Introduction to Point Set Topology, (Part I)
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Module - 07
Lecture - 07
Examples

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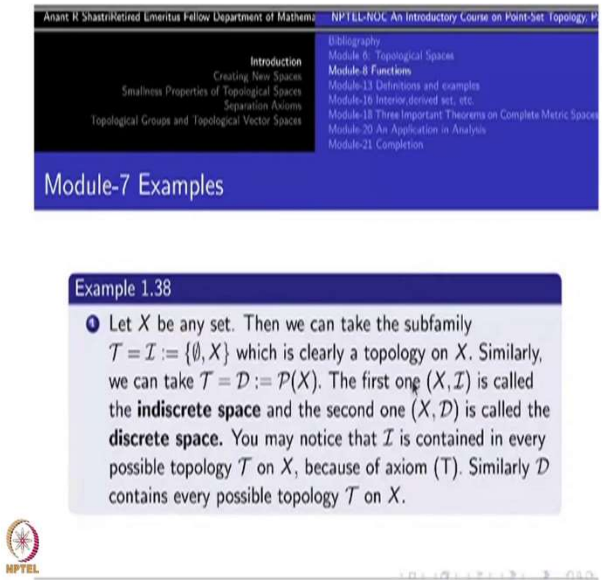

Welcome to module 7. As promise last time let us now take up one by one a few examples of topological spaces directly, which may or may not arise from a metric space. We shall begin with a few examples of topologies which do not arise naturally as metric topologies ok. So, that may be the end of today's lecture. Just some more examples.

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Let X be any set then we take a subfamily namely consisting of empty set and X. We cannot do more stringently than that because the first one first axiom says empty set and X must be there right. So, put that and do not do anything more that is it. The other two conditions are automatically satisfied.

So, what is subfamily of this? Either it is just empty or just X or both X. What is the union? If you get an empty family, union will be the empty set. If you take singleton X, then union will be just X. If you take the intersection it is always empty if you take only singleton X as the set then intersection is X again. So, (AU) and (FI) are automatically satisfied by this one, ok.

Similarly, there is another one, well you may say these are disinteresting. In some sense they are disinteresting but in some other sense they are extremely interesting also. So, take other way around, namely, put all elements of $\mathcal{P}(X)$, all subsets of X. That one, we are going to denote this by $\mathcal{D}$, here this is $\mathcal{I}$ for indiscrete space, $\mathcal{D}$ for discrete space these are the names ok.

Again I am checking that this $\mathcal{D}$ is a topology. It is very straightforward because we have made no restrictions at all every member of a power set is there the power set automatically satisfies all these properties ok.

What is happening here is this I the indiscrete space this topology is the smallest topology. If you take any topology it will contain this $\mathcal{I}$.

Similarly, this $\mathcal{D}$ is the largest topology every topology is subset of this by definition, because $\mathcal{D}$ is $\mathcal{P}(X)$ ok? there is no more. There is nothing bigger than $\mathcal{P}(X)$ on X alright. So, the least one and the biggest one that is what we have observed. Now, these things did not actually come from metric space, right we have created it without reference to any metric, but maybe they are metrizable. That is a different problem altogether.

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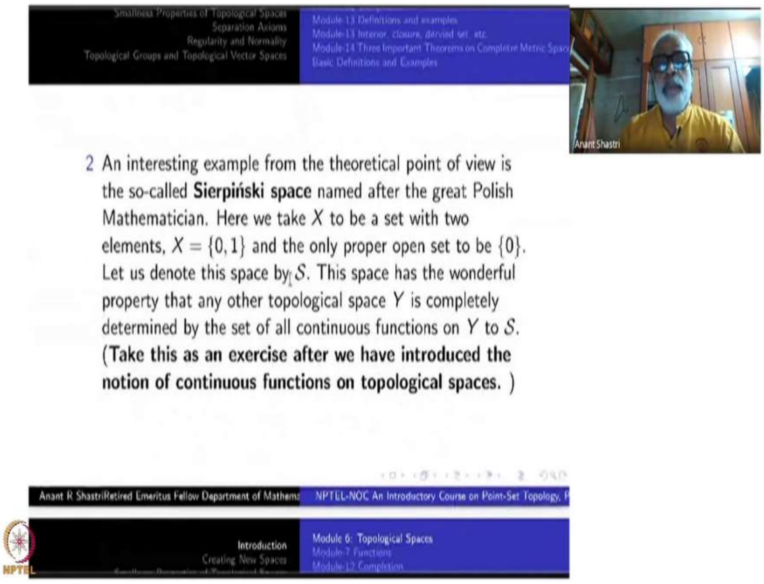

So, let us examine that also later on. Presently, let me give you another important example here. By the way this fellow is also a very important person. He has done a lot of work not only in topology, but in number theory, geometry, all sort of things. So, he was a Polish mathematician Sierpinski.

Maybe you have to pronounce it as Sierpinski? An interesting example from theoretical point of view is the so called Sierpinski space. Here we take X to be a set with two elements. So, any two elements set, usually we can denote the members by by $0, 1$ or $-1, 1$ depending upon what kind of algebra you want to do ok. Here we are not going to do any algebra so, you choose whatever symbols you want. Any a, b also will do. I do not care. Ok?

Take any two element set say $0, 1$. Now take the only proper open set to be one of the singletons. When I say proper open set I mean of the one which is not empty nor the whole set. The whole set is always there the empty set is also there. They are not called proper open sets they are open always. So, proper means not equal to empty set not equal to whole set.

In between them there is only two of them namely $\{0\}$ and $\{1\}$. but don't take $\{1\}$. Just take $\{0\}$ ok. Only one of them you take. That is the meaning of this that will be a topology. Let us denote it by this symbol $\mathcal{S}$? Verification that its topology is very straight forward ok.

Why I am interested in this? One of the properties which you can verify later on because right now we have not done enough terminology here, is the following. This space has a wonderful property: take any other topological space Y. You will know it completely if you know all continuous functions from Y to $\mathcal{S}$?. So, right now you do not know what is the meaning of continuous function from topological space to the another topological space.

So, right now you just take it for granted, but once you know what is a continuous function you take it as an exercise and solve it ok? Knowing all continuous functions the same thing as knowing Y means what? you should be able to describe what are all the open sets here in Y or what are all the close sets here that is the meaning of knowing the topology on Y, ok?

So, for that reason, that is one reason, but there are other things too. This example will serve as illustrating examples or what you may call as counter examples ok. So, there is a method in this. This Sierpinksi space construction is a method that I am going to describe now.

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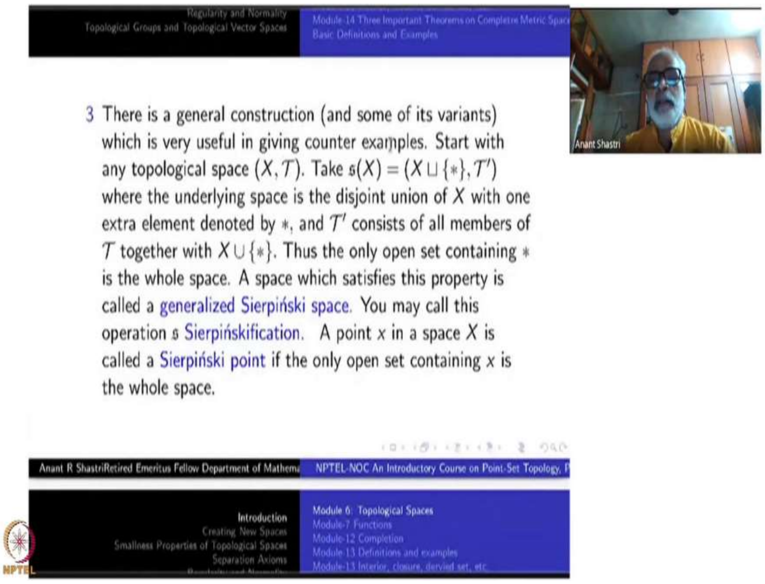

The general construction and then there are variants also. One part, I will tell you, is very useful in giving counter examples. Start with any topological space $(X, \mathcal{T})$ ok? Take this $s(X)$ what I have put, this notation, I mean because I want to celebrate this guys name Sierpinski, this is the Sierpinski operator. So, what does it do? It adds an extra point to the space X. So, like what we have done in the extended complex plane, like that, ok? X disjoint union an extra point. And a topology $\mathcal{T}'$ ok?

The underlying space is a disjoint union of X with an extra element star and this $\mathcal{T}'$ topology consists of all members of $\mathcal{T}$ they are there, anything which is subset of X and open in X (open in $\mathcal{T}$) is already in $\mathcal{T}'$. So, all members of $\mathcal{T}$ are there. Further, it has one extra element namely $X \cup \{*\}$ itself. That is the only open set containing this $*$. The whole space.

So, that is the property so, this star which has the property that look at any open set containing it is the whole space. Such a point is called Sierpinski point ok? Such a point is called Sierpinski point and such a space here what I have called as $s(X)$ which generalizes Sierpinski space. So, starting with the X if you obtain this $s(X)$ you can call it Sierpinskification. For example, in this example what is the Sierpinski point? it is 1.1 is a Sierpinski point ok.

Because take a look at any open set containing 1, it is the whole space for 0 there is an extra one 0 singleton 0 is an open set. So, 1 is a Sierpinski point here ok.

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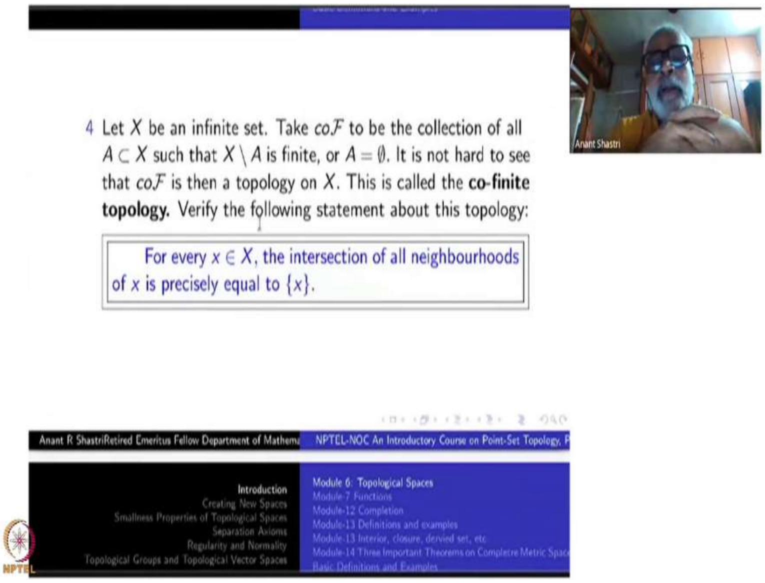

Another interesting topology which is quite celebrated, which is quite central than Sierpinski is the following. For this, I have to start with an infinite set. Then this $co\mathcal{F}$ corresponds to, you know, 'compliment of finite', so, that notation itself is indicative, to be the collection of all subsets A of X such that the complement of A is finite. Of course, A equal to emptyset should also be included, though the complement of the emptyset is not finite. I have to allow empty set also just to take care of the axiom (T).

It is not hard to see that this is a topology on X ok? As soon as $X \setminus A$ is finite for one member in the unions of several of them, X minus the union will be also finite. That it is very easy to see ok. If A_1 and A_2 are there $X \setminus A_1$ is finite and $X \setminus A_2$ is finite then $X \setminus (A_1 \cap A_2)$ will be also finite because it is the union of these two, by De Morgan law ok?

So, you can easily verify that it is a topology this topology is called cofinite that is why I writing it $co\mathcal{F}$. co-finite topology. One of the statements about that is that if you take all neighbourhoods of a given point ok? Take any point this this is going to be true for all the points, take the intersection of all neighbourhoods a point x, it will precisely equal to the $\{x\}$.

So, this is going to be a characterization of this space in some sense. So, take it as an exercise think about it. Later on, we will give you several characterization of this topological space again and again this will be used ok different way of looking at this same space.

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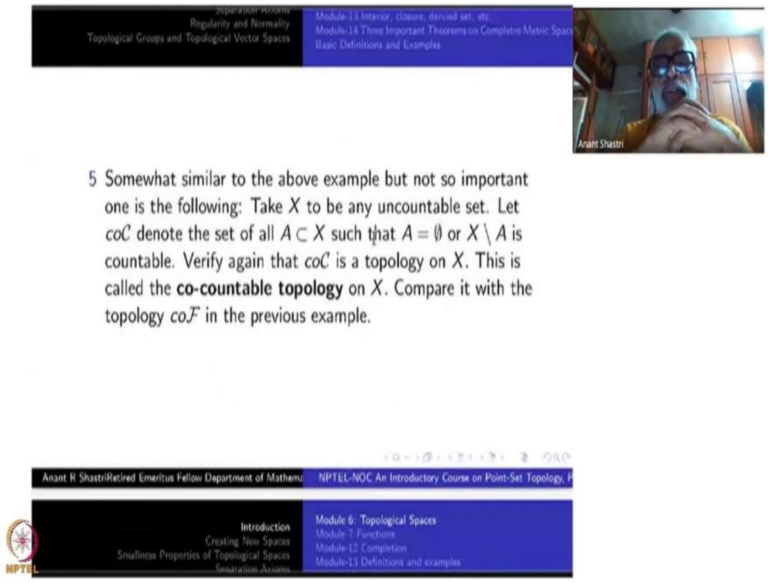

Somewhat similar to the above example, but not so, important is the following. Instead of taking a just a infinite set now you take an uncountable set ok. Let this be denoted by $co\mathcal{C}$. $co\mathcal{C}$ denote the set of all A such that either A is empty or $X \setminus A$ is countable. Countable includes finite also, but it could be countably infinite ok? That case is also there. So, this will be larger in some sense. That is why I have started with an uncountable set to put this condition.

Once again verification that this it is a topology is similar. I mean you have to use De Morgan's law ok. So, this will be called co countable that is why $co\mathcal{C}$, co countable topology on X. So, to understand what is going on you can compare it with the earlier example number 4 ok.

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Now, I come to more familiar things ok? These things were just to tell you that metric is not always necessary. There are many other ways of getting topology and interesting topologies. So, now I am coming back to the real numbers. I again look at the metric $d(x,y)$ is here is $|x-y|$ right? And it gives you a topology and that topology we are calling the usual topology or Euclidean topology right?

So, it is denoted by $\mathcal{U}$, almost all authors curly U for the usual topology or you may call it as Euclidean topology. It can be described in a slightly different way. That is my object now. Instead of looking d as a metric, now I am looking at the order in $\mathbb{R}$: $x \leq y$. So, it is an order in $\mathbb{R}$ right? there is a total order in $\mathbb{R}$. Consider all the open intervals; when I say intervals, that refers to the order not to the distance right?

The interval means what? All points lying between any two given points right? So, I hope you know what is the definition of an interval ok? Then take the collection of all subsets which are unions of members of this, namely, union of open intervals.

Of course empty set is also allowed. So, this is similar construction here. Similar to what we did for $\mathcal{T}(d)$. There we took the balls with the metric ok. With the real numbers you do not have to refer to the metric, but you can use the order. If you take for example, an interval

(a, b) can be also thought of as an open ball with its center $a + b/2$ and radius equal to be $b - a/2$, ok?

So, in some sense I am cheating, but no. You see I am never mentioning the metric here I am just looking into that order here. Such order is not available if you go to $\mathbb{R}^2, \mathbb{R}^3$ and so on so. It is available only inside $\mathbb{R}$. So, that is why I am using it. It is not even available for complex numbers you see. Ok so, that is the important thing here. So, you can describe this topology as follows: an open set is union of open intervals.

Then it is not very difficult to prove that the third property that intersection of two intervals is either empty or it is again an interval ok? Open interval intersection with another open interval is open interval that you can prove very easily. So, that is the part of the proof here to show that this forms a topology ok?

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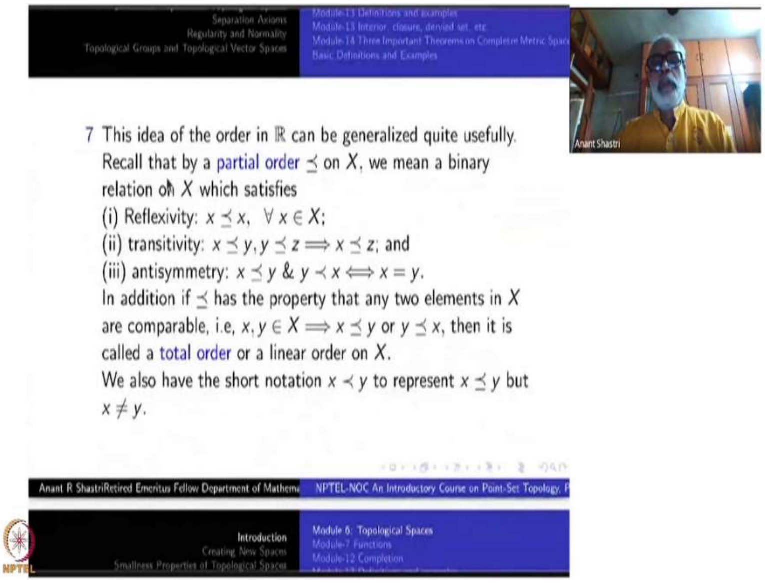

Now, I come to the point why I have introduce this one. This idea of order in $\mathbb{R}$ can be generalized quite usefully ok? Recall that by a partial order on X, we mean a binary relation on X which satisfies reflexivity, transitivity, i.e., $x \preceq y, y \preceq z$ implies $x \preceq z$, right? And antisymmetry if $x \preceq y$ and $y \preceq x$ then it should imply $x = y$ ok? In addition, if this partial order has the property that any two elements in X are comparable, so, I am putting this extra condition here; that means, that it is a total order. So, I am recalling these concepts. I hope you already know these things. If you do not know, right now you can learn it ok? This is called the total order or linear order ok?

We also have a short notation $x \prec y$ just implies $x \preceq y$ that was the relation, but x is not equal to y ok. Now, I have deliberately denoted this one by this curly symbol because right now x is not the set of real numbers ok? So, not to confuse it with the standard less than equal to relation inside $\mathbb{R}$, I have denoted by this one this $\preceq$ symbol alright?

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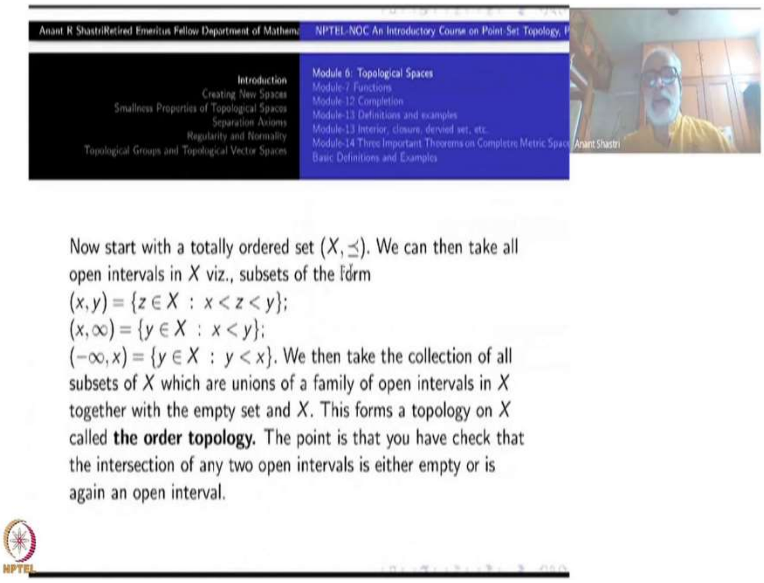

Now, start with a total ordered set $(X, \preceq)$. We can then take all open intervals in X. So, let me redefine what are the open intervals here, now carefully. Given x and y in X you can look at this this notation (x, y) this is also unfortunately the standard notation for ordered tuple ok? Here it is an interval all point z inside X which will lie between x and y, that is $x \prec z \prec y$ ok.

So, again here there is a typo. I want to write this curly less than, curly thing because that is the notation here, never mind. So, the interval (x, ∞) is the set of all y in X such that x is less than y. See this infinity is not a symbol whereas the interval (x, ∞) is a symbol. The ∞ does not make sense. We do not have ∞ as an element of X. Nor it implies that we have an increasing sequence such as $1, 2, 3, \ldots$ in X.

There is no convergence or anything like that its (x, ∞) is an interval just a way to denote the set of all y in X which are bigger than x ok? Similarly, $(-\infty, x)$ is the set of all point y which are less than x. Take all these subsets. They will form a what? They will form a topology on X. Where is the emptyset? You can take x equal to y then this will be an empty set ok? There is no z between them; strictly between them. That is the point ok. So, that is a topology and that topology is called the order topology. So, what you have to check here is that intersection of any two open intervals is again an open interval if it is non empty. So, do not use any other properties of real numbers, other than the total order. That is the whole idea.

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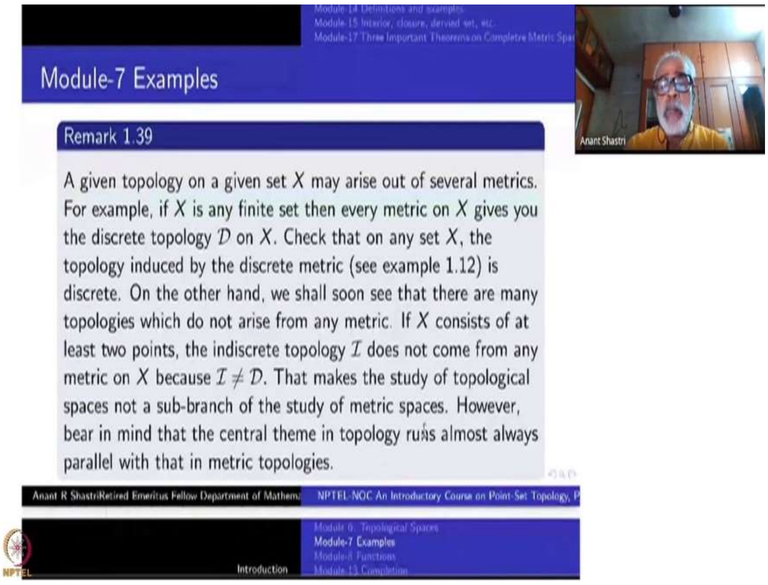

So, it can proved that a topology on a given set may arise by several metrics or it may not arise out of any metric right. If you take a finite set for example, then every metric on X gives you the discrete topology, which I denoted by $\mathcal{D}$ right? On any set X the topology induced by the discrete metric is again discrete.

On the other hand, we shall soon see that there are many topologies which do not arise from any metric. Of course, in a simple example, we have already seen this. Namely, you take a set with exactly two points, but take the indiscrete topology ok. Then it cannot be got out of any metric because as soon as you put a metric which induces the topology the topology will be a discrete topology. On the two-point set, discrete and indiscrete topologies are different already.

In fact, it is true that if you take any indiscrete topology on any set with more than one point, it cannot come out of any metric ok. That may need some more seeing. Once we understand what happens to metric topologies in general. So, having said all these, we still want to stick to a lot of ideas and lot of results from metric spaces. Why? because the central theme in topology is always parallel to the results that we get from metric spaces. So, we should not forget metric spaces or discard them.

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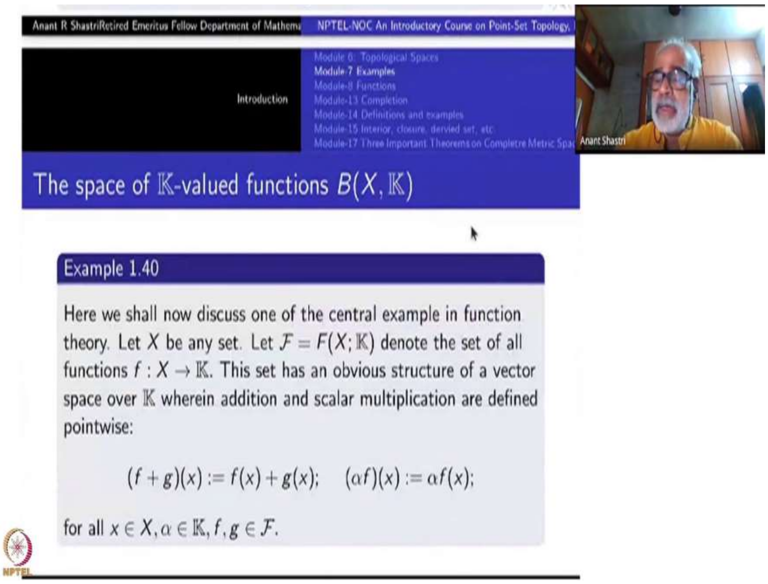

Now, let me give you one of the important examples. It looks as if I am doing algebra now, but those algebras are very familiar. Algebraic structure that we have on the complex numbers or the real numbers the field $\mathbb{K}$ in the field $\mathbb{K}$ you can add you can multiply addition and multiply have some inter relations, associativity, commutativity law of distributively and so, on.

These are the things which make the formal definition of a group and a ring and so on. So, you do not have to worry about it too much, about those structures. So, whatever I am going to do you can take that as definitions. If I introduce new notation new term you can take that as definition it is that much accurate there ok.

So, having said that take any set and look at all the functions from that set to $\mathbb{K}$. Point wise addition and point wise multiplication is the key. Take two functions f and g. You define $f+g$ by taking their values and then adding them $f+g$ operating upon x is $f(x)+g(x)$. Similarly, αf operating on x is $\alpha f(x)$ ok. So, this already makes the set of all functions from x to $\mathbb{K}$ itself a vector space, the vector space structure actually comes from that of $\mathbb{K}$, ok?

So, that is a vector space structure, moreover I can also multiply two functions because I can multiply two numbers right? $fg(x)$ is $f(x)g(x)$. So, that multiplication will have again the same kind of relation with the addition namely it is distributive. So, what one can say is: the multiplication is bilinear with respect to the vector space structure $f(g+h)$ is $fg+fh$ and you can pull out α wherever you like $(\alpha f)g$ is same thing as $f(\alpha g)$ right?

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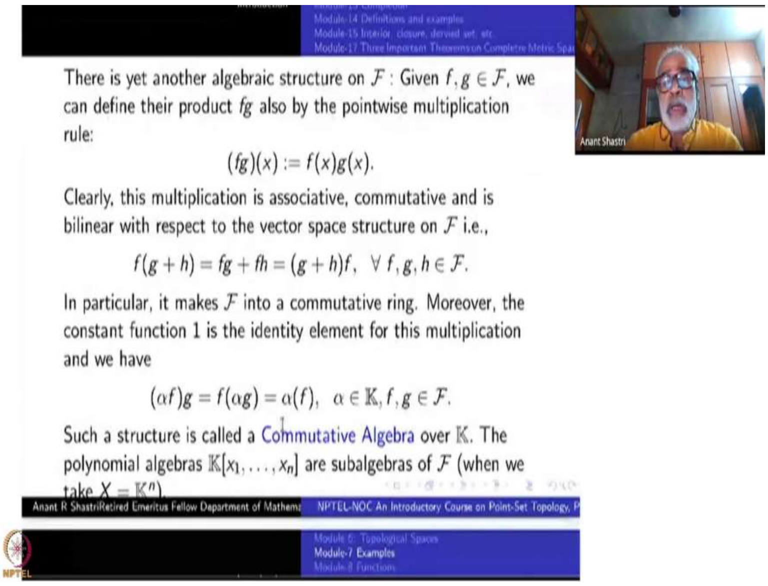

So, that is the vector space structure. So, you can have linear maps on them. What it means to say bilinear? Linear in both the variables. Not only that even on the other side also $(g+h)f$

is the same thing as $gf + hf$ which is same thing as $fg + fh$ and so on. That is because multiplication is commutative also ok? In particular, any such structure is called a commutative ring. So, this $\mathcal{F}$, the set of all functions becomes a commutative ring.

It is a commutative ring and it is a vector space right? Such a thing is called an algebra. So, this algebra has another important property namely, the constant function one plays the role of multiplicative identity. $(f1)(x)$ is $f(x)$ into one. So, $f(x)$ into $(1f)(x)$.

So, that is why this is called a commutative algebra over k with an identity element. If you want to know other examples, the most famous examples, the most useful examples are polynomial algebras. Take real or complex numbers as coefficients take one variable polynomials they have all these properties.

Take real coefficient take one variable they are all these properties. More generally you can take n variables and take $\mathbb{K}[x_1, x_2, \ldots, x_n]$ which denotes the set of all polynomials in the variable $x_1, x_2, \ldots, x_n$. It becomes what is called a polynomial algebra. It has exactly similar properties. Actually this polynomial algebra is a subalgebra of our is great $\mathcal{F}$ how do you see that? The special case namely take the set X as just just you know $\mathbb{K}^n$.

So, there are all functions on $\mathbb{K}^n$, ok, do not take all functions only take polynomial functions that is all that is a sub algebra ok.

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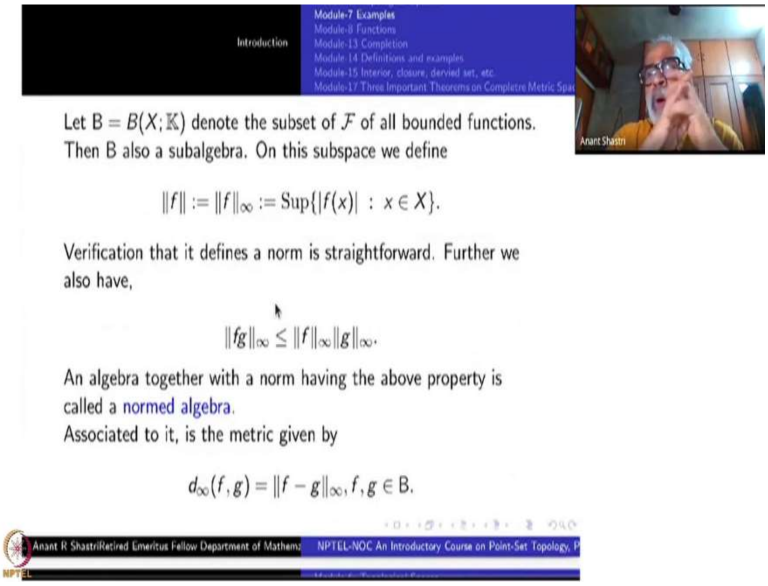

Now, I am going to give you another important sub algebra here namely I am writing it as $\mathcal{B}$. $\mathcal{B}$ stands for bounded functions ok? Only take bounded functions into $\mathbb{K}$. Sum of two bounded functions is bounded. The scalar multiple of a bounded function is bounded product of two bounded functions is bounded.

So, this is all easy to see. what is the meaning of bounded? As x varies $|f(x)|$ is bounded by a single number say M for all x. This is the condition. So, for each f, we may have different M. If f is bounded and g is bounded then you can find a common bound for $f+g$ as well as fg as well as for αg and so on. So, the set of all bounded functions forms a sub algebra, subspace as a vector space and a subring. It is sub algebra.

On this subspace we can do some geometry. We can put a norm on it. So, the norm is if you do not want rigorous notation the short notation is just norm, but it is the infinite norm this notation is used for supremum norm. What you do? Take $|f(x)|$ as x range over X take the supremum of that set it is a bounded set, the supremum is a finite number. Then the verification that it is a norm is exactly similar to what we have done in other cases, ok?

Only triangle inequality takes a little bit of time. Unless you spend that time yourself you would not know what is going on. That much of time you have to give ok? On your own, you verify that this is actually a norm.

Moreover there is one extra property here, namely the norm of fg, the product, is less than or equal to norm of f into norm of g. This not a part of the definition of a norm. For the norm you have norm of f is 0 if and only if f is 0 ok, norm of αf will be $|\alpha|$ times norm of f and triangle inequality. But this is an extra thing because, there is an extra structure. You can multiply two functions here f into g. Its norm is less than or equal to norm of f into norm of g. This property of the norm you may call, it is respecting the multiplication.

So, that makes it a normed algebra, the norm and the algebra structure are not just on their own. They have some relation and that is the relation describe above. In that case, it is a normed algebra ok. Once you have a norm, of course, as usual you can take the distance function corresponding to that, namely distance between f and g is norm of $f - g$. So, that will become a metric space that is what I meant by geometry here.

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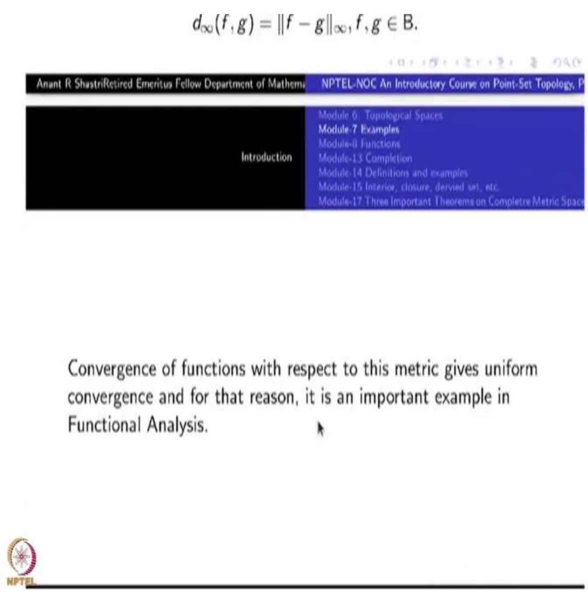

Now, we can talk about distance between two functions and so on ok. Now, we can talk about convergence ok? For convergence, you have to look at the metric. With respect to that metric, we can talk about convergence of these functions. First pointwise, if f_n is a sequence of functions, at a point $x \in X$, you get a sequence $f_n(x)$ of real number or you get a sequence of complex numbers. You can take the convergence of that that convergence will be pointwise convergence.

Whereas the convergence here with respect to supremum norm always gives you what is known as uniform convergence ok? This uniform convergence is exactly same thing as what you have studies in your calculus course. If you have not done it yet, we will do that whenever we use it more seriously. We will tell you what it is ok? but this is not part of topology but is actually analysis. You must have learnt it ok?

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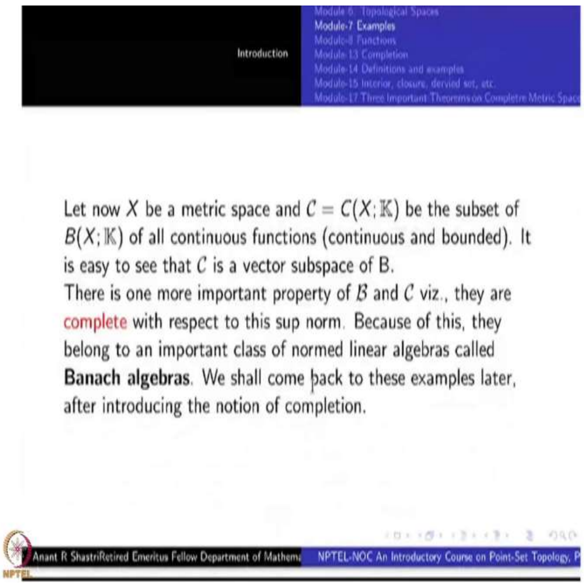

The algebra $\mathcal{B}$ has another interesting subalgebra but only when you take X as a metric space or a topological space and so on ok, to take X as a metric space. Remember $\mathcal{B}$ consists of all bounded functions ok, but now I want to put an extra condition, viz., consider the set of all those elements in $\mathcal{B}$ which are continuous also.

Now, continuity makes sense because here I have a metric space as the domain and as the codomain, I have another metric space you can think of this as metric space always when you do not mention anything it should be taken as the metric space with the usual topology namely the modulus ok.

With respect to that if you look at all continuous functions ok? We have the $\epsilon - \delta$ definition or you may use sequential continuity definition. Look at all continuous functions. Then you can show that, exactly same way as in your calculus course, that if f is continuous and g is continuous then $f + g$ is continuous αf is continuous and fg is continuous etc.

So, what does that mean; that means, that this $\mathcal{C}(X, \mathbb{K})$ is a sub algebra like $\mathcal{B}(X, \mathbb{K})$ is a sub algebra of $\mathcal{F}$, $\mathcal{C}(X, \mathbb{K})$ is also sub algebra of $\mathcal{B}(X, \mathbb{K})$. Now, not only that, one of the most important property of both $\mathcal{B}$ and $\mathcal{C}$ is that they are complete with respect to this norm. Now, what is the meaning of completeness? Similar to what you have done in other cases namely in the case of $\mathbb{R}, \mathbb{C}$ and so on. Every Cauchy sequence is convergent.

That property makes it a Banach algebra. Otherwise you would have called it just a normed algebra ok, here normed algebra. These two are $\mathcal{B}$ and $\mathcal{C}$, they are actually Banach algebras because they are complete. So, for this part, we will come back to it, we will study these things more thoroughly later on ok. So, this is an important example we may come back to it again and again. Let me now go back to giving you more examples of topologies ok.

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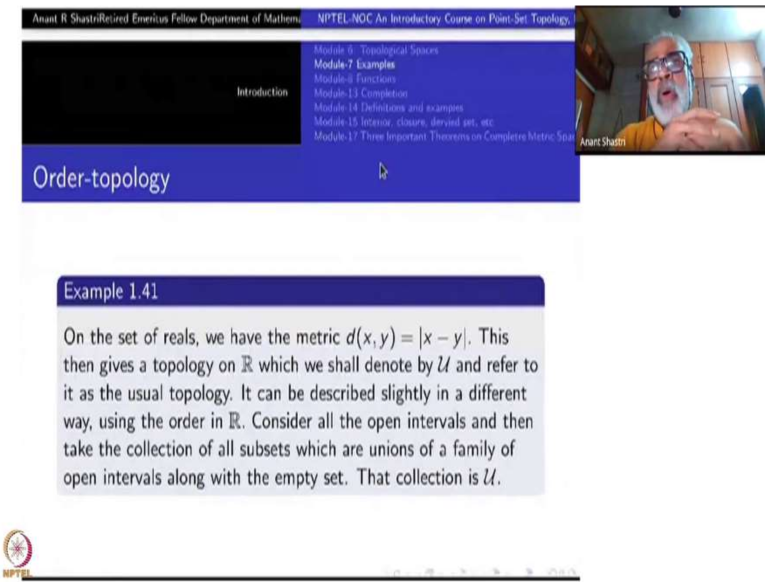

Once again, look into $\mathbb{K}$ with the modulus function modulus function I have told you gives you the usual topology right? the so called usual topology or Euclidean topology. This topology now can be expressed in a different way when $\mathbb{K}$ is $\mathbb{R}$ ok not for complex numbers when you take only the real numbers you can describe it in a different way, namely look at the order there each number can be compared with another number right there is an order there.

So, using that order we can talk about intervals the open interval (x, y) means all those points lying between x and y strictly and so on. So, we have open intervals right now you take collection of all open intervals ok. Then just like we have done in the metric space in general way, what you do here, take unions of all open intervals ok? I mean not all intervals only open intervals, any family of open intervals ok.

Take all such possibilities, open intervals two of them three of them infinitely many of them also you can take unions of such things. Each of them you put them together in one single collection that will form a topology, what is that topology it is precisely the same as given by the metric namely, open balls are the same open intervals. Ok?

But now I do not want to refer to the metric, but I just want to talk about intervals. I am looking at different side of the same coin. that is all alright. So, that collection is exactly same as $\mathcal{U}$, the usual topology alright, but why bother about looking in a different way, that is the whole idea.

Now you can just forget about all the additive structures multiplicative structures and so, on in the real number just look at the order itself that is enough for me to decide the topology there right. Therefore I would like to look at other structures it may have just the order only. So, that is what I am going to do now ok.

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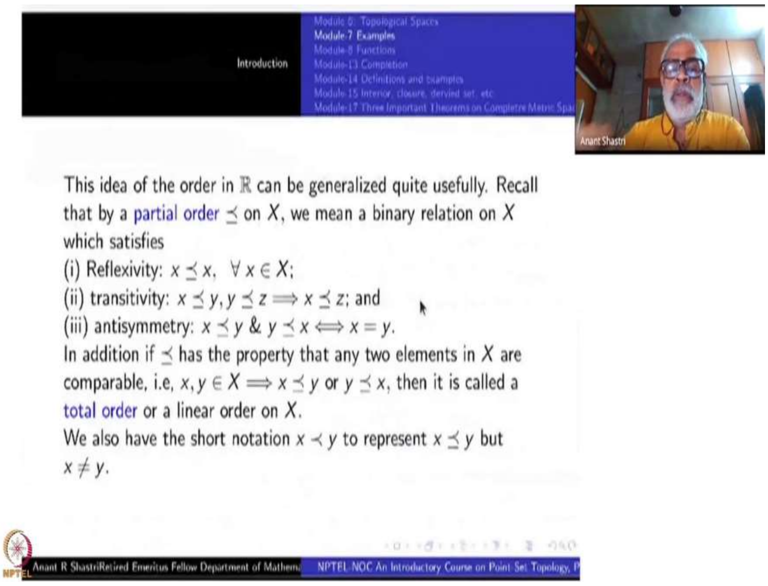

The idea of the order in $\mathbb{R}$, it can be generalized and usefully they will be something useful. It is not just for generalization sake. We recall that by a partial order if you have not done what is partial order I am going to to recall here now ok. So, we mean a binary operation on a set X which satisfies reflexivity namely x is always related to x ok. Let us this one as $\preceq$ just to distinguish it from the usual less than or equal relation OK? $x \preceq x$ for every $x, x \preceq y$ and $y \preceq z$ implies $x \preceq z$ this is transitivity. Antisymmetry if x is smaller than or equal to y and y smaller than or equal to x then x must be equal to y so, the same thing here $x \preceq y, y \preceq x$ implies and implied by $x = y$.

So, these three properties define a partial order, in addition there is one extra property I am going to assume namely given any two elements ok given any two elements in X that is x and y belong to X they must be comparable what is the meaning of these x is less than or equal to y or y is less than equal to x you hold then it is called total order or a linear order.

So, this property is also there with real numbers. So, I am going to stick to only this these properties. Later on I may put more and more properties ok? Let us just take a totally order set ok, let us have one more notation namely when you write x strictly less than y what is the meaning of these x is related to y, but x is not equal to y. So, that is the meaning of this one x is strictly less than y means $x \prec y$ here x is not equal to y.

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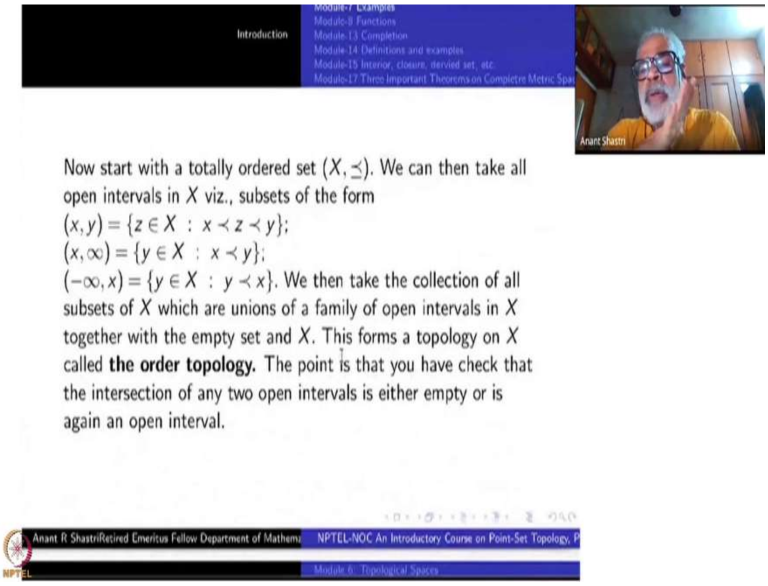

So, this is notation ok with this notation let us do something now some meaningful things start with a total order then you define intervals just the way you define it for real numbers ok, but these are not numbers no need; (x, y) is all points z in X which are strictly between x and y. $x \prec y \prec z$, but x not equal to z and z not equal to y, ok that is what.

Similarly you can define the open ray (x, ∞) is all point y such that x is less than y which means y is bigger than x ok. Similarly $(-\infty, x)$ all point were which are y is less than x. So, I have defined the intervals here now you take collection of all intervals and unions of all them.

Now ok I am looking at all subsets of X which are unions of open intervals along with empty set of course, I do not have to say that because if you take x equal to y the open interval x equal when x is y, (x, x) that is empty there is nothing in between right. So, that is logic, but let us just make it clear that empty set is also there ok and the whole set X is also there ok.

So, let us take this collection of union of all open interval. Then it will be a topology on X. Now, you have to verify it. you cannot rely on just your intuition. Anyway, we have done it for a real numbers. What is that is the property that we use for the real numbers, that property should be here in the just order topology if you use something else that will be wrong.

For example, suppose I have stopped just with partial order I do not assume total order and I do the same thing intervals can be defined and I can take this one it will not be a topology. You understand my point, the total order is necessary to assure that this collection becomes an order becomes a topology ok.

So, take a minute verify this one by yourself ordered topology has to be ensured. What you have to do just like in the case of open balls you will have to check that intersection of two open intervals if it is non empty must be again a union of intervals if it is interval it is fine that is stronger. You have show in here that is what you have show then only union is coming, but the third property finite intersection you have to verify ok yeah.

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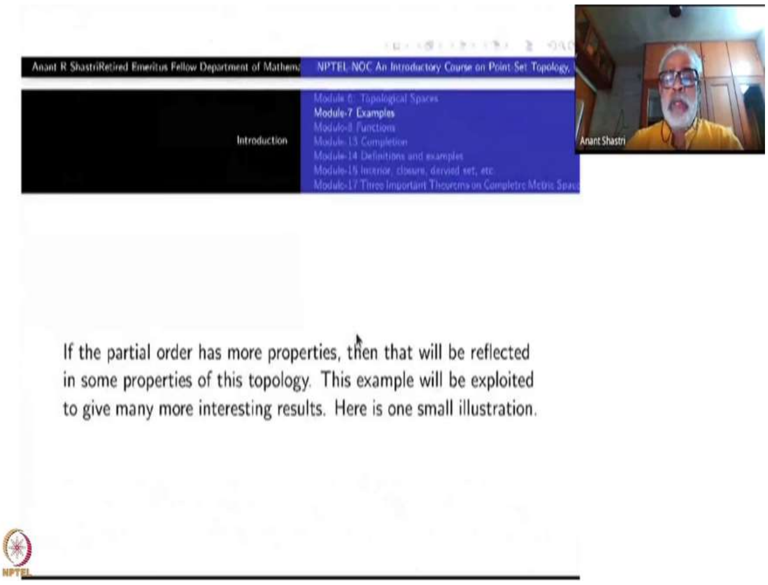

So, let us carry on. So, once more one more ah remark I can make, if you assume little more properties for this totally order set that will be reflected in a topology and vice versa. So, you may get more and more interesting things when you put topology more ah hypothesis on that ok. So, this example will be again met you know may be at the end of a course or maybe at some other time, but it will keep coming again ok.

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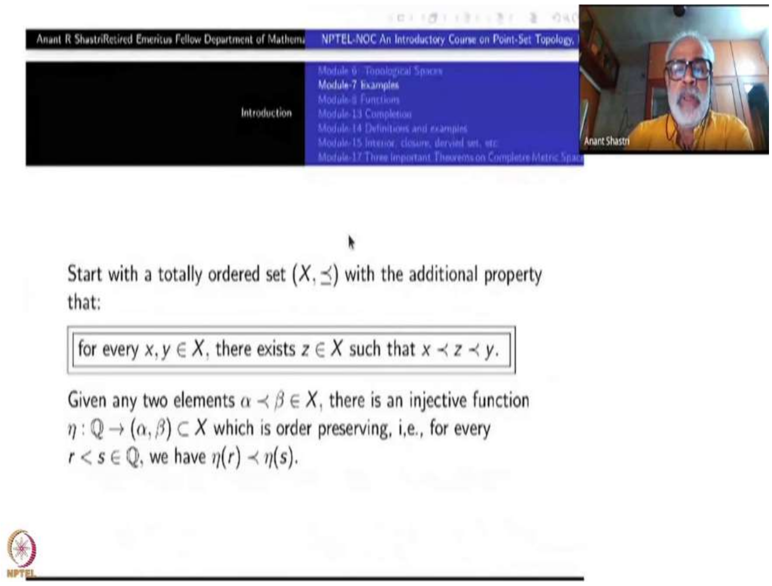

Just to emphasize this one if I put more properties then it will have more you know interesting things happening. Let us just put one example let us take namely I start with an order topology ordered relation totally ordered set and ordered topology ok, I have taken X now I have fix here.

Now, assume one more property of this order namely given any (x, y) let us say x is less than y ok then their exists z such that x is less than z less than y. So, in between any two element there must be a third element. See the third element it is not is no it is different from both x and y ok.

I am not talking about real numbers here. I can take $x + y/3$, that is not allowed here, but I am saying that this property is true. Let us assume this property ok. One can give a name, but let us not bother about the name right now ok. So, given any two elements α and β belonging to X, I want to say that there will be an injective function from set of all rational numbers to this interval (α, β) inside X ok.

So, the whole of $\mathbb{Q}$ is sitting inside (α, β) through this map this is an injective map such that is η its order preserving. So, what is the meaning of order preserving there is an order here there is an order here ok. So, it must be preserving and it must be injective what does it mean it is strictly monotonically increasing function $r < s$ in $\mathbb{Q}$ implies $\eta(r) \preceq \eta(s)$, ok.

So, this is a claim now, now I want to show you something just wonderful thing just imagine what is the meaning of this one this means that whenever it is such a thing you just one element is there between any two elements these hypothesis, from this one you can conclude that.

Inside every non empty interval you can find representatives like the entire rational number the whole of rational numbers would be there inside that, in case one one correspondence and its order preserving ok. So, such a wonderful thing will come out of that now alright? it is not very difficult, but the proof is educative it can be the proof itself the idea can be used elsewhere that is why I am doing it ok

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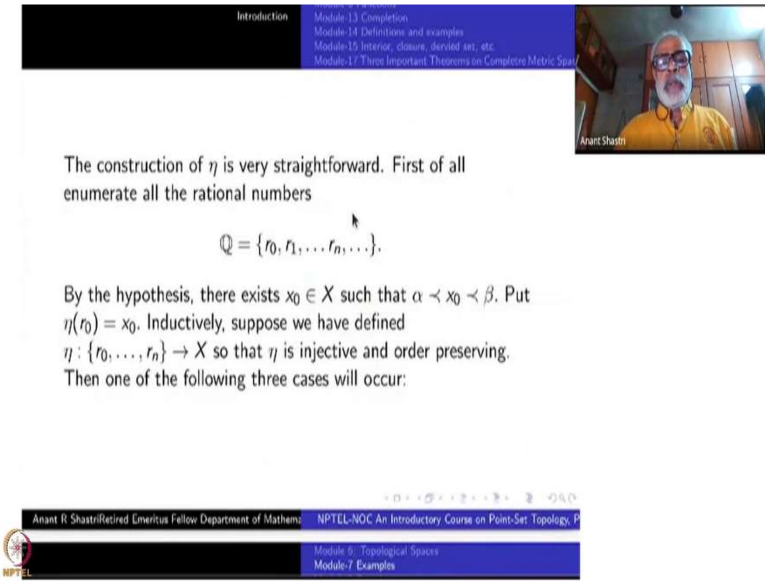

The construction of η is very straight forward for all first of all let us start with an enumeration; enumeration means what? Labelling of the all the rational numbers rational numbers are countable. So, I can put $\mathbb{Q}$ equal to $\{r_0, r_1, \ldots, r_n, \ldots\}$. I am not saying now that this is smaller than this one that is not possible it is not an total order some enumeration of $\mathbb{Q}$ we start with that ok.

By the hypothesis, there exist x_0 belonging to X such that α is less than x_0 less than β right. Because α and β are given any two numbers are given there any any two elements are given there inside X and X has this properties that between any two elements, there is a third element. So, take that one you start mapping this r_0 to x_0 now.

So, $\eta(r_0)$ is x_0 inductively. So, the definition of η is completed by induction. Suppose we have defined η up to n, $r_0, r_1, \ldots, r_n$ ok. So, n could be 0 just one we have defined. So, that whatever you have defined that is already order preserving ok.

Now I am looking at extending this function to r_{n+1}. So, I am going to define $\eta(r_{n+1})$, ok. So, r_{n+1} look at its status inside $\mathbb{Q}$, that $\mathbb{Q}$ is totally ordered already right? So,r_{n+1} look at $r_0, r_1, \ldots, r_n$ it will be somewhere in between. So, where exactly it is that is what you have to know.

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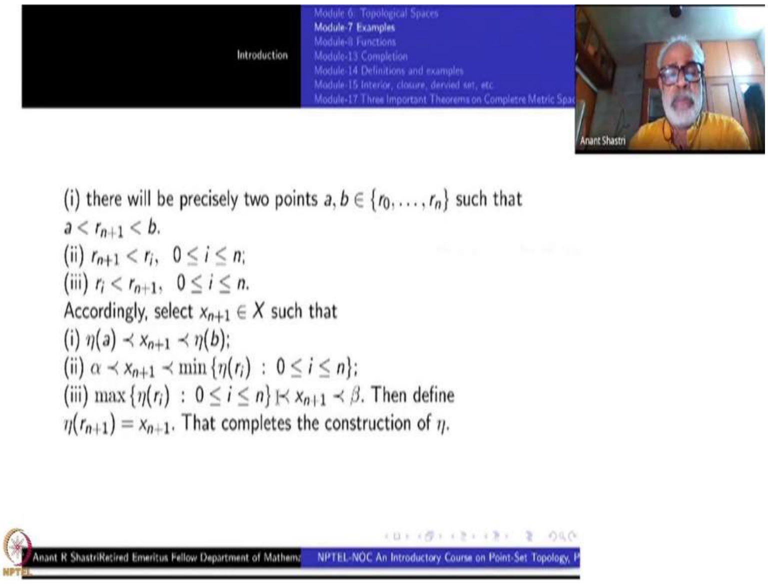

So, I am making three cases here, namely, first case is that there will be precisely two points a, b belonging to $r_0, r_1, \ldots, r_n$ such that r_{n+1} sits between them $a < r_{n+1} < b$. a and b are already elements of $r_0, \ldots, r_n$. So, somewhere it is there alright.

The second case is that it is not inside anywhere here ok? r_{n+1} is smaller than all the r_i. The third case is that r_{n+1} is bigger than all $r_i's$. So, this is the that out of these three one of them has to happen ok.

So, accordingly what we will do? Select x n plus one belonging to X such that in the first case select it between eta a and eta b which are already defined, must be bigger than eta a because eta is order preserving these two are defined already. So, we pick up one element between them.

See x_{n+1} has to be picked up by us now and this x_{n+1} is going to be $\eta(r_{n+1})$ ok how do I pick at pick up this one between this because r_{n+1} is between these two ok. In the second case, r_{n+1} is smaller than all the r_i is therefore, look at α it is smaller than all the $r_i's$ therefore, it is smaller than the minimum of them. So, between them there will be x_{n+1}.

So, pick up one element here the third case you do other way around namely β is bigger than all the all these elements therefore, it is bigger than maximum. So, between the maximum of all these r is you pick up not $\eta(r)$ is you pick up x_{n+1} between them and β defined $eta(r_{n+1}) = x_{n+1}$ that is all ok. So, let us stop here next time we will continue with more examples and better things.

Thank you.