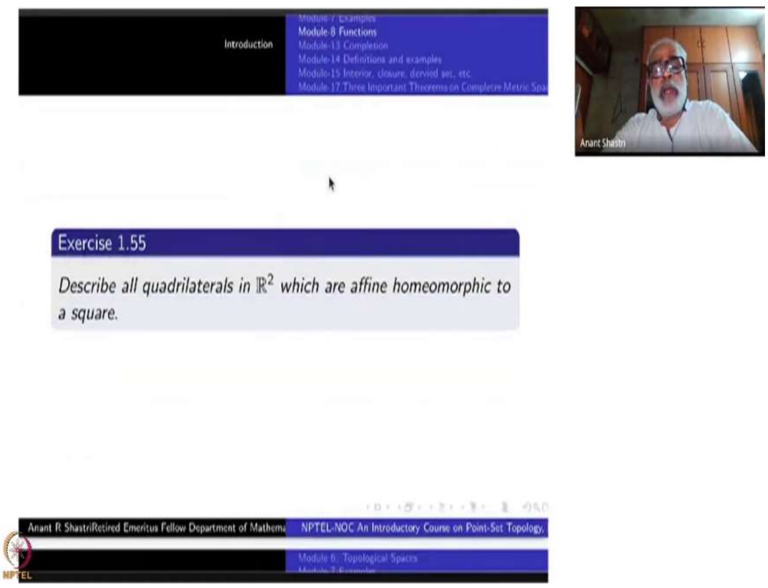


We shall later on show that the unit disc in $\mathbb{R}^n$ with respect any norm, remember we have several norms here all these ℓ_p norms and so on right? This is homeomorphic to the unit disc in the in the $\mathbb{D}^n$ means what now the ℓ_2 norm ok. The Euclidean norm here they are all homeomorphic to each other, but that will take time we will do it next time, maybe little later not next time.

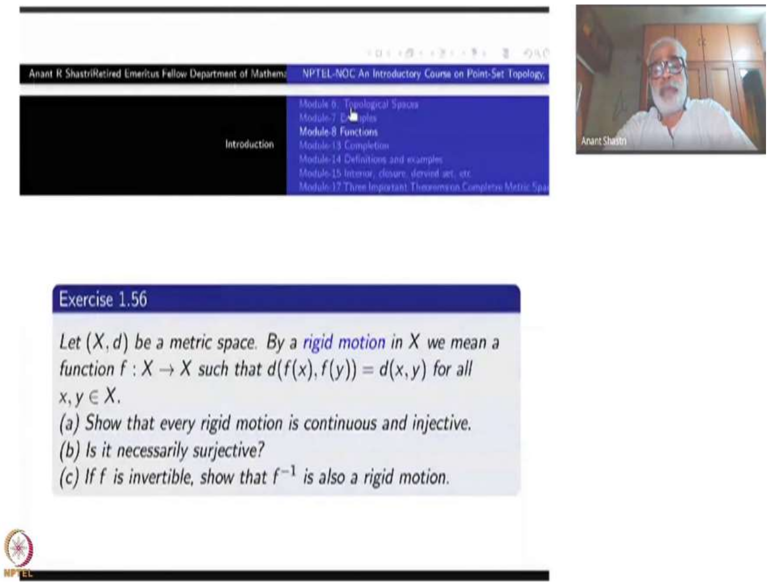
(Refer Slide Time: 34:35)



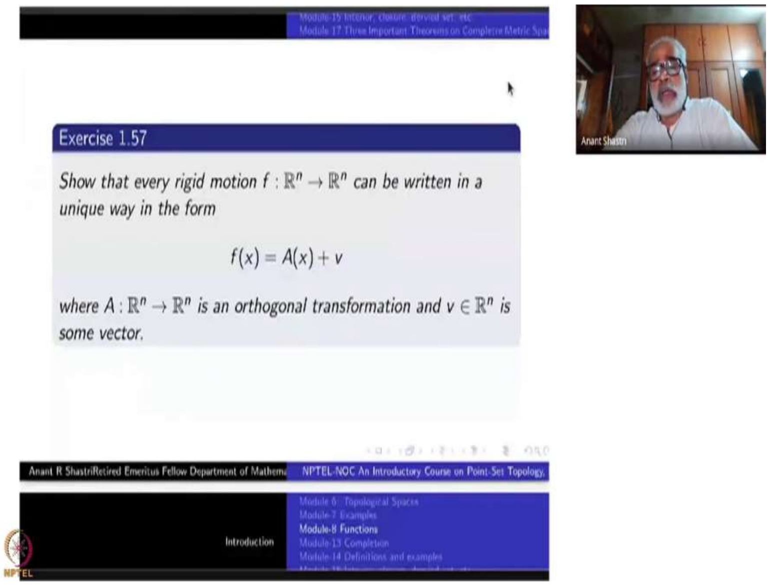
(Refer Slide Time: 34:40)



(Refer Slide Time: 34:48)



(Refer Slide Time: 34:53)



There are some exercise here which you can go through by yourself, they will be given to you as they will appear as assignments now. Also they will be in the notes. So, you do not have to worry about that, you try your hand at these exercises ok? So, that is it.

Thank you.