Introduction to Point Set Topology, (Part I)
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Lecture - 60
Topological Vector Spaces-continued

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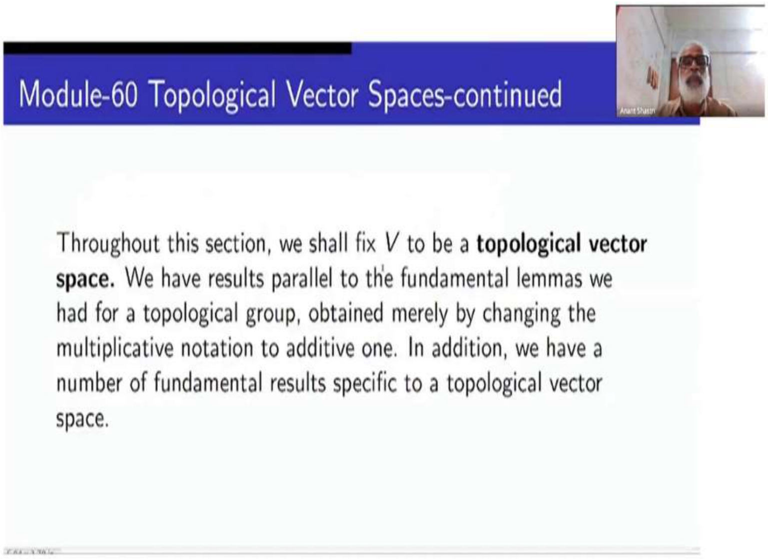

Welcome to module 60 of Point Set Topology, Part I. We shall continue the study of Topological Vector Spaces today. As earlier throughout this section, we shall fix V to be a topological vector space. I will not use this ah notation for anything else.

We have results parallel to the fundamental lemmas we had for a topological group, obtained merely by changing the multiplicative notation to additive one. Because, if you have topological vector space V, together with the operation $+$ and the 0 element there, will give you a topological group which is abelian.

So, we shall denote them by additive notation plus instead of the multiplication dot, the multiplication will be there only not from $V \times V$ to V, but only from $\mathbb{K} \times V$ to V now. So, all the statements which were multiplicative for the topological group with that we have seen, will now be obtained by just changing them to additive notation.

In addition, because of the scalar multiplication and the continuity of that, we will have many other important results ok. So, for all those things, the new ones, I will give you hint or a full proof of those things, but those which we have already got for a topological group as such, those I will only restate if at all.

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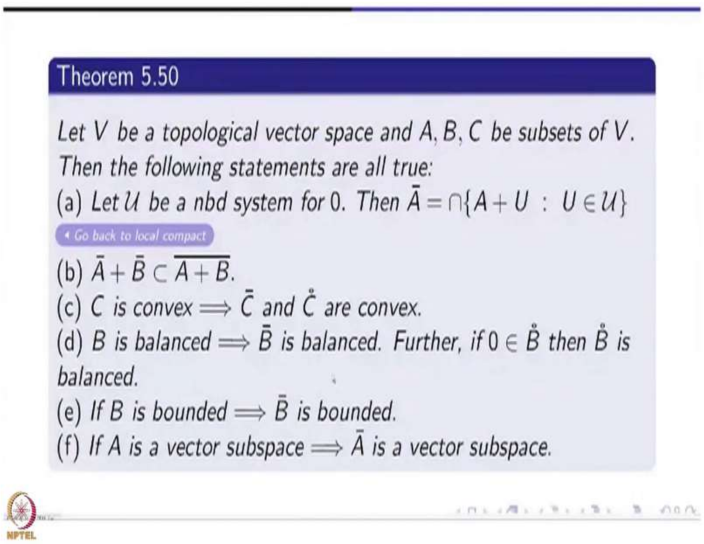

So, with this, the first theorem here is again a list of statements: (a), (b), (c), (d), (e), (f) which are all easy consequences of the definition of a topological vector space.

Let V be topological vector space. Start with any $A, B, C \subset V$. Then all the following statements are true:

(a) Let $\mathcal{U}$ be a neighbourhood system of 0, then for any subset $A, \bar{A}$ will be equal to the intersection of all $A + U, U$ inside $\mathcal{U}$. Remember there was exactly similar result even stronger intersection of AU multiplicative notation in the case of topological groups AU or you can take UA and so on ok.

And, you do not have to take all the neighbourhoods, just a neighbourhood system will do ok? So, that is also an easy consequence anyway ok.

(b) So, the next one is $\bar{A}+\bar{B}$ is contained inside $\overline{A+B}$. Once again the corresponding thing for topological groups was that $\bar{A}\bar{B}$ is contained inside $\overline{AB}$. So, same thing is true for this one, I do not have to elaborate this one.

(c) The next one is new to topological vector spaces only. Because, now I am using convexity, this convexity notion was not available in a topological group. Let C be a convex set then both $\bar{C}$ and interior of C are convex.

(d) B is balanced implies $\bar{B}$ is balanced; further if 0 is in B interior (which is same thing as saying that B is a neighbourhood of 0), then interior of B is also balanced. See here unless you have 0 here you cannot claim that one ok?

(e) The fifth one; if B is bounded then $\bar{B}$ is bounded. So, I have taken convexity, balancedness and boundedness ok.

(f) If A is a vector subspace, then this closure is also vector subspace ok.

So, statements (c), (d), (e), (f) are new to topological vector spaces; (a) and (b) are old one only right?

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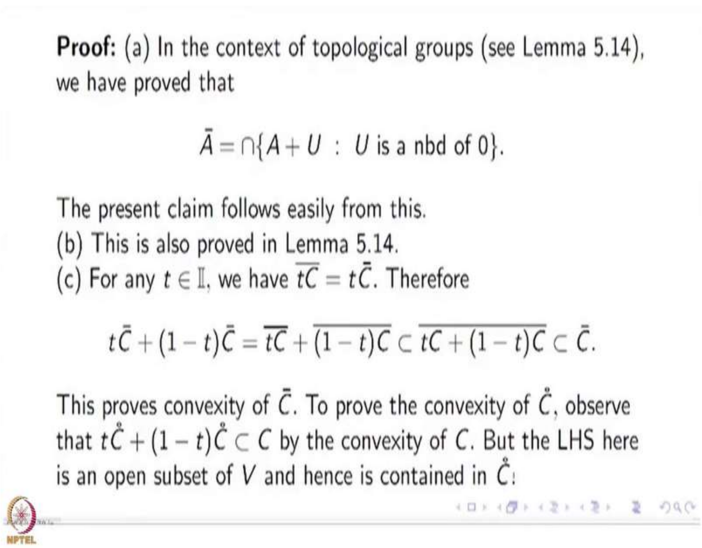

Proof: (a) In the context of topological groups (see Lemma 5.14), we have proved that

$$\bar{A} = \cap\{A+U \ : \ U \text{ is a nbd of } 0\}.$$

The present claim follows easily from this.
(b) This is also proved in Lemma 5.14.
(c) For any $t \in \mathbb{I}$, we have $\overline{tC} = t\bar{C}$. Therefore

$$t\bar{C} + (1-t)\bar{C} = \overline{tC} + \overline{(1-t)C} \subset \overline{tC + (1-t)C} \subset \bar{C}.$$

This proves convexity of $\bar{C}$. To prove the convexity of $\mathring{C}$, observe that $t\mathring{C} + (1-t)\mathring{C} \subset C$ by the convexity of C. But the LHS here is an open subset of V and hence is contained in $\mathring{C}$:

Let me just go through them again. In the context of topological groups, you can check Lemma 5.14 ok. We have proved that $\bar{A}$ is equal to intersection of $A+U$, such that U is a neighbourhood of 0.

If you take all the neighbourhoods then this is just in the multiplicative notation you have done; so, it is true for this one also. But, then you can just restrict it to a neighbourhood system that will be a smaller family of same members of this family itself, that is smaller family. So, intersection has to be larger ok.

But, here the smaller family has the property that each member is contained in some member here therefore, the equality occurs. So, the B is directly from Lemma 5.14 for products namely $\bar{A}\bar{B}$ is contained inside $\overline{AB}$, which will become now $\bar{A}+\bar{B}$ contained inside$\overline{A+B}$, ok.

Now, come to statement (c). If C is convex I must show that $\bar{C}$ is convex. First of all, for any $t \in [0,1]$, we have $t\bar{C}$ is equal to $\overline{tC}$. If t is a homeomorphism and therefore $t\bar{C}$ is equal to $\overline{tC}$.

In particular, this is also true for $1-t$ in place of t. So, I can take sum of these two which is nothing, but $\overline{tC}$ plus $\overline{(1-t)C}$. now we can use (b) to conclude that this is contained in the bar of the sum. Finally, since C is convex, we have $tC+(1-t)C$ is equal to C. No problem ok. So, this proves the convexity of $\bar{C}$, ok?

Now, prove the convexity of the interior: Observe that tC interior plus $(1-t)C$ interior is already contained inside C. Because, C is convex. But, now tC interior is an open set and similar;y, $1-t$ times this one is also an open set. Maybe one of them may collapse to a single point that is the caution. In that case, either $t=0$ or $1-t=0$, it is not an open set, but then the containment of LHS in RHS is obvious. So, we can assume that $0<t<1$.

Therefore, this part is always an open subset of V and hence it is contained inside the maximum open subset that is interior ok. LHS is contained inside C, but this is open therefore, it is contained inside C interior. So, C interior is also convex.

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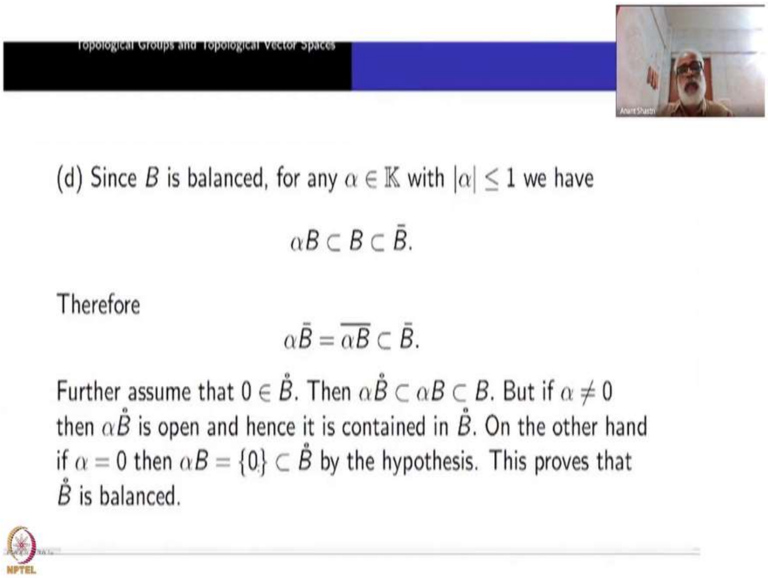

(d) Now, let B be balanced. Then for any scalar α such that $|\alpha| \leq 1$, we have αB is contained inside B. So, it is contained inside $\bar{B}$ also ok. Now, $\bar{B}$ is a closed set right? Therefore, if you take $\overline{\alpha B}$ which is same thing as $\alpha \overline{B}$ ok, $\overline{\alpha B}$ is the smallest closed subset containing αB must be contained inside $\bar{B}$ ok. So, that proves that $\bar{B}$ is balanced.

Further, now we assume 0 is in the interior of B. Then αB interior is contained inside αB, that is contained inside B. If $\alpha \neq 0$, then αB interior is open subset and hence it is contained inside interior of B.

On the other hand, if $\alpha = 0$, then this will reduce to single point 0, and is there in B interior by assumption, (otherwise there will be problem). So, you are done. So, this proves that B interior is also balanced ok, αB interior in both the cases is contained inside inside B, but then it is contained inside B interior. So, B interior is also balanced.

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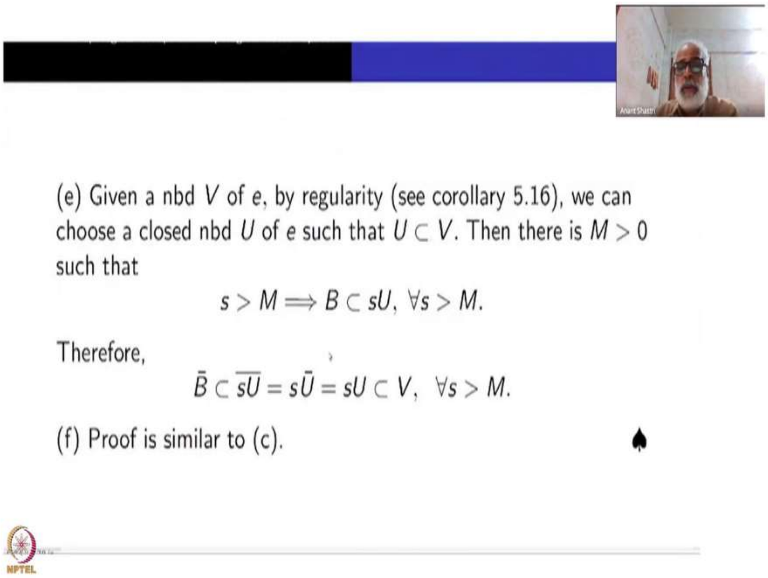

Now, come to (e). Let us go back at that statement (e) here: If B is bounded then $\bar{B}$ is bounded That is what we have to prove. What is boundedness? Given a neighbourhood V of identity ok, I must produce $M > 0$ such that $s > M$ implies $\bar{B}$ is contained in sV. That is what I have to do. $\bar{B}$ is bounded means that ok.

So, first I apply regularity of the topological vector space which we have proved earlier ok. Given a neighbourhood V of e, by regularity we can choose a closed neighbourhood U such that e is contained inside U contained inside V. Given any neighbourhood there is a closed neighbourhood contained inside that. You take this one to be a closed neighbourhood ok.

Now, apply the boundedness of B to get an M such that B is contained inside sU for every $s > M$, ok. Therefore, $\bar{B}$ is contained inside bar of sU which is nothing to $s\bar{U}$. But U is closed and so $\bar{U}$ is U which is contained in V by the choice. Therefore, sU is contained inside sV for every $s > M$ ok. So, the proof of that $\bar{B}$ is balanced, it is done ok.

(f) Now, the last one, it is a statement that if something whatever notation I have taken let me use the same notation.

So, A is a vector subspace then $\bar{A}$ is a vector subspace. The proof is similar to what we had for topological groups also ok. But, here, you can look at this one, the proof of (c). Here it is only for the scalar between 0 and 1. But, the entire discussion you know, can be applied to any scalar here ok because, now I am assuming that instead of C, I am assuming A is a vector space already. So, λA will be already inside A; therefore, the same proof will work here for (f). If A is closed, A is a closed under addition and scalar multiplication that is a vector subspace, then $\bar{A}$ is also closed under addition and scalar multiplication is what we have to see.

If u and v are in A, $\lambda u + \mu v$ is there, that is what you have to show right? Instead of $tu + (1-t)v$. So, the proof is the same. With the extra assumption that instead of C, an arbitrary convex set so, we now have a vector subspace A. That compeltes the proof of the theorem.

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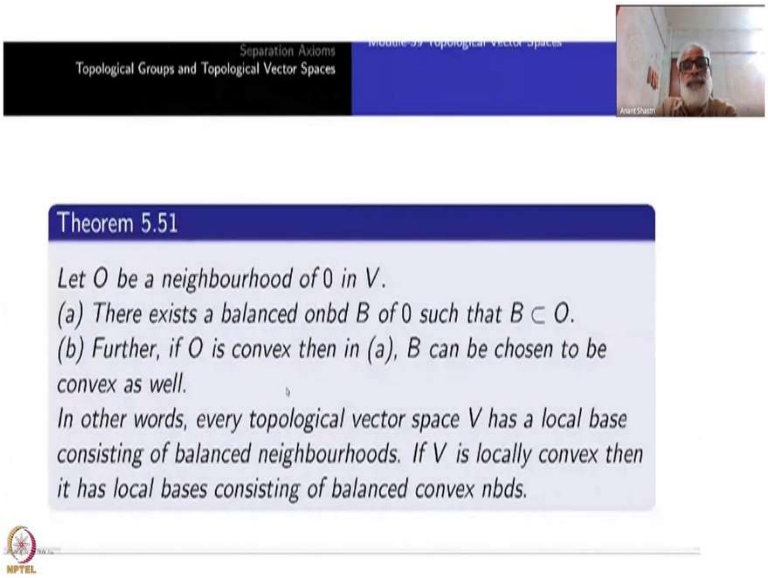

Let us go ahead, to the next theorem. Let O be a neighbourhood of 0 in V. The first claim is that there exists a balanced open neighbourhood B of 0 such that B is contained inside O. So, from an arbitrary neighbourhood, we can improve it to become balanced open neighbourhood, openness is obvious anyway, balanced neighbourhood that is a whole idea. Further, if this O is convex then in statement (a), we can choose B to B convex as well; that means, B is balanced and convex ok.

That is every topological vector space V has a local base consisting of balanced neighbourhoods, that is the statement (a). If V is locally convex; that means, it has convex neighbourhood like this, then it has a local base consisting of balanced and convex neighbourhoods. Here the important thing is you just assume one convex neighbourhood then there is a whole system of convex neighbourhoods ok? That is the whole idea here.

Student: Sir. This balanced open neighbourhood, this concept is it something similar to symmetric neighbourhoods in topological groups?

Teacher: It is just symmetric is just one single thing right, inverse corresponds to -1 here right. So, that is.

Student: Yes, sir.

That is then much weaker there. In a topological group you could not do anything more. Scalars allowed are only 1 and -1, that is all right, inverse is same as minus right. So, here is a stronger symmetry right, you may say only for unit scalars also that is also symmetry, but this is much more stronger ok.

It does imply symmetry, but it is much much stronger ok. So, I told you that this is almost like bringing concept of the balls inside a metric space ok. So, these things will play the role of open balls in a metric space. So, we are bringing them through a back door, it is like that ok. So, they will play the role of the open balls; so, that is much more stronger than just symmetry ok?

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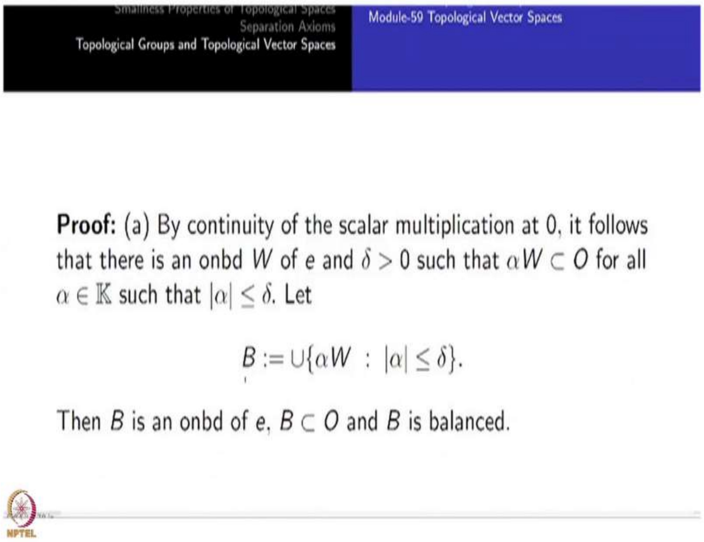

So, by continuity of scalar multiplication at 0, the moment you have some neighbourhood, there exist an open neighbourhood W of e such that if δ is positive such that αW is contained inside O for all α with $|\alpha| \leq \delta$, ok? O is a neighbourhood, 0 goes to 0 under scalar multiplication ok. So, you can control the scalar that is the whole idea.

There is a positive δ such that $|\alpha| \leq \delta$ would imply the whole image αW is contained inside O. So, this is by continuity of scalar multiplication alright. So, all that we are doing is the use of scalar multiplication here. Now, let B equal to union of αW, where $|\alpha| \leq \delta$. The first part says that this family is non-empty, there is at least one W.

Now, you take all such W which satisfies this and $|\alpha| \leq \delta$. There may not be any right? So, you have taken all such things where at least one such that αW is contained inside O, ok. Now, you take union of all of them, then B is an open neighbourhood of V ok, such B is contained inside O and B is balanced. We must put that $|\alpha| \leq \delta$, I would like to have this one contained inside O, but that is the condition here on alpha ok?

So, I would like all this αW's also contained inside O. So, in any case being the union of all open subsets αW's, B is open. B itself will be now balanced, why? Because, if if multiple say beta with $|\beta| \leq 1, \beta B$ will be contained inside union of $\alpha(\beta B)$, but if multiply α by β where $|\beta| \leq 1$, $|\alpha\beta|$ will be automatically less than equal to δ.

So, those things will be contained inside this one again, that is the whole idea. So, B is balanced and it is an open subset contained inside O and containing 0.

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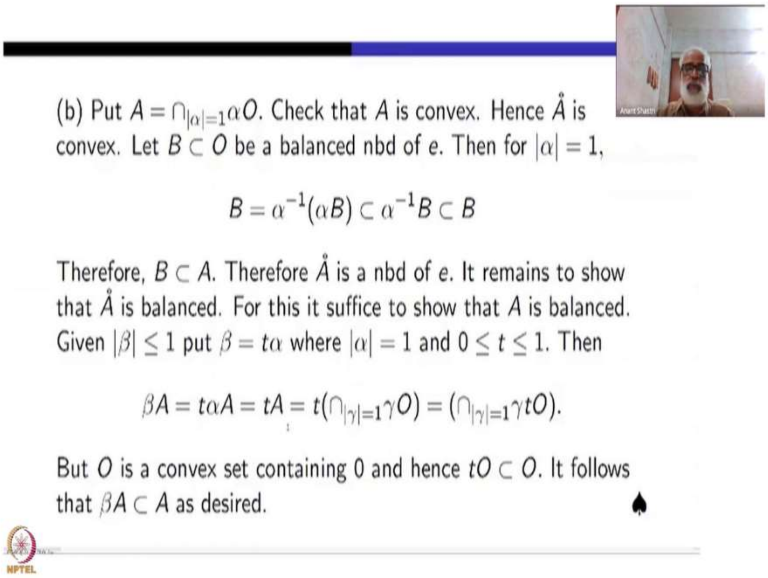

Now, in the second part, I want it to be what? In the second part, what is there? I want it to be a convex subset as well right?

So, suppose one of them is convex, that O itself is convex here already. Now, you take A to be the intersection of all alpha O, where $|\alpha| = 1$ ok. So, in the case the vector space is real vector space, this just means that O and $-O$, that is the way we have done to get a symmetric neighbourhood remember that?

So, here you have to take $|\alpha| = 1$ only to saturate it, intersection of all of them ok. Now, check that A is convex because O is convex to begin with. So, this we have to check that this will be also convex ok. Hence, interior of A is convex that is what we have seen right, if C is convex, interior of C is convex was one of the part of the previous theorem.

Now, let B contained inside O be a balanced neighbourhood of e. In part (a), we can take a balanced neighbourhood of e. Then, if $|\alpha| = 1$, then you can write B as $\alpha(\alpha^{-1}B)$, α and α^{-1} cancel out, but $\alpha^{-1}B$ is contained in B, because B is balanced and $|\alpha^{-1}|$ is 1. So, B is contained αB, but αB is contained in αO. Since this is true for all such α, it follows that B is contained in A which is the intersection of all αO's.

Therefore, A neighbourhood of 0 ok, because B is a neighbourhood 0. So, if you take the whole of interior A that will be a neighbourhood of 0.

It remains to show that interior of A is balanced alright? Interior is always convex. Now, interior is balanced is what I would show. For this it suffices to show that A itself is balanced, since we shown that if A is balanced, then A interior is balanced ok.

So, to show that A is balanced take $|\beta| \leq 1$; write $\beta = t\alpha$, where $|\alpha| = 1$ and $0 \leq t \leq 1$. Any β can be written as $e^{2\pi it}$ right times t, that is why you can write t, where $|\alpha| = 1$ and t will be between 0 and 1 non-negative ok.

Once you write like that βA is $t\alpha A$, it follows that βA is equal to $t\alpha A$ is equal to (you can push $t\alpha$ inside the intersection) intersection of $t\alpha\gamma' O$, where $|\gamma'| = 1$. That is equal to intersection of $t\gamma O$ where $\gamma = \alpha\gamma'$ and hence is of modulus 1.

But, O is a convex set to begin with ok, and containing 0 and hence tO is contained inside O, tO consists of all the line segments from 0 to points of points of Otz_1. So, all the tO will be also inside O. So, βA is contained inside instead of this t, I can just write O, right. Once, I write O, this whole thing will be nothing, but A right, βA is contained inside A.

Thus, what we have shown that, in general balanced neighbourhoods form a fundamental system of nbds of 0. If V is locally convex then convex and balanced neighbourhoods form a fundamental system of nbds of 0. ok?

These are fundamental results. Now, we slowly arrive at some concrete results.

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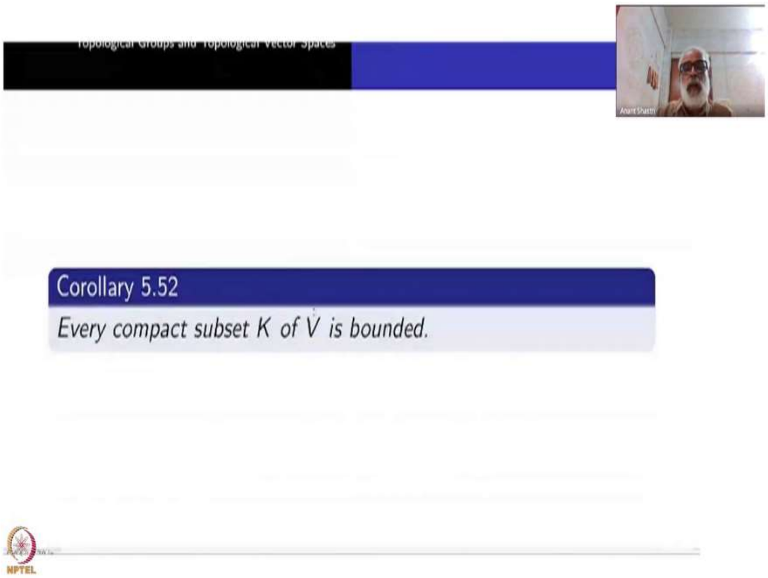

Every compact subset K of V is bounded. So, you see I told you that these balanced neighbourhoods are playing the role of balls in a metric space.

So, now, we can talk about compact subsets being bounded just like in a metric space ok. Of course, our notion of boundedness is also different here, this boundedness is stronger. That is what we have already remarked the last time.

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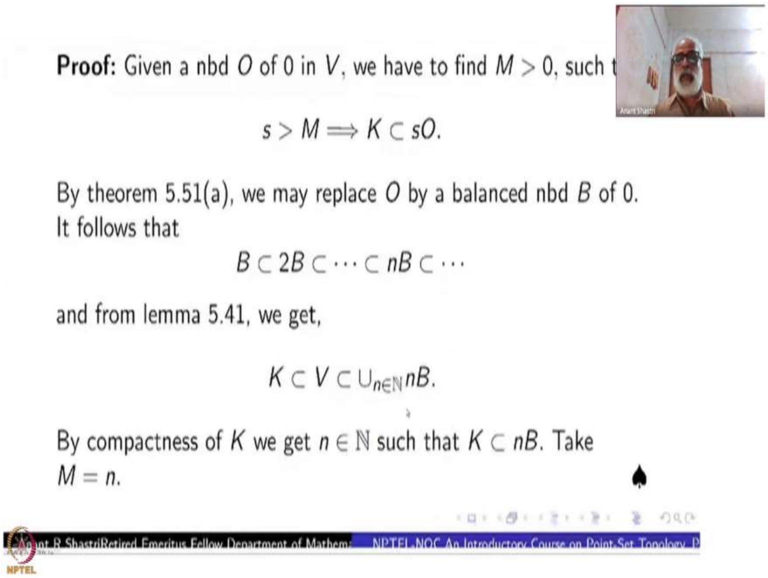

What is the proof? Proof is very easy, given a neighbourhood O of 0 inside V, we have to find some m positive such that $s > M$ implies K is contained inside sO.

For any neighbourhood, you find an M this this property that will mean that K is bounded, this is the definition of boundedness ok. Now, the previous theorem part (a) says that I may assume O is balanced neighbourhood. Suppose, I do it for a smaller neighbourhood, which balanced neighbourhood smaller than O, the same statement will be true for O also ok?

So, I can assume that O itself is balanced. But I am just using this B, B is a neighbourhood of 0 which is balanced and contained inside O. So, we shall now prove find an M such that $s > M$ implies K is contained inside sB. If you prove that then it will be contained inside sO also ok.

Now since B is a balanced neighbourhood, now B will be contained inside $2B$ contained inside $3B$ and so on.

So, this will be an increasing sequence. Just like expanding balls (centered at the origin). See in $\mathbb{R}^n$ what you would have done? You would have taken the ball with radius 1 say, it will given you a sequence concentric balls right? one contained inside twice of that contained in 3 times that and so on. This is precisely the property that I was hinting at from lemma 5.41 or whatever.

So, remember, if we have an increasing union of any neighbourhoods, where these numbers go to infinity then this whole thing is covering the entire vector space V. Just like in the case of $\mathbb{R}^n$ and so on. So, this is what is happening in a topological vector space also, that is from 5.24.

Now K is a subset of V anyway, but V is contained inside a union of nB's; so, that is 5.24. But, K is compact and these are open subsets. So, you will get one of them covering the whole thing because, it is increasing union ok. So, we get n such that K is contained inside nB. Now, you take M to be equal to this n, if $s > M$ again using the fact that B is balanced, we get K will be already inside nB, K will be also inside sB ok, because $s > M$ here. So, this M can be chosen as n. That completes the proof of the theorem.

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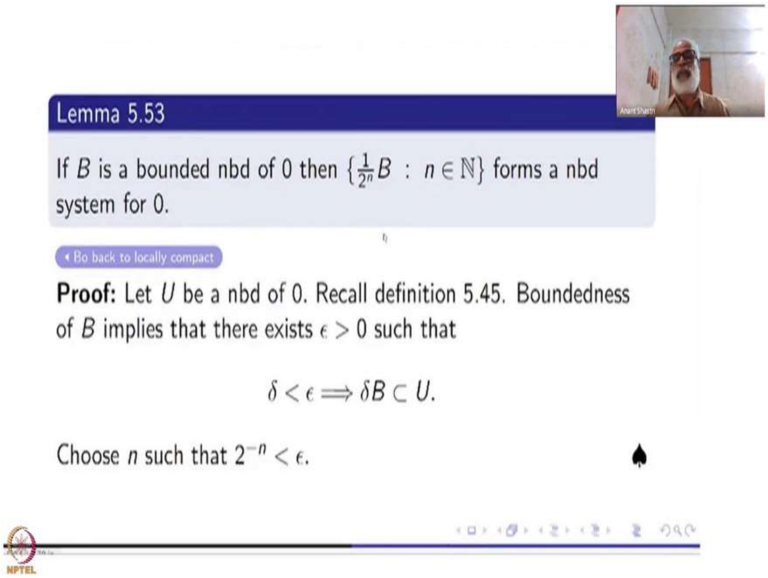

So, one way, you have seen the usefulness of the increasing union. And now there is another parallel concept. Decreasing union, decreasing intersections, balls of smaller and smaller radius right. So, similar thing here if B is a bounded neighbourhood of 0, then $(1/2^n)B$, where n ranges over all the natural numbers forms a neighbourhood system for 0.

It is just like the balls B_ϵ, where $\epsilon > 0$, is a neighbourhood system. Instead of that you can write $(1/n)B$ also or then you can $(1/2^n)B$ also right. So, similar to this is what we are going to prove $1/n$ etc epsilon into 0 etc are more difficult.

Just this lemma is nice and easy and that is enough, because this sequence `converges' to 0 ok? So, let U be any neighbourhood, recall, by definition of boundedness, boundedness of B implies that, there is another way of looking at it, instead of there exist M such that $s > M$ etc, you can invert the whole thing right?

There exist $\epsilon > 0$ such that $\delta < \epsilon$ implies δB is contained inside U ok, if B is contained inside sU, where $s > M$ is one definition, it is equivalent to $\delta < \epsilon$ implies δB is contained inside U. This ϵ can be chosen like this. Once you have that, all that you have to do is choose n such that $2^{-n} < \epsilon$, ok.

Then $(1/2^n)B$ for some n large, that will be contained inside U. For every neighbourhood some member is here inside U, means that this is a neighbourhood system over ok? That is the lemma.

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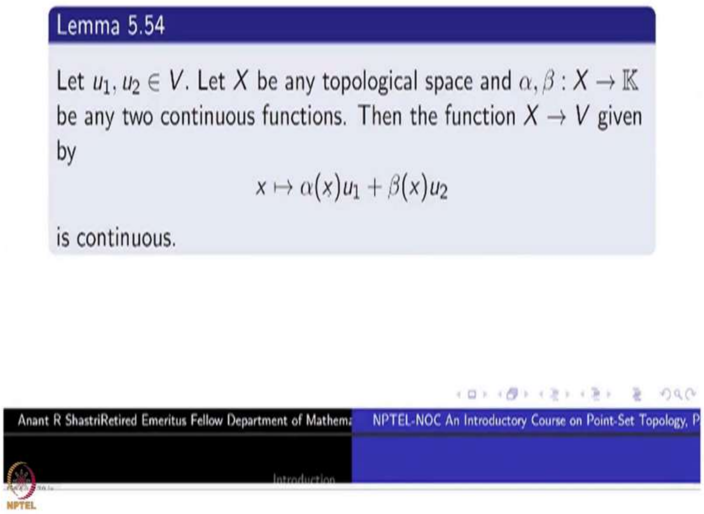

So, we shall use all these things in a neat way to, you know, to derive some nice results. Another easy and fundamental result here is: Take any two vectors ok, take any two vectors in V. Let X be any topological space and α, β from X to $\mathbb{K}$ be any two continuous functions ok. So, α and β are continuous functions into the scalars, $\mathbb{K}$ is a scalar field on which V is a vector space ok? Then you take the linear combination of $\alpha(x)u_1$, (that is a vector inside V) plus $\beta(x)u_2$.

This is a linear combination right, but it is a function now as x varies on X, into V. So, this function is continuous. These are elementary things which we have observed inside $\mathbb{R}^n$ and so on right. So, same thing we are observing in any topological vector space. What is the proof? Proof is precisely this one, namely scalar multiplication and additions are continuous that is all.

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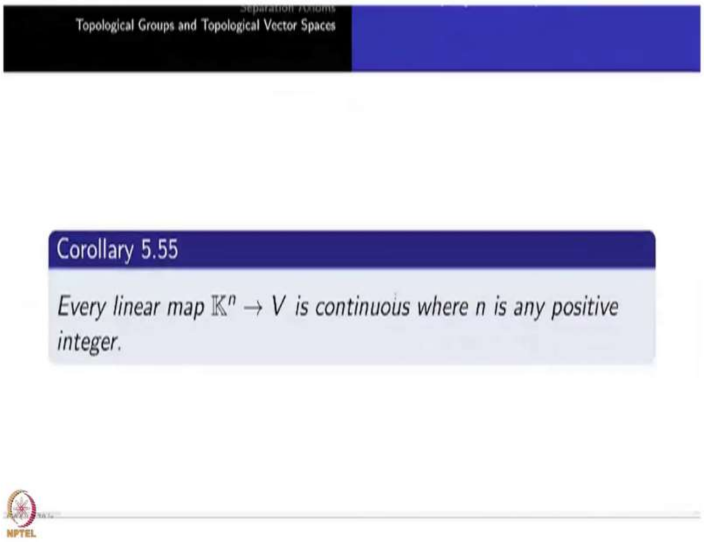

Now, we have an easy corollary here. Every linear map $\mathbb{K}^n$ to V is continuous, where n is any positive integer. This is a special case now ok, Earlier X was an arbitrary space ok. Given α and β continuous functions, then we proved the linear combination is continuous. Now, I want to say that every linear map from $\mathbb{K}^n$ to V is continuous. So, to talk about continuous linear maps, you need to have both domain and codomain as a topological vector spaces.

However, you cannot replace the domain here by an arbitrary topological vector space. From an arbitrary topological vector space to an arbitrary topological vector space, a linear maps may not be continuous ok? That is why this theorem, this corollary is important. It says that $\mathbb{K}^n$ which is nothing, but a finite dimension vector space over $\mathbb{K}$, then every linear map is continuous. Understand the importance of this one. If you replace this one by arbitrary topological vector space then it will makes sense, but it will not be true.

Proof is easy. Similar to what we do for $\mathbb{R}^n$ and any normed linear spaces, once we have that lemma ok?

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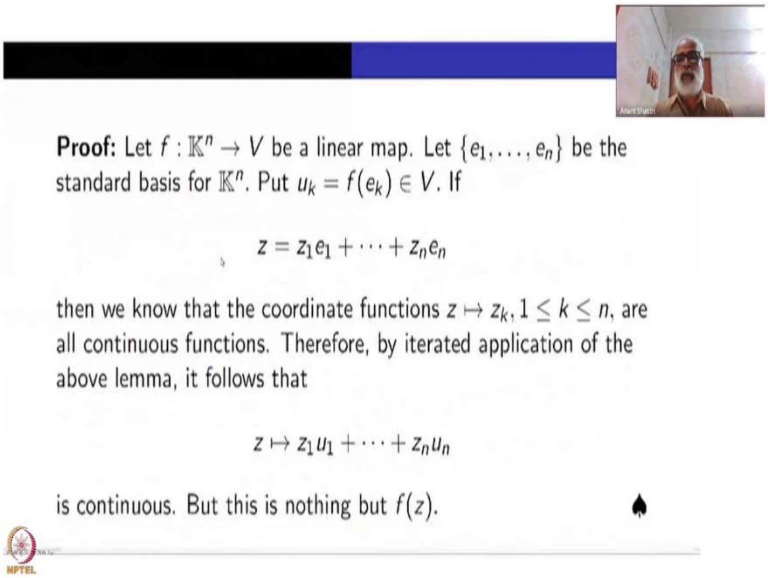

Let f be any linear map alright? Any linear map into a vector space from a vector space is determined by its values on a basis. So, let $e_1, e_2, \dots, e_n$ be the standard basis. (You could have chosen any basis, no problem.) Then, I will look at the images of $e_1, e_2, \dots, e_n$ under f.

So, put $u_k = f(e_k)$ ok? Now, every element z belonging to $\mathbb{K}^n$ can be written as $z_1 e_1 + \cdots + z_n e_n$, where z_i's are scalars inside $\mathbb{K}$ right. Then, we know that by linearity $f(z)$ is nothing, but $z_1 f(e_1) + \cdots + z_n f(e_n)$, right. Not only that the coordinate functions inside $\mathbb{K}^n$ to $\mathbb{K}$, z going to z_k, this is the k^{th} coordinate of z, right? They are also continuous functions. Therefore, all that I have to do is iterate the previous lemma which is for two functions.

z going to z_1 is a continuous function times e_1 plus z going to z_2 is continuous times e_2; so, add them. So, first we get sum of two terms is continuous, next third one is continuous and so on. What you get is z going to $z_1 e_1, e_1$ is what? $f(u_1)$ etc, $z_n e_n$ is what? $f(z_n u_n)$.

You know as a student, because we are all studying finite dimensional vector spaces only, linear maps are always continuous. But, suddenly when it goes infinite dimension and linear maps may not be continuous, that was a realization ok? So, it took some time, I really thought that I can prove that every linear map is continuous, but that is not the case ok. So, I want to emphasize that fact here.

So, let us stop here today. So, next time we will reap a good harvest and prove three important theorems and that will be the end of the course. So, today we will stop here.

Thank you.