Introduction to Point Set Topology, (Part I)
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Module - 08
Lecture - 08
Functions

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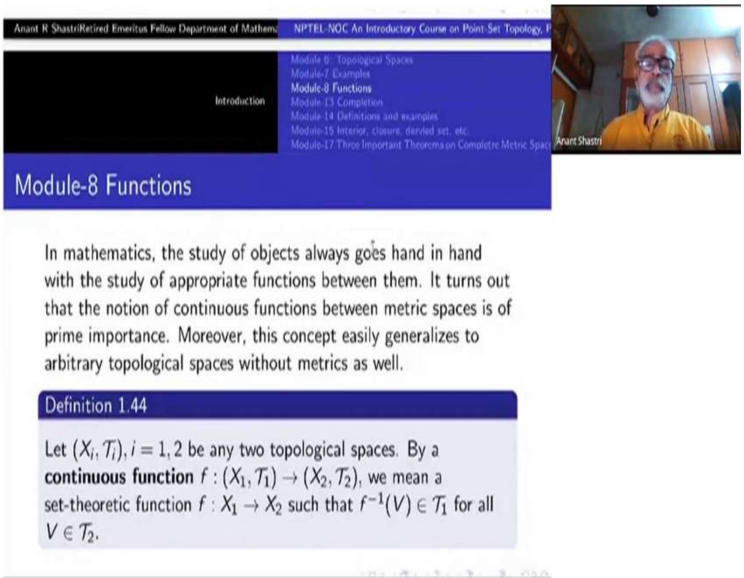

Welcome to module 8 of Point Set Topology course. So, in mathematics the study of objects always goes hand in hand with the study of appropriate functions between them. Like, when you are studying vector spaces you take linear maps, when you are studying groups you will take group homomorphisms, right. Like this, when you are studying metric spaces we started with continuous functions $\epsilon - \delta$ definitions.

So, we want to extend that definition to encompass all topological spaces now, ok. So, the new definition that we are going to make always should encompass the older definitions, ok that must be the motivation of keeping this and it should give you more. So, that is the whole idea. So, let us make this definition first, namely of continuous functions between two topological spaces.

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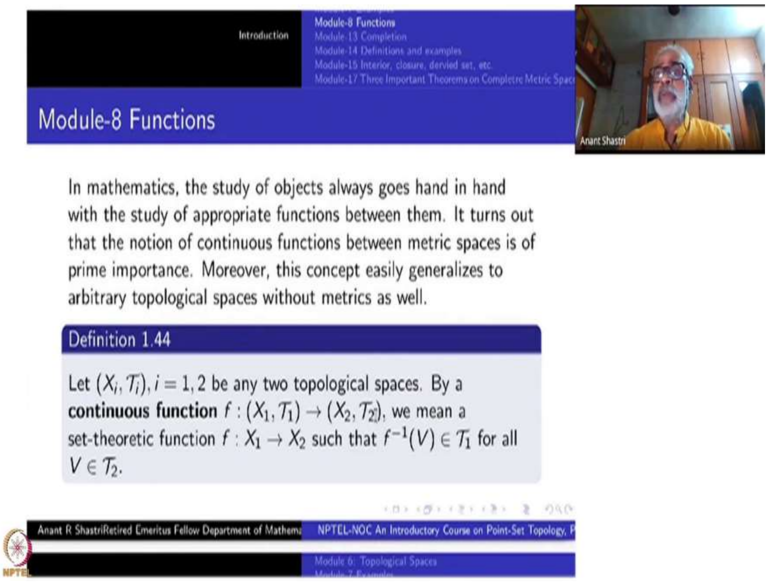

X 1 tau 1, X 2 tau 2 are two topological spaces. A function from X 1 to X 2 that is a set theoretic function, it will be said to be continuous, ok. So, it will be called continuous function provided inverse image of every open set in tau 2 namely in X 2 should be open in X 1.V is in tau 2, f inverse of V must be in tau 1, ok.

So, for the first time you may see that why things are happening the other way round, but soon you will realize that this is most natural way to define. It is not open sets equals to open sets, inverse image of an open set is open. So, that is the most natural thing. There is a concept which this open set goes to open set that becomes a subsidiary concept which is not so important as continuous functions, ok.

So, there is another one also. So, we will come to that one later. This is the correct thing in terms of epsilon delta definitions of our metric spaces. So, let us see how that is true, ok. So, that will be justification for making such a definition in the case of topological spaces.

This tau 1 and tau 2 are just topological spaces they might not have come from any metric, but suppose they come from a metric, then you have two different definitions. One is for this one whatever you have given just now as topology, but there is already something continuity coming from metric definition.

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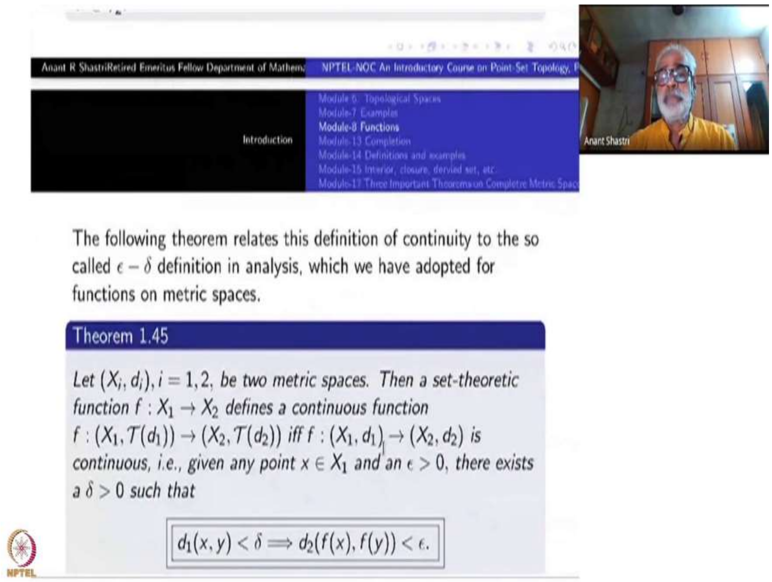

$(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ are two topological spaces. A function from X_1 to X_2 that is a set theoretic function, it will be said to be continuous, ok, it will be called a continuous function provided inverse image of every open set in $\mathcal{T}_2$ namely in X_2 should be open in X_1. V is in $\mathcal{T}_2$ should imply $f^{-1}(V)$ must be in $\mathcal{T}_1$, ok.

So, for the first time you may see that why things are happening the other way round, but soon you will realize that this is most natural way to define. It is not open sets going to open sets, inverse image of an open set is open. So, that is the most natural thing.

There is also a concept in which this `open set goes to open set' becomes a subsidiary concept which is not so important as continuous functions, ok. So, there is another one also. So, we will come to that one later. This is the correct thing in terms of $\epsilon - \delta$ definitions of our metric spaces. So, let us see how that is true, ok. So, that will be a justification for making such a definition in the case of topological spaces.

This $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ are just topological spaces they might not have come from any metric, but suppose they come from a metric, then you have two different definitions. One is for this one whatever you have given just now, a topological definition, but there is already something continuity coming from metric definition.

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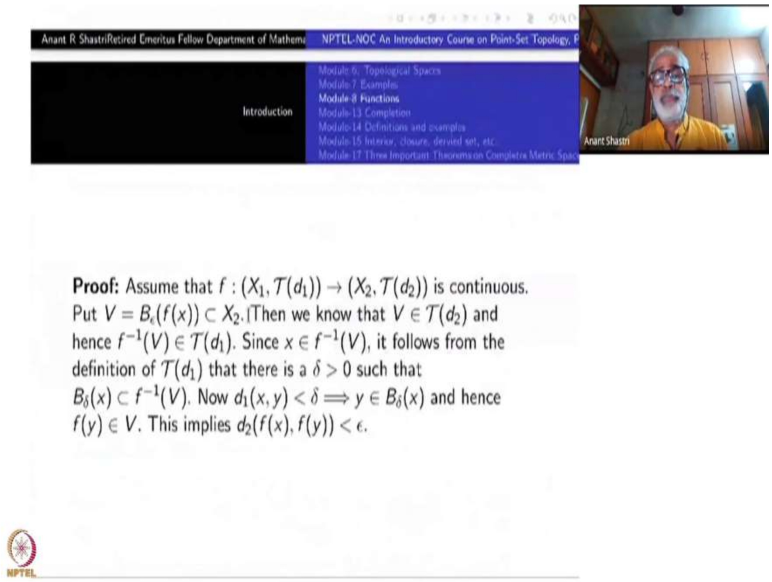

So, what we want to say is that these two things are coinciding, ok. So, that is the next theorem here. Let (X_1, d_1) and (X_2, d_2) be two metric spaces. Take a set theoretic function from X_1 to X_2 as usual. that will be a continuous function in the topology $\mathcal{T}(d_1)$ to $\mathcal{T}(d_2)$ here, now, these are topological spaces, if and only if as on metric spaces f from (X_1, d_1) to (X_2, d_2) it is continuous. Namely, there it is $\epsilon - \delta$ continuous at every point. Given any point x belonging to X_1 and an $\epsilon > 0$, there must exist a δ such that $d_1(x, y) < \delta$ implies $d_2(f(x), f(y)) < \epsilon$.

So, this, I am just recalling, I am not making yet another definition here, of course. So, this is the $\epsilon - \delta$ definition for a function between metric spaces. It is equivalent to the continuity of the same function on the corresponding topologies induced by the metric, ok. So, once we prove this one there will be a full justification for the new definition, alright.

So, let us just do this one which is not at all difficult. Assume that f from $(X_1, \mathcal{T}(d_1))$ to $(X_2, \mathcal{T}(d_2))$ is continuous according to the new definition, ok. Now I want to prove it for $\epsilon - \delta$. So, x in X_1 is given, epsilon is given, right. Look at $B_\epsilon(f(x))$, that is an open subset say V, inside X_2, right? This $B_\epsilon(f(x))$ is the member of $\mathcal{T}(d_2)$.

So, f inverse of that must be inside $\mathcal{T}(d_1)$. So, in $\mathcal{T}(d_1)$ it is open, but now look here, I have taken $f(x)$, here $f(x)$ is a point here. So, in the inverse image x will be there. Therefore, x is in $f^{-1}(V)$ and $f^{-1}(V)$ is open so, it follows that by the definition of this topology, that there is a $\delta > 0$ such that the δ ball around x, this is an open ball is contained in $f^{-1}(V)$, right?

Because, $f^{-1}(V)$ is the union of open balls inside this metric space $\mathcal{T}(d_1)$ in the metric space (X_1, d_1), alright. So, it must be union of such balls, $B_\delta(x)$ must be in contained in $f^{-1}(V)$.

Now, $d_1(x, y)$ is less than δ implies y is in $B_\delta(x)$, right. $d_1(x, y) < \delta$ means is y is inside here. So, y is inside $f^{-1}(V)$ means $f(y)$ is inside V. Now $f(y)$ inside V means what? Look at this one this V is the same $B_\delta(f(x))$, $f(y)$ is inside V means $d_2(f(x), f(y))$ must be less than ϵ. One way is done.

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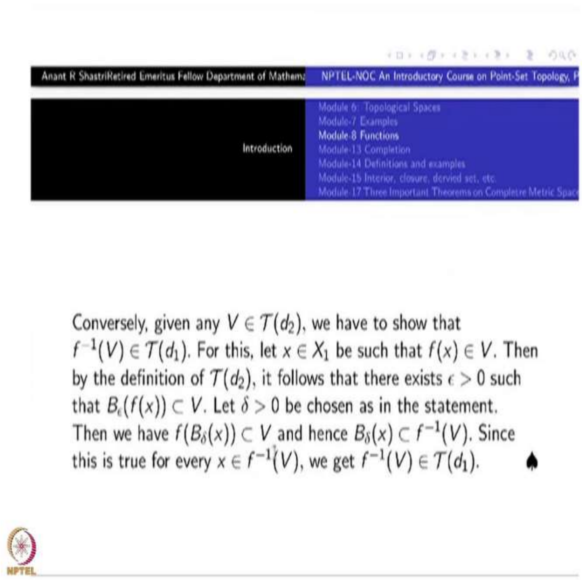

Conversely, start with any V inside $\mathcal{T}(d_2)$, we have to show that $f^{-1}(V)$ is open namely it is inside $\mathcal{T}(d_1)$, is what you have to show. So, to show that this is inside $\mathcal{T}(d_1)$ take any point x inside X which is inside $f^{-1}(V)$; that means what $f(x)$ is inside V.

But, V is open and $f(x)$ inside V means, you must have an $\epsilon > 0$ such that $B_\epsilon(f(x))$ is contained inside V. Therefore, there is a δ positive such that, now that is the ϵ definition as in a statement; that means what difference between x and y is less than δ would imply $f(y)$ is inside this open ball, ok.

So, chose such a δ then $f(B_\delta(x))$ is inside V. Therefore, $B_\delta(x)$ is inside $f^{-1}(V)$. So, this is true for every x. Therefore, $f^{-1}(V)$ is open in X_1, namely, it is in the element of $\mathcal{T}(d_1)$, ok? Go through this proof carefully.

So, here I have used the fundamental property of $\mathcal{T}(d_1)$ and $\mathcal{T}(d_2)$ that around every point inside an open set there is a ball around that which is contained inside that. So, this is the property I have used here, ok?

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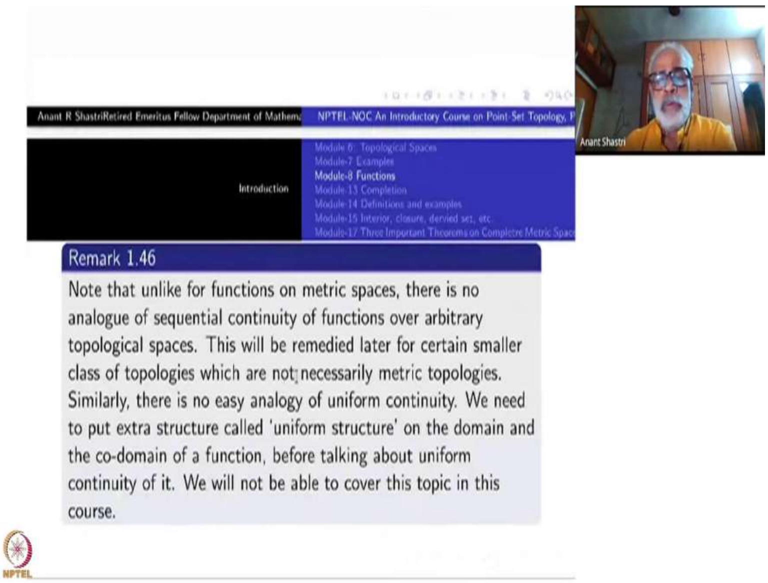

Now, one remark here is: unlike for the metric spaces, wherein we have the notion of uniform continuity we do not have such uniform continuity concept in topological spaces, in general, ok. There is no such notion except you have to work hard namely we will do that later on, if time permits, for some smaller class of topologies which are not necessarily metric topologies, if it is a metric topology of course we have uniform continuity, ok. We need to put an extra structure called uniform structure on the domain and co domain. So, they are not ordinary topological spaces, but satisfying some special conditions, ok? Indeed, that will not be done in this course. Uniform continuity is not a main thing, it is a side topic. So, we will not have time for that.

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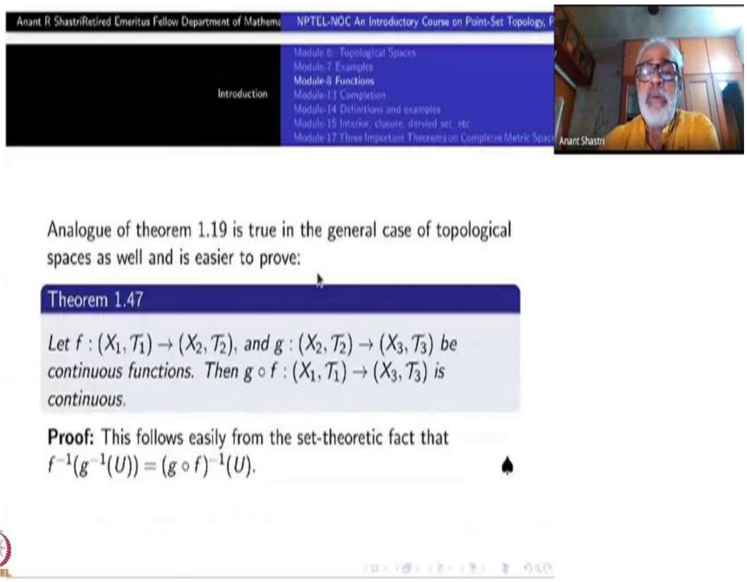

Analogue of theorem 1.19 is true in the general case of topological spaces as well and it is easier, this 1.19 is nothing but the theorem on composites, ok.

So, this remark was as a negative thing that is all, this remark was a somewhat in a negative tone, but rest of them will be now very happy things everything positive. So, composite of continuous functions is continuous is theorem 1.19 for metric spaces. The same thing is here true and proof is much easier now. See, it is much easier what you have what you have to do.

There is a function here, there is a function here the composite is there, right. Take an open set here inverse image open here it is what you want to show. Inverse image here under g first comes here, but that is an open set because g is continuous. Now, you take the inverse of that that will be the full inverse image of under $g \circ f$ of inverse ok; $(gf)^{-1}(U)$ is $g^{-1}(U)$ and then f inverse, right.

This is the set theoretic property; this is purely set theoretic property. So, U is open this is open, this is open f inverse of that is open, ok. So, it is easier to show that composite of two continuous function is continuous in the case of topological spaces, alright. Therefore, this also proves now whatever we proved for metric spaces. See we need not have proved that we have used that, ok.

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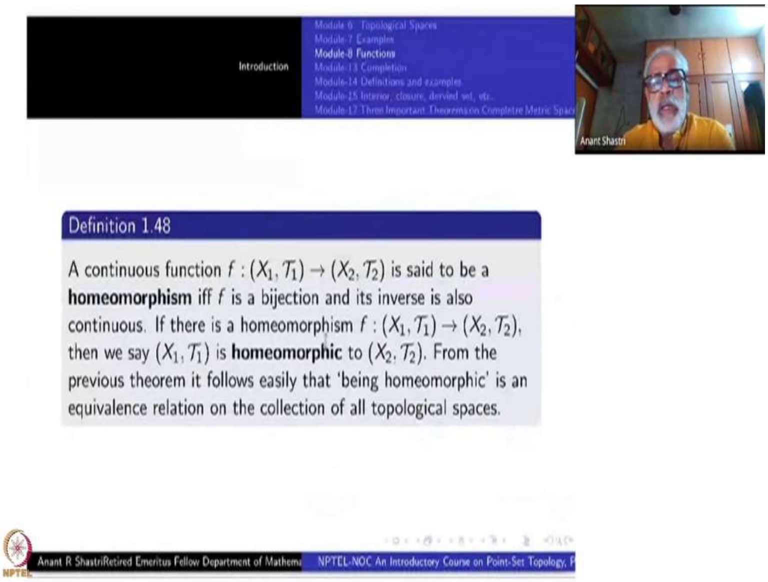

A continuous function $(X_1, \mathcal{T}_1)$ to $(X_2, \mathcal{T}_2)$ now arbitrary topological spaces, ok said to be a homeomorphism, this is the word I am going to use now, ok? What is a homeomorphism? f is a bijection and it is inverse is also continuous; f is a continuous function, it is a bijection therefore it has an inverse and that inverse is also continuous, ok?

Suppose, you have a homeomorphism from one topological space to another one; then the two topological spaces are called homeomorphic to each other. This is homeomorphic to this ok? From the previous theorem it follows easily that being homeomorphic is an equivalence relation on the collection of all topological spaces.

One thing is clear by the very definition of homeomorphism, inverse is also homeomorphism. There is no need to work, because it is a bijection, inverse is a bijection inverse is there inverse is continuous f is continuous. So, inverse of inverse of f is f itself. Therefore, if there is a homeomorphism like this, f^{-1} should be homeomorphic from the other way round. So, symmetry comes.

Any topological space is homeomorphic to itself because, identity map is always continuous and bijection of both ways from the same topological space to same topological space, ok. Identity map is always continuous no problem.

Finally, transitivity is what you have to prove. It is precisely this theorem. this one is homeomorphic this one this one is homeomorphic to that one X_1 to X_2 to X_3 so, X_1 to X_3 you will get, ok.

So, homeomorphism is a function, being homeomorphic is a relation on topological spaces. So, on topological spaces this in equivalence relation, ok. So, this equivalence relation is of profound interest to us. Here are examples.

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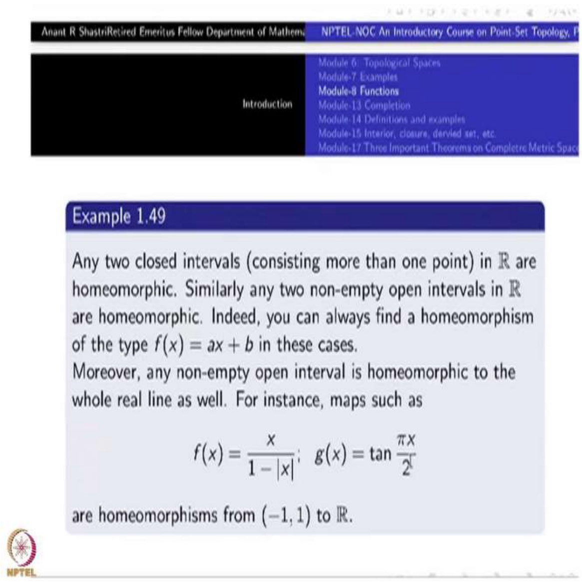

Any two closed intervals consisting of more than one point. A closed interval could be a singleton. So, you should avoid that. Singleton is a singleton. If there are more than one point, then that closed interval no matter what it is all of them are homeomorphic to each other this is the statement.

Similarly, any two non empty open intervals. If you take one of them empty and the other one nonempty, they will not be homeomorphic ok, empty set is never bijective with respect to any non emptyset ok. So, any two non empty open intervals in $\mathbb{R}$ are homeomorphic. So, there are many homeomorphisms. actually, but you can find something which is very nice namely of the type $f(x)$ equal to $ax + b$ is a linear polynomial, i.e., polynomial of degree one.

So, when is this a homeomorphism, it must have an inverse, right. So, it is very clear that a must be non zero, b could be anything. And then, you can write down it is inverse, these are linear maps, ok. For any non empty open interval is homeomorphic to the whole line itself, the interval is bounded, whole real line is not bounded, but still they are homeomorphic to each other. This is what one has to see and there are several ways you can see it. Some standard maps are the following.

Look at $f(x) = x/(1 - |x|)$. So, where I am going to define this? From the open interval $(-1, 1)$ to the whole of $\mathbb{R}$. Suppose, I have proved this one is a homeomorphism. Then, I know that any open interval finite like this will be homeomorphic to any open interval (a, b) by this method so, all of them are homeomorphic to the whole of $\mathbb{R}$, that is what I get.

So, how do you get this homeomorphism? Very easy. Look at its inverse it is nothing but $x/(1 - |x|)$, ok? You can compute it. The standard method is to put y equal to this and solve for x in terms of y. So, because there is a modulus you may have to make two different cases, x non negative and x negative, ok.

So, if x is positive what is this; this is $1 - x$ then you can find y equal to $x/(1 - x)$, you can rewrite it in terms of x equal to something purely in terms of y and so on. So, that is the way to check that this one is a homeomorphism. Easy way. Directly right down the formula for the inverse, ok.

Here is another one from trigonometry: $tan(\pi/2x), x$ is ranging from $(-1, 1)$. So, the domain when you put -1, it goes to $-\infty$, if 1, it will go to ∞, 0 goes to $0. tan$ is a strictly monotonically increasing function from $(-1, 1)$. Because, you can look at its derivative blah-blah-blah, it is trigonometry and some calculus you may have to do, ok? To show that it is stricty monotonic, then you can take the limits to see that both -1 and 1 go to $-\infty$ and ∞ respectively. Those points are not there in the domain. But, the entire open interval is there and the function all the values in $\mathbb{R}$. It is surjective map. Because, 1 goes to ∞ and -1 goes to $-\infty$, everything in between must be there by intermediate value theorem.

There is so many different things you can use to see why this is homeomorphism. I am telling you. That is why you can write tan^{-1}, it is just a justification for writing tan^{-1}, ok?

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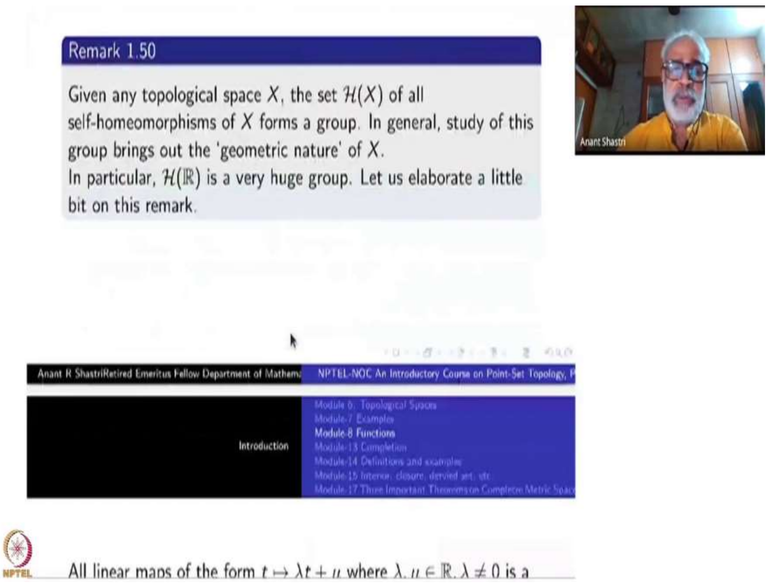

Given any topological space X you can look at all self homeomorphisms. Say from $\mathbb{R}$ to $\mathbb{R}$, or from $\mathbb{R}^2$ to $\mathbb{R}^2$ and so on. Take any topological space X, take a map from X to X which is continuous and a bijection so that its inverse is also continuous. Look at all them. You can compose two of them, again that will be a homeomorphism. You take the inverse and that is again a homeomorphism, identity map is always homeomorphism. These three things together constitute what? What do they make? They make a group; that is the definition of a group. In fact, the group of automorphisms of spaces and such things, they are the harbingers, they are the originators of group theory, ok.

The set of all self-homeomorphism forms of a space is a group. Unfortunately this group is too huge. Unlike in group theory, in the beginning you get to know you know small groups or nice groups like integers and so on, ok.

So, in general, the study of this group namely $\mathcal{H}(X)$, brings out the geometry inside X, ok. In fact, people have gone to the length of defining geometry as the study of the groups of homeomorphisms, groups of automorphisms, groups of isometrics and so on or sub groups of this groups.

What is happening in this group, that is a geometry, ok? So, this group is quite huge. Let me elaborate what is the meaning of this 'quite huge' a little bit ok? Such study cannot be completed in any semester course, ok?

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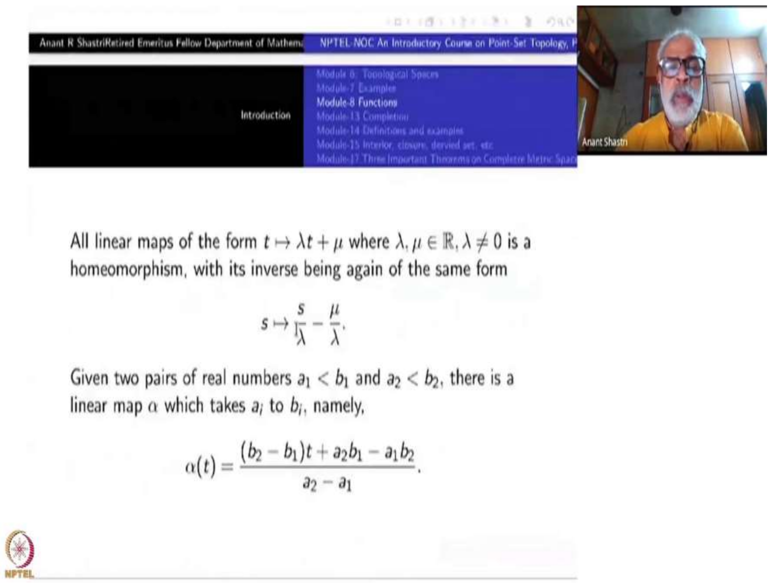

So, let us look at some examples here. Take all linear maps $t \to t + \mu$ or $\lambda t + \mu$, like $ax + b$. I have written there λ and μ are real numbers λ must not 0, that is all. That you need to assure that this map is invertible.

Now, I am writing the inverse also. Inverse will look like s going to $s/\lambda - \mu/\lambda$ we can check it that this is the inverse of that, ok. So, they are all there, they are all homeomorphisms of $\mathbb{R}$ with itself.

Given two pairs of real number $(a_1, b_1), (a_2, b_2)$ ok, a_1 less than b_1, a_2 less than b_2, or the other way round whichever way you want, you assume that. There is always a linear map alpha which sends a_1 to b_1 and a_2 to b_2. You just write down a linear map $\alpha x + \beta$ and solve for α and β by putting this condition.

So, that is that is third standard stuff. Solving two simultaneous linear equations. So, that will give you the formula for this function itself, ok? Namely this α is now the function which takes a_1 to b_1 and a_2 to b_2; check that there may be some errors here you should check that, ok?

So, if there is b_1 here in place of b_2 that is not all that serious error, we have to check that and correct it if at all. If it is correct it is fine. So, what is claimed is that you can solve for this αt is something like $at + bat + b$.

I am finding this a and b here. What is the condition, a_i should go to b_i, a_1 should go to b_1, a_2 should go to b_2 this is very straight forward, alright. Once we have done that this is already geometry. See I can find a map which is a bijection one one mapping, one one correspond from any interval to any other interval right; that has been this one now, ok. But, there is more than that ok, more geometry is coming out of this.

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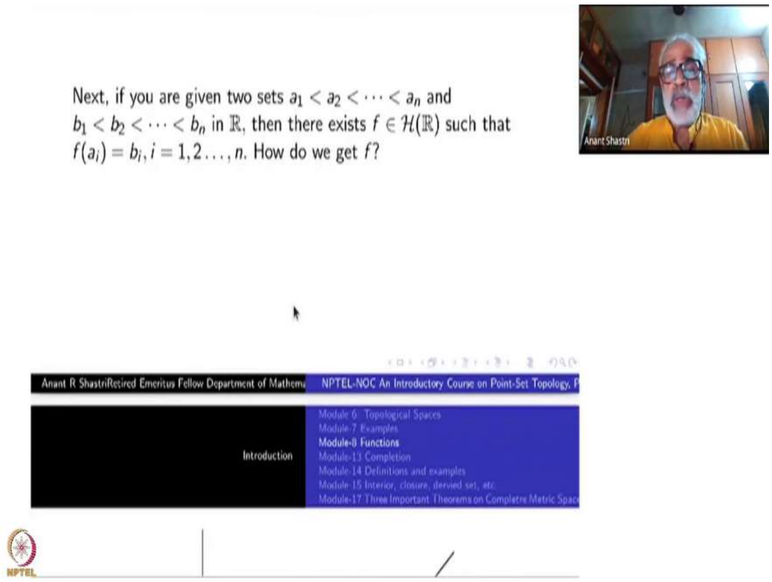

Take any set of real numbers finite set of real number, put them in an order. Take another set with the same number put them in the order. The order is important here. When we are taking only two of them even the order was not important. You could have mingled here a_1 going to b_2 and b_1 going to a_2 that is also possible here, but here it is important that you should have the same order.

Then, you can find a homeomorphism which takes a_1 to b_1, a_2 to $b_2, \ldots, a_n$ to b_n, ok? How do you get that? I will explain it to you. I do not want to write down the full formula. If you want you can write down. Actually I have written down that also, but first I will explain it to you.

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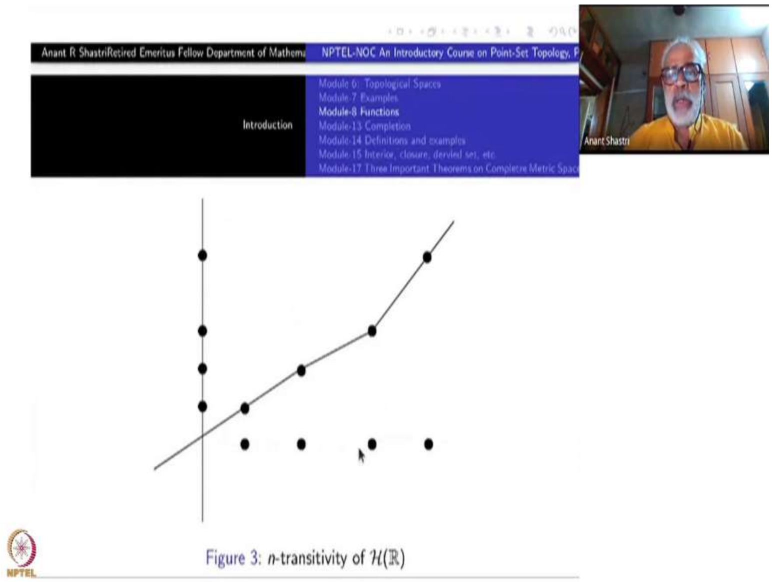

These are a_1, a_2, a_3, a_4 those are b_1, b_2, b_3, b_4 there, ok? I want a_1 to go to b_1 means graph of that the point would be here right this is a graph, a_2 goes to b_2. So, point would be here a_3 will go to b_3 point would be here like that.

So, these points I have been given then I have joined them by straight lines, because I know there is a linear map which takes a_i to b_i, a_{i+1} to b_{i+1}. Concentrate on each each interval here, two points at a time right; take two at a time in the order namely so, this interval, that interval, that interval.

First get the map which takes this one to this point and this one to this point so, this is this map and this is the line segment. Here there is no condition. So, extend the same line segment do not disturb it at all. Here there is no condition beyond that extend that line the same way all the way from last line, but in between join them by the line segments determined by those points.

This is a graph of the function. You can write down the formula for the function now no problem. So, in each a_i to b_i, there will be a different formula, ok? So, once you know this we can write down, I have written it down here you can check it; there may be some errors here from a_i may be we see in a_2 may be seen something i maybe become $i+1$, all that you have to check, ok.

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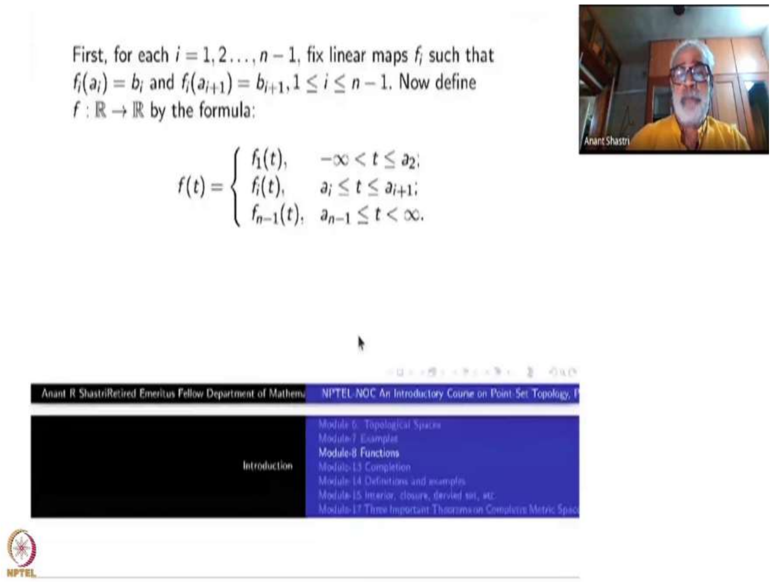

So this is not a big game, you have to know of course but, what you should now understand this, how to get this one, alright? So, I have defined the function in three different ways, but these middle ones give you all, between intervals. The last two gives you what you have to do namely do not worry this part we have to extend it as if we have defined here.

Similarly, whatever function comes here you extend it here. In between use the i^{th} function from i^{th} to $i+1$ point use this formula. So, that is what I have done here ok, $f_i(t)$ Where is $f_i(t)'s$ are there right, we send a_i to b_i. So, I have taken this f_i is function $f_i(a_i)$ to b_i and $f_i(a_{i+1})$ to b_{i+1}. For each i, there is an f_i, if you change i of course they will change right that is correspond to these having different slopes here, ok.

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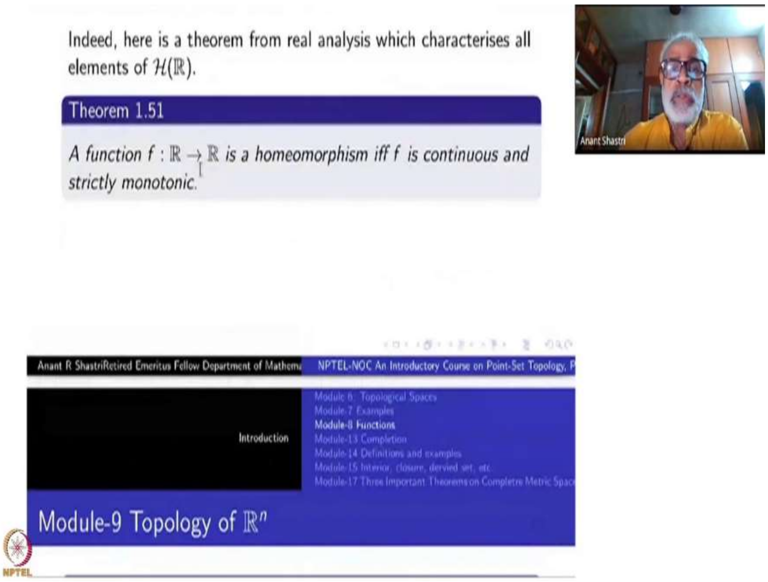

So, indeed here is a theorem from real analysis which characterizes all elements of $\mathcal{H}(R)$, the group of homeomorphisms. What are all the homeomorphisms? The characterization may not be much helpful, but it is quite helpful: a function f from $\mathbb{R}$ to $\mathbb{R}$ is a homeomorphism if and only if it is continuous and strictly monotonic. Strictly monotonic because, you want what? you want it to be one one mapping right? like that, ok.

It may not be on to, but if you put onto condition also it will be homeomorphisms onto $\mathbb{R}$, Otherwise, it will be homeomorphisms on to the image, ok. So, I am using this word here homeomorphism, in a slightly more general sense that is all. See I have not put ontoness here I should put ontoness here also, ok. Yeah. So, let us stop here today, in next time we will do more on not just on $\mathbb{R}$, but now$\mathbb{R}^n$ that will be the next topic.

Thank you.