Introduction to Point Set Topology, (Part I)
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Lecture - 61
Topological Vector Spaces-continued

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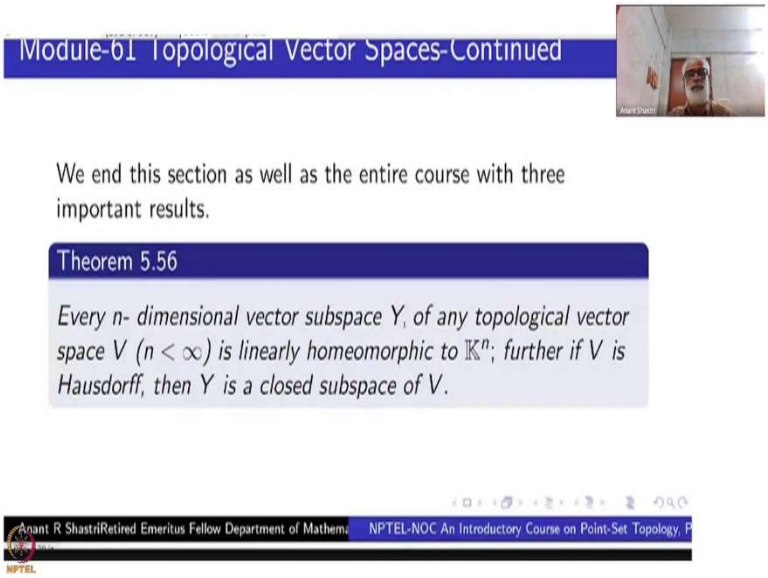

Welcome to the last module of Point Set Topology course part 1. So, we shall continue the study of Topological Vector Spaces. Several important basic results have been derived already. So, in this last module, I will prove three important results about topological vector spaces.

The first one is every finite dimensional vector subspace Y of V is linearly homeomorphic to $\mathbb{K}^n$.

Further if V is Hausdorff, which is same thing as just assuming T_0, then Y is a closed subspace of V also ok. Every finite dimensional vector subspace is linearly homeomorphic, is linear isomorphism to $\mathbb{K}^n$. Without the assumption of Hausdorffness, it need not be the closed subspace, because you can always take indiscrete topology on any topological vector space.

Then the only closed subsets will be empty set of the whole space ok. So, that is very easy to see that, the condition is necessary here ok.

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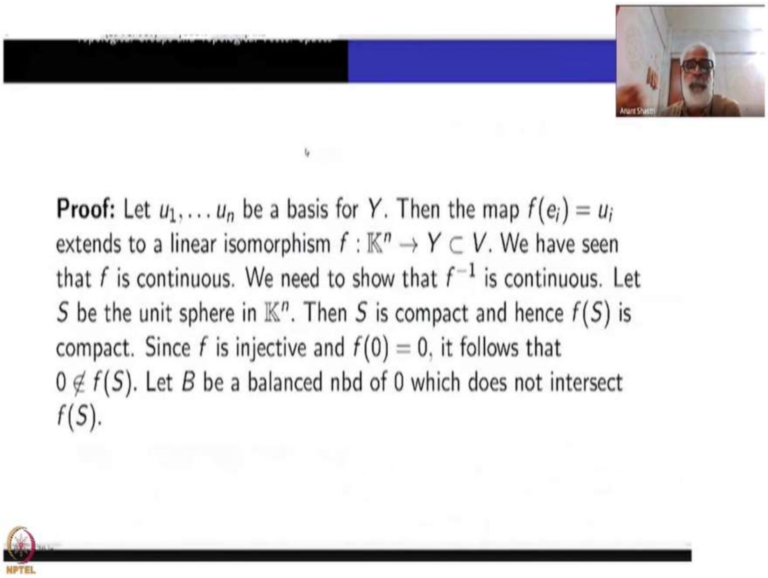

So, here is a proof, start with a basis for Y, say $u_1, u_2, \ldots, u_n$. Then the map $f(e_i)$ going to u_i extends linearly to an isomorphism from $\mathbb{K}^n$ to Y right? So, linear part is done, already we have seen that in the last corollary of the previous module, that any linear map is continuous. So, f is continuous.

The only thing that is missing is f inverse should be also continuous, from Y to $\mathbb{K}^n$ ok. So, this is what we have to show. Now, you take S to be the unit sphere in $\mathbb{K}^n$ ok, I am taking $\mathbb{K}^n$ which has the Euclidean norm, so I can talk about unit sphere there; there is no problem. Then we know that S is compact. Under a continuous function, its image is compact, so, $f(S)$ is compact.

Now, f is injective, so $f(0)$ goes to 0. So, non-zero vectors will not go to 0; that means, 0 is not in $f(S)$ ok. Now, let B be a balanced neighbourhood of 0, which does not intersect $f(S)$, see $f(S)$ is a compact subset and 0 is not there, so you can choose some neighbourhood here which is disjoint from that, because of regularity or whatever, but I want to choose a balanced neighbourhood also alright.

A compact subset and a disjoint closed subset can be separated, that is what you have already seen. So, for that you do not need Hausdorffness and so on. So, see slowly I am using all those results that we have proved in the past two lectures for topological vector spaces.

Choose a balanced neighbourhood B of 0 which does not intersect $f(S)$, alright?

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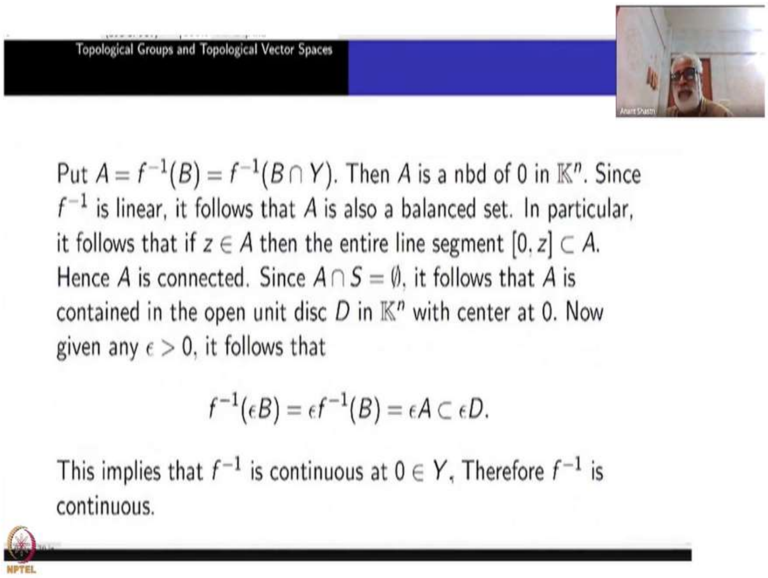

Put $A = f^{-1}(B)$ which is same thing as $f^{-1}(B \cap Y)$, because the image of f is inside Y, that is why it is $f^{-1}(B \cap Y)$ alright. A is a neighbourhood of 0. Because f is continuous, since f inverse is linear also, linearity there is no problem what we are trying to prove is continuity of f inverse. It follows that this A is a balanced set also ok.

Some some scalar multiplication times something is contained inside etc, you have to check this. Since B is balanced neighbourhood, it follows A is also balanced. So, scalar multiplications comes here because f is a bijection ok.

α times some vector is alpha times f of some vector so that is all I have to use, in order to show that B is balanced implies A is balanced.

In particular if z is a point inside A, then the entire line segment $0, z$ will be inside A. This is the property of balanced set we have seen all the time ok. So, this implies that A is star shaped at 0, all the line segments are there therefore, in particular A is connected ok.

Now, $A \cap S$ is empty, because I have started with $B \cap f(S)$ is empty ok. So, you know that the unit ball inside $\mathbb{K}^n$ ok, whether it is $\mathbb{C}$ or $\mathbb{R}$, the unit ball always separates the wholes whole $\mathbb{K}^n$, ok. Namely those things which are strictly inside the ball and those things strictly outside. So, there are two disjoint open subsets right? So, since $A \cap S$ is empty and A is connected, it follows that A must be contained in the open unit disc D in $\mathbb{K}^n$ with center 0 right?

Now, given any $\epsilon > 0$, it follows that if you take $f^{-1}(\epsilon B)$ that will be $\epsilon f^{-1}(B)$ by linearity of f inverse ok. But, this ϵB, that $\epsilon f^{-1}(B)$ is $f^{-1}(B)$ is A, that is ϵA because ϵA will be contained inside ϵD. Because A is contained inside this unit ball D ok? (I can write this as D^n , but that is not correct because I do not know whether $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$.)

So, it depends. It may be D^{2n} or D^n. So, I have just written as D, unit disc here. This implies that, now we see $f^{-1}(\epsilon B) = A$ is contained in ϵD. This implies f inverse is continuous at 0 belonging to Y.

See how connectivity finally has been used to show that the inverse is continuous alright?

So, this is one important theorem.

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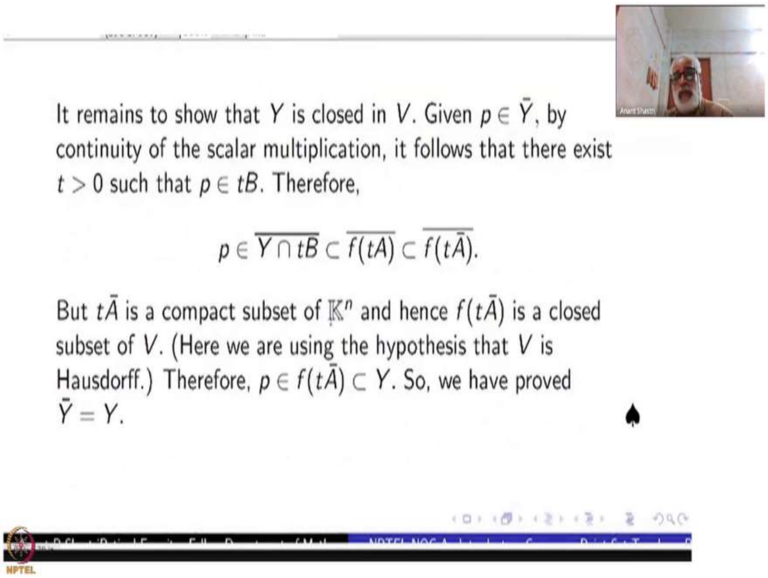

Well, next I have prove the other part, namely, suppose V is Hausdorff. Then I want to show that Y equal to $\bar{Y}$ namely Y is closed also, that is what I have to show. Given p belonging to $\bar{Y}$, since B is a balanced nbd of 0 in Y, it follows that there is a non zero t such that p is in tB. After all mutliples of B will cover the whole of Y. Since scalar multiplication by $t \neq 0$ is a homeomorphism, it follows that tB is an onbd of p.

Therefore, p will be inside $\bar{Y} \cap tB$ contained in the $\overline{Y \cap tB}$. But $Y \cap tB$ is equal to $f(tA)$, because A is nothing but $f^{-1}(B)$ ok. So, if we apply f, f of f inverse is what the whole thing right. Y is nothing but $f(\mathbb{K}^n)$.

But, this $f(tA)$ is contained inside, $\overline{f(tA)}$ here. and then take the closure ok. But $\overline{tA}$ is compact. See compact subset of $\mathbb{K}^n$, right, see A is a bounded subset already inside the unit disc its closure is compact ok. So, it is compact subset of $\mathbb{K}^n$. See once $\bar{A}$ is compact $t\bar{A}$ is also compact, because t is anyway some scalar.

So, $\overline{f(tA)}$ will be a closed subset of V, because it is compact and V is Hausdorff. So, this is where we are using first time that V is Hausdorff, that compact subsets are closed that is what you have to use ok. So, therefore, p is inside $\overline{f(tA)}$, there is no need to take the closure here, p is inside $\overline{f(tA)}$, but $\overline{f(tA)}$ is after all inside Y, because $f(\mathbb{K}^n)$ is Y, ok.

So, p is inside Y, that is what we have to do. So, $\bar{Y}$ is contained inside Y therefore, $\bar{Y}$ is equal to Y, alright. So, closure property follows by again Heine-Borel theorem, we have used here, viz., that closed and bounded subsets of $\mathbb{K}^n$'s are compact. Many important things have been used here alright.

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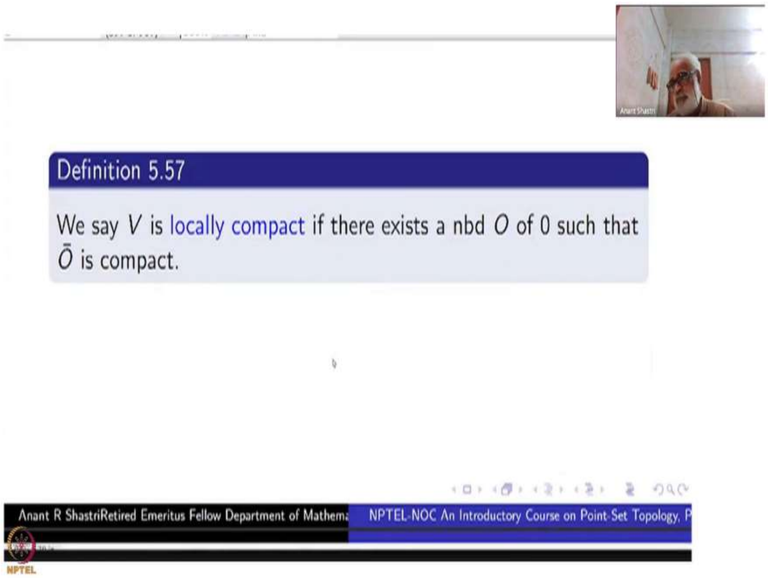

So, let us go to the next result, for which I will make a temporary definition of local compactness. Later on, in part 2, we will study local compactness on its own for arbitrary topological spaces.

Here it is a tentative definition of local compactness, for a topological vector spaces.

V is locally compact if there exists a neighbourhood O of 0 such that the closure of O is compact. So, there is a compact neighbourhood for 0 element, that is all, ok. So, that is the meaning of local compact alright?

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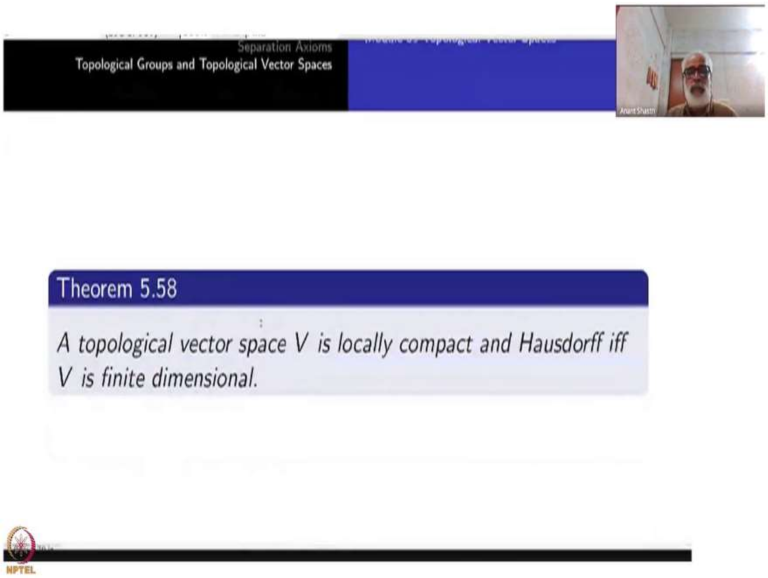

With this definiton, we make this final result about topological vector spaces and finite dimensionality. You see we have already proved that a normed linear space is finite dimensional if and only if the unit sphere in it is compact.

In the topological vector space situation, there is no unit sphere, there is no way, there is no concept of unit sphere because there is no metric. So, the local compactness is an indirect way of bringing that ok. Once unit sphere is compact the disc was compact. I think I have been telling that for normed linear spaces, unit sphere is compact is equivalent to saying that 0 has a compact neighbourhood.

And that compact neighbourhood makes sense, that is the definition of local compactness here. So, we have brought back the ideas from metric spaces, a compactness property can be always generalized. So, its local compactness has come here ok. So, that is what it is and the expected theorem here is that now every locally compact vector space is finite dimensional and conversely. Converse we have already proved.

How and where? Because any finite dimensional vector space we have just proved that it is isomorphic to $\mathbb{K}^n$, linearly isomorphic to $\mathbb{K}^n$. So, in particular it is homeomorphic to $\mathbb{K}^n$ and therefore, we know that it is locally compact ok.

So, we have to prove that if it is locally compact then, it is finite dimensional. Now, the proof of this one maybe a little longer I am not very sure, but it is much more elegant, than the proof for normed linear spaces ok. It is more or less canonical proof here, so just observe that ok?

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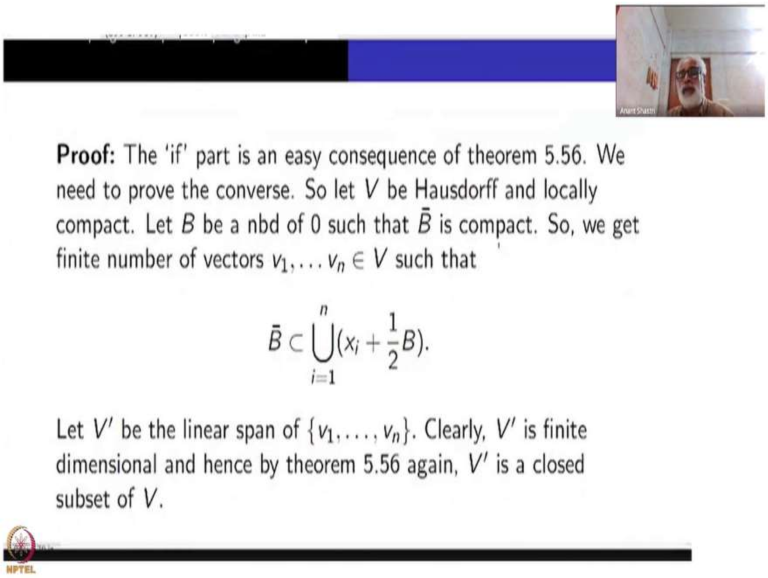

The 'if' part is an easy consequence of an earlier theorem that we have proved, that every finite dimensional vector space in a topological vector space is isomorphic to $\mathbb{K}^n$. So, let us prove the converse. Now, let V be Hausdorff and locally compact ok, did I mention Hausdorffness in the hypothesis? Hausdorff is yeah it is Hausdorff and locally compact, ok?

Hausdorffness is a must here ok, locally compact and finite dimensional vector space are automatically Hausdorff right? So I do not have to worry about that. So, choose a neighbourhood B, such that $\bar{B}$ is compact that is local compactness. So, we get finite number of vectors $v_1, v_2, \ldots, v_n$ belonging to V such that $\bar{B}$ is contained inside finitely many translates of half $\bar{B}$. Like half the balls around zero.

v_i plus this one would be a neighbourhood of v_i, v_i's are taken all over V they will cover $\bar{B}$, but then you can choose finitely many $v_1, v_2, \ldots, v_n$, so set i going to 1 to n, $v_i + (1/2)B$ will cover the whole of $\bar{B}$, ok. So, let V' be linear span of $v_1, v_2, \ldots, v_n$. Then V' is finite dimensional, because it is spanned by only n vectors. Therefore, by 5.56 theorem just we have proved V' is a closed subsets of V, here we have used Hausdorffness of V.

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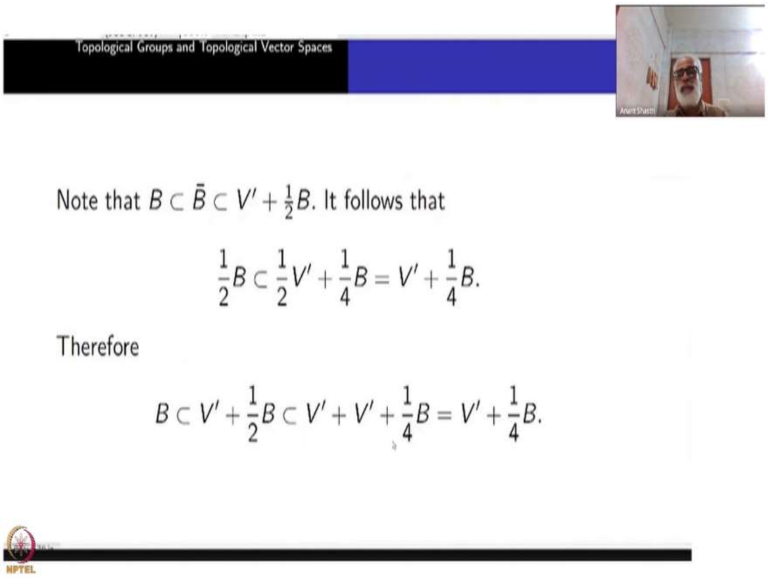

V' being finite dimensional subspace, is a closed subset, this is the previous theorem, part of previous theorem alright. Now, look at B, which is contained inside $\bar{B}$ obviously, and that will be contained inside $V' + (1/2)B$, because what is V' contains all the v_i's.

So, this whole thing I can replace it by one single element, V' which is very huge actually compared to what I have written there. $\bar{B}$ is contained $V' + (1/2)B$, right?

Now, we keep using the property that V' is a vector subspace, which is a strong property. It follows that if you take half of B, that will be contained in half of V' plus half of half of B right? But, half of V' is V' itself, because it is a vector space right?

So, $(1/2)B$ is contained inside $V' + (1/4)B$. Therefore, start with B which is contained in $V' + (1/2)B$ which in turn is contained in $V' + V' + (1/4)B$. Now, I can combine $V' + V'$ into just V', again because V' is a vector space. to get that B is contained in $V' + (1/4)B$.

You repeat this process now. Next time what we will get? B is contained inside $V' + (1/8)B$. And next time it will give $(1/2^n)B$. So, this is just repeating this process that is all, right.

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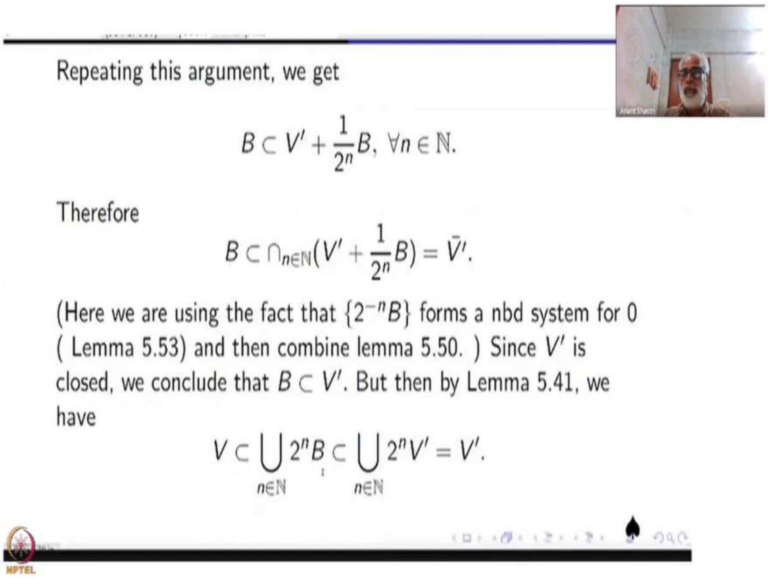

So, what is this? This is that we get B is contained inside $V' + (1/2^n)B$ for every n. That means of B is contained inside the intersection of all these.

This sequence $(1/2^n)B$, is a decreasing sequence of sets right? So, it is intersection of all these things, because it is contained inside everything, but you remember that this forms a neighbourhood system for 0, this is what we have proved last time. Therefore, this intersection is nothing but the closure of V', but the closure of V', what is that? Closure of V' is V' itself right, since V' is closed. We conclude that B is contained inside V'.

The moment an open subset, non-empty open subset is contained inside a vector subspace, that vector subspace must be the whole of the space ok. So, this follows from our lemma that we have we already used, namely V is contained inside powers of 2 times B, union of all those things. Any increasing sequence of numbers will do this job ok, increasing sequence I mean strictly increasing converging to infinity, I should say ok.

Well that is same thing as now, it is contained inside union of all $2^n V'$ because B is contained in V', but $2^n V'$ is V' itself, so it is union of V' is just V'. The entire vector space is contained inside the subspace; that means, they are equal.

So, what we have to done? We started with V, we produced some finitely many elements with certain properties and then we took the linear span V' of those elements and then we showed that V is equal to V'.

Anyway; so what we have done is that V is finite dimensional ok under the assumption that it is Hausdorff and locally compact, alright? So, two very important theorems we have done, very nice and elegant theorems.

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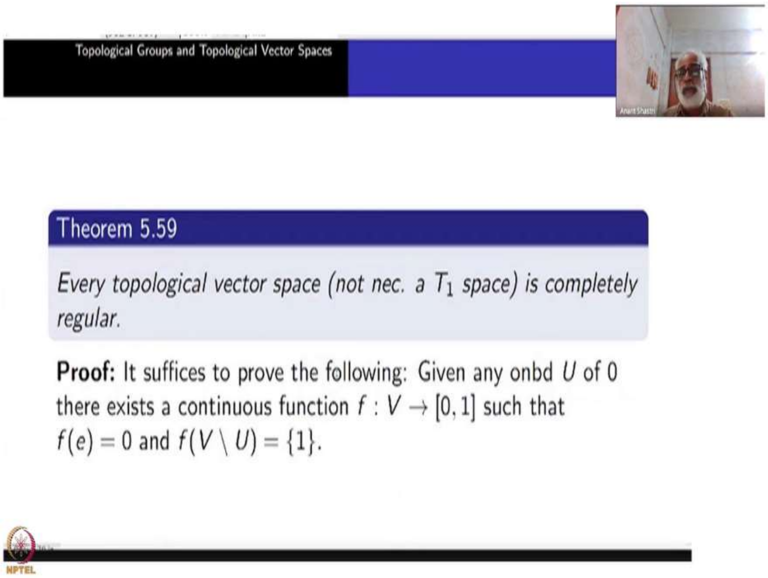

The third one is again curiosity ok, see you have proved that a topological group is a regular space without any further assumption. Now what is the best thing you can say about a topological vector space without any further assumptions?

So, here is one, namely every topological vector space is completely regular. By the way it is very important, complete regularity, though in this course, you could not see any of these importance.

One is about embedability theorems. Because continuous functions are coming there right. Another one is what is called a uniform topology. So, this we will not be able to discuss here. So, there is something important in saying that something is a completely regular and that is it. So, every topological vector space is completely regular.

If we assume T_1, it will be $T_{3\frac{1}{2}}$ ok. So, this is just for a little short of normality, if we would have proved the normality then that would have been even great, but that is not true generally perhaps ok.

So, I am not very keen on that one. Now, let us prove this one ok it takes a little more time, but let us prove that in a topological vector space V, given any point and a closed subset I must find a continuous function. But, you do not have to do this for arbitrary points, because we are working in a topological vector space.

A point can always be specialized to be 0 ok, given any neighbourhood U of 0, we must find a continuous function f on the entire vector space V to the closed interval $[0, 1]$ such that $f(0)$ is 0 and $f(U^c)$ is singleton one.

The entire complement of the open set is goes to one point, the 0 goes to another point. Those two points can be chosen anything arbitrarily, but we will conveniently choose them 0 to 1 and the interval also the domain to be also $[0, 1]$ ok. You could have denote (A, B) and so on, there is no problem this is what we have to prove ok.

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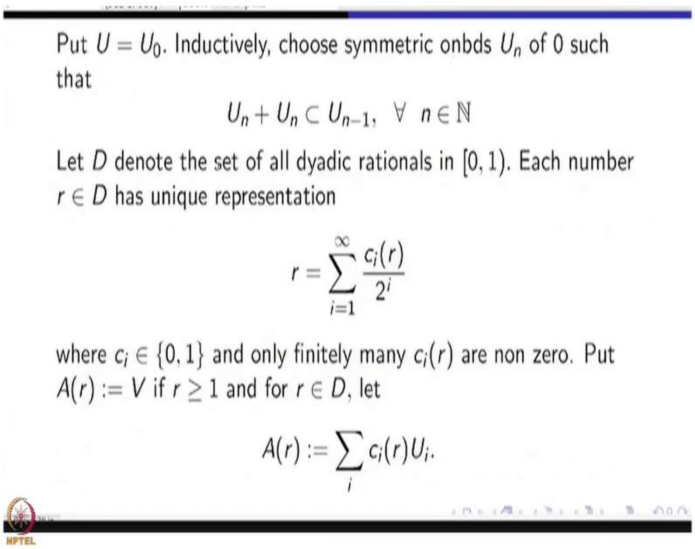

So, somewhat like in the theorem of Urysohn's, we have to work, but it is easier than the proof of Urysohn's lemma. Start with U_0 equal to U. Inductively, we are going to find a sequence of open subsets ok. Now, we are using topological vector space condition here. So, it is easier than doing it in an arbitrary normal space.

Inductively choose symmetric open neighbourhoods U_n of 0, such that $U_n + U_n$ is contained inside U_{n-1}. Remember symmetric means U_n inverse is U_n, but here inverse is $-U_n$. So, I could have written $U_n - U_n$ also, it will be same thing ok? $U_n + U_n$ or $U_n - U_n$ is same thing because U_n is symmetric. Symmetric here with respect to the addition ok?

So, $U_n + U_n$ is contained inside U_{n-1}, for every. How to get that? Start with U_0, take a U_1 which has this property $U_1 + U_1$ is contained inside U_0 and keep doing that ok.

Now, let D denote the set of all dyadic rationals in the interval $[0, 1)$ including 0 but not 1. What are dyadic rationals? Each $r \in D$ has a unique representation as a $\sum_{i=1}^{\infty} c_i r/2^r$ ok? So, it is a unique representation here ok, where all these $c_i(r)$, the denominators are either 0 or 1 ok, and only finitely many $c_i(r)$ are non-zero. This is my definition of these dyadic rationals. I have given you complete description of this, what dyadic rationals in $[0, 1)$ mean.

Now we define a collections of subsets $A(r)$ of V indexed by some non negative numbers. Put $A(r)$ equal to V if $r \geq 1$ ok? And if r is inside D contained in $[0,1)$, then put $A(r)$ equal to sum $c_i(r)U_i$; look at these numbers $c_i(r)$ either 0 or 1, they are the scalars right? So, I can put $c_i(r)U_i$ this makes sense because we are working in a topological vector space. The U_i's are subsets of V, 0 times U_i is just the 0, that is a 0 vector and if $c_i(r)$ is 1, then it is just U_i, ok? Then you are taking a finite sum of all these subsets. So, what I should mention here is, only just finite sums make sense.

Only finitely many $c_i(r)$'s are non zero. I have put infinity here, as if convergent sequence and so on. But, for dyadic rationals r, only finitely many c_i's are non-zero ok. Sicne I do not know how many that is why the upper range is put infinity. So, this is just 0 to n and then n going to infinity all of them are there, that is all.

So, c_i's are 1 and most of them are 0 here. So, this is a finite sum ok? So, this make sense, and since each U_i is open and each $A(r)$ will be open subsets, alright.

We first make a few observations on the subsets $A(r)$.

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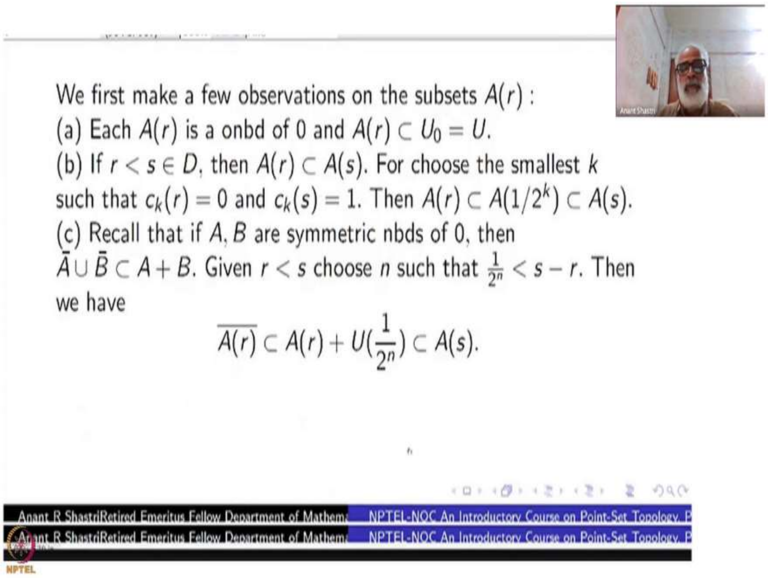

Each $A(r)$ is an open neighbourhood of 0 and $A(r)$ is contained inside $U_0 = U$ that is by definition. See $U_1 + U_1$ is contained inside $U, U_2 + U_1$ is also contained inside U_1 contained in U_0 and so on. So, $A(r)$ is contained inside U_0, this is important.

If $r < s$ inside D then $A(r)$ will be contained inside $A(s)$ ok. So, these are similar to what you have done when proving Urysohn's lemma somewhat similar, but easier steps here ok. So, how I see this is true? For each fixed r, there are one $c_1, c_2, \ldots, c_k$'s right? Choose the smallest k, so that $c_k(r)$ is 0 and $c_k(s)$ is 1.

Among the two sequence of $c_k(r)$ and $c_k(s)$, you keep comparing the corresponding terms, the first they differ, we must have $c_k(r) = 0$ and $c_k(s) = 1$, because $r < s$. Then $A(r)$ will be contained inside $A(1/2^k) = U_k$ and $A(s)$ will contain U_k, it is at least as big as that. That means, $A(r)$ is contained in $A(s)$.

Next, recall that if B is symmetric neighbourhoods then for any subset A, we have proved that $\bar{A}$ is contained in $A + B$. This is what we have seen for topological groups itself ok. Given $r < s$, choose n such that $1/2^n < s - r$ which is some positive number. So, $r < s$, $s - r$ will be some positive number. So, choose n sufficiently large.

Then $A(\bar{r})$ will be contained inside $A_r + U(1/2^n)$ ok. (See what is this notation? U_0, U_1 and so on we know. I could have just put a suffix here, but I have put a bracket here, just because I have indexed it with $1/2^n$ rather than n, just for emphesis. Correct notation would be U_n.) Anyway, $\overline{A(r)}$ is contained in $A(r) + U_n$ which is contained in $A(s)$. For to get s from r you have add at least $1/2^n$ and hence U_n will come as a summand in $A(s)$, alright?

So, (a), (b), (c) we have proved.

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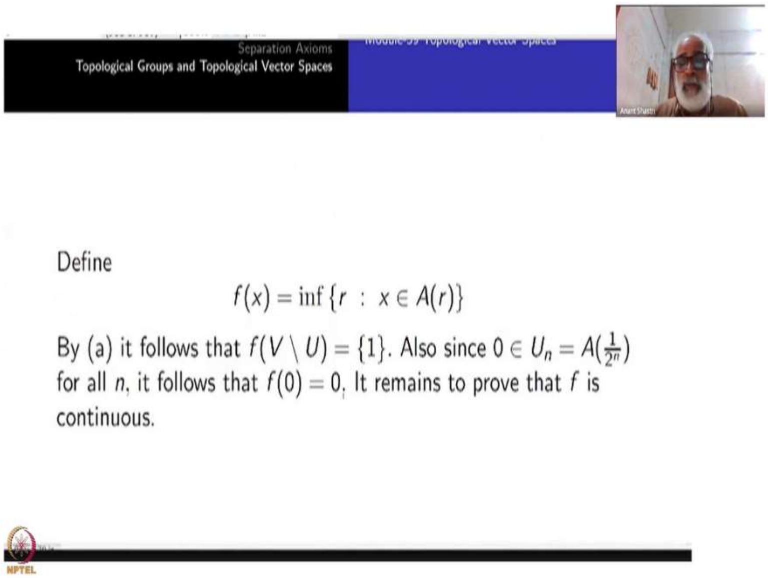

Now, we define the function f on V. For x in V, let $f(x)$ be equal to infimum of r; for r in $[0, 1]$ such that x is in $A(r)$ provided this is set is non empty. You see if there is at least one dyadic rational r such that x in $A(r)$ then $f(x)$ will be some real number in $[0, 1)$. Otherwise, we have defined $A(r)$ to be the whole of remember this will be the whole of V and x will be equal to 1. In either case, f makes sense.

Therefore it follows that $f(V \setminus U)$ is 1 because the moment it is outside U, there will not be any element $r \in D$ such that x is in $A(r)$ and $A(1)$ is the whole of V. So, that infimum of this will be 1. So, this is by definition.

Note that U_0 contains U_n, and U_n is $A(1/2^n)$ ok?

$A(r)$ is the sum of all these, but when $r = 1/2^n$, there is only one set on the RHS there right. So, that will be U_n. U_n is $A(1/2^n)$, for all n this is case. Therefore, if x belongs to U_n for all n, infimum will be 0. So, $f(0)$ is 0.

It remains to prove that why f is continuous. We have proved that $f(0)$ is 0, $f(V \setminus U)$ is 1, that is fine. Now, continuity of f is what we have to do. Note that there is a complete analogy between Urysohn's lemma and this one, if you had not understood Urysohn's lemma you will not have been able to prove this one ok. I do not claim that if you have understand Urysohn's lemma you would be able to prove this one, but now you have adopted those ideas in the case of topological vector spaces. So why f is continuous?

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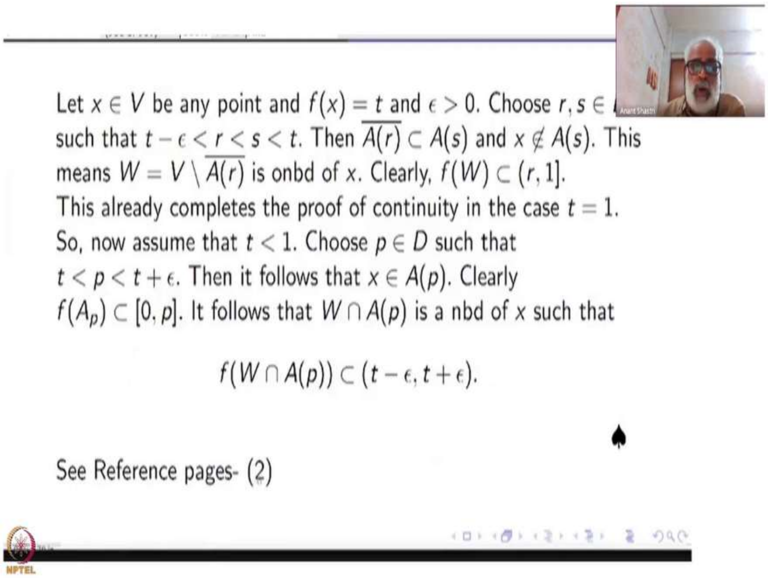

Ready? Start with any point x in V and put $f(x) = t$. And take $\epsilon > 0$ ok. Assume $t > 0$. Choose r and s belonging to D, such that $t - \epsilon < r < s < t$. What is the definition of $f(x)$? It is the infimum of on some set ok. So, therefore, you will have r and s inside the dyadic rationals, such that $t - \epsilon < f(x)$ right, which less than $r < s < t$. So, between r and t, t is larger, s is between them.

So, first you choose, r here between $t - \epsilon$ and t and then you can choose s between r and t. This is possible because of the definition of f and density of dyadic rationals. There is nothing more than that ok. So, then $\overline{A(r)}$ will be contained inside $A(s)$ and x is not inside $A(s)$ ok. Because $f(x)$ is t, which is bigger than s. So, x cannot be inside $A(s)$. If it is inside $A(s)$, the value of $f(x)$ would be smaller than s, right? This means that if you take W equal to $V \setminus \bar{A}(s)$, ok, then W is an open neighbourhood of x ok and $f(W)$ is contained inside the interval $(r, 1]$. So, the set cannot have anything less than s. So, the infimum has to be bigger than or equal to s, so it is in the interval $(r, 1]$.

It cannot be bigger than 1 anyway, all of them are less than equal to 1 only ok. $f(W)$ is contained inside $(r, 1]$, ok.

This already completes the proof of continuity in the case $t = 1$, we have found a neighbourhood as required.

I do not say $t = 1$, I start with x, put $f(x) = t$ that is all ok? It may happen that this $t = 1$, then this argument says that already the continuity of the function f at points x such that $f(x)$ equal to 1 is proved ok. That is a special case alright.

Now, you assume $t < 1$; Then we are inside dyadic rationals ok. Choose p belonging to D such that $t < p < t + \epsilon < 1$. (You can assume ϵ is sufficiently small so that $t - \epsilon$ is positive and $t + \epsilon$ is less than 1.) Given any ϵ, I must produce an appropriate neighbourhood ok, that is what I have to do. Then it follows that x is inside $A(p)$ because $t < f(x)$. Now, for points of $A(p)$, f is taking values between 0 and p. Once x is already in $A(p)$, $f(x)$ is smaller or equal to p. If it follows that the same W as before, the same thing here intersection $A(p)$ is a neighbourhood of x, such that $f(W) \cap A(p)$ will be contained inside the interval $(t - \epsilon, t + \epsilon)$.

Of course the case when $t = 0$ remains, but then that is similar to the case when $t = 1$. We then ignore W and take the nbd $A(p)$. It follows that $f(A(p))$ is contained in the open interval $[0, \epsilon)$.

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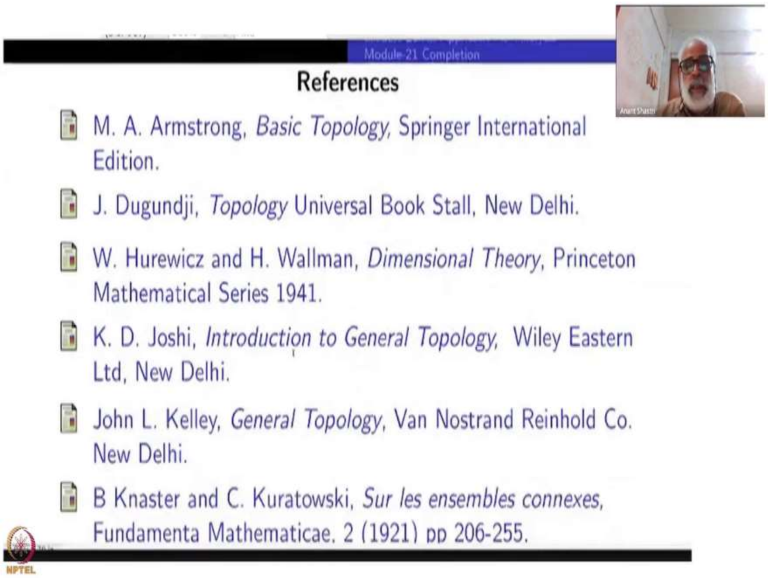

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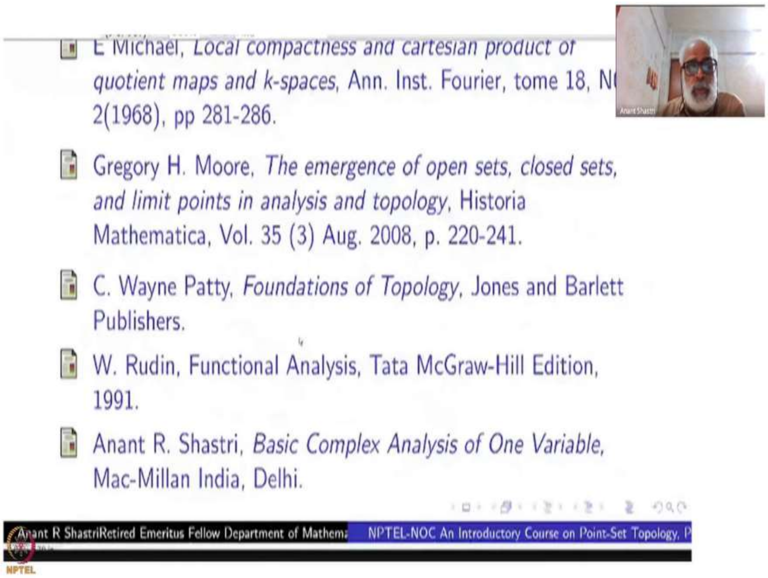

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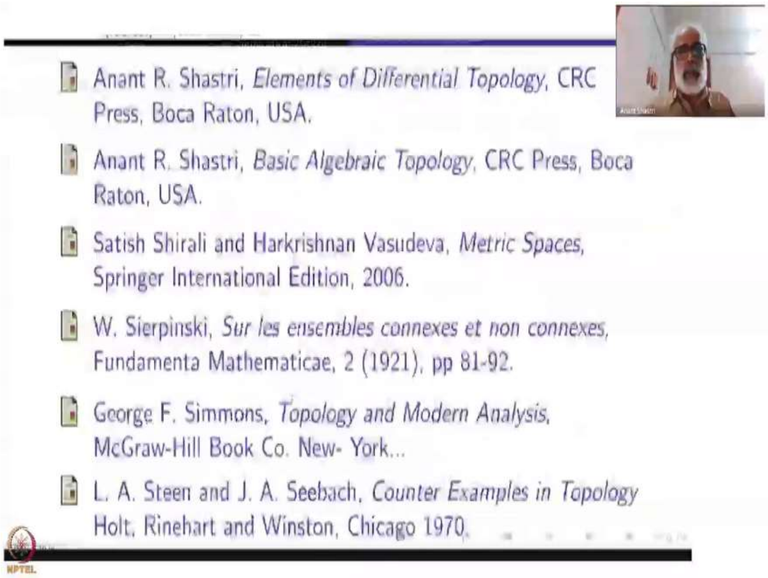


So, that completes the proof of the theorem and the whole plan whatever I have made for part I of this course.

I just repeat what were the reference books here. I was completely influenced as a student and even today by the book of Simmons; George Simmons. And in spirit, though not in content, I have followed this book. For example, the definition of T_0 space, T_1 space normal space and so on. Normal and T_3, T_4 and so on. My definitions of these concept differ from that given in Simmons book.

So, be cautious about that, I have already told you so earlier. So, be cautious about that.

The book by Steen and Seebach. This book I have refered to is Counter Examples in Topology. This book is now available online and from whatever I had seen it has expanded much. So, since 40 years back ok, it has expanded quite a bit. So, so this is good not necessarily for learning topology, but as a reference book it is nice that is what I have said.

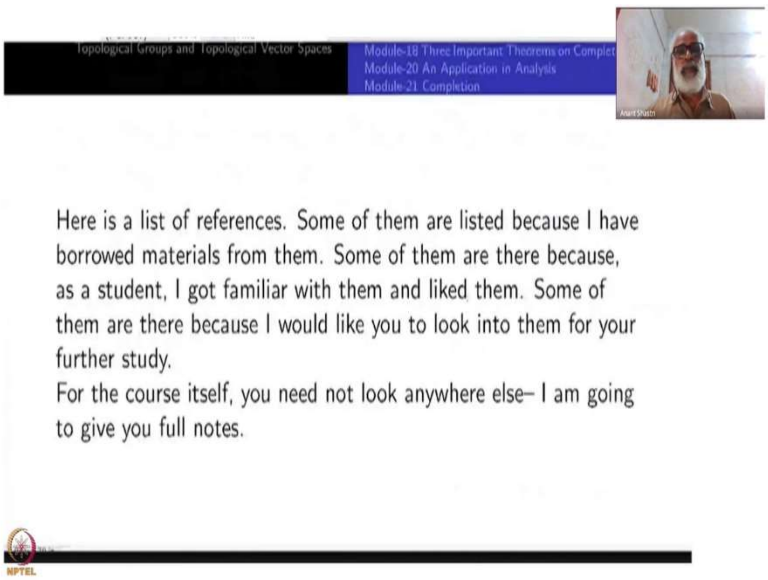
So, my second favourite book in topology, we may say is Kelley's book which is a very fantastic book, but difficult to read as compared to Simmons. Simmons' was a pleasant one but I enjoyed Kelley's book also.

Later, when I joined IIT Bombay, I came across a book my colleague had written, a nice book written by K D Joshi, an Introduction to Topology. So, I have borrowed material from this book also. I have high regard for this book. So in fact, the existence of this book stopped me writing a book on my own on point set topology alright.

Before that at TIFR, I had come across another book, it was my third favorite book in topology, Hurewicz and Wallman's Dimensional Theory. This is a very specialized book, but it teaches you topology like anything, so this is a wonderful book ok? It deals with dimension theory. And then there are some nice books, which I have just browsed through. I have not studied them and so on.

One is by Armstrong. Another one by C Wayne Patty ok. So, here you can see Rudin's Functional Analysis, which I used for topological vector spaces and so on ok. So, that is roughly what it is; so you can also look into some papers here, from which I have borrowed material. All these books they are good books Dugundji and what are the other ones, here well that is it. Satish Shirali's nice book on Metric Spaces ok.So, that is what it is.

However, I want to tell you, I remind you that I am going to give notes and you have already the notes with you, Just to get through this whole course you do not need anything else. So, you just go through the notes properly, work out the assignments and you are done, ok. So, yeah, so that is all I wanted to tell you about this.(Refer Slide Time: 46:57)



So, finally, I would like to end up this one with a lot of thanks to the NPTEL team, to my own team of tutors, whose help has been extremely useful, extremely great help, moral support and so on in bringing out this course to you. And also big thanks to all of you, if you have stayed with me so far. I hope I will see you in the 2nd part also ok.

Thank you.