# Introduction to Point Set Topology, (Part I)
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**Module - 06**
**Lecture - 06**
**Topological Spaces**

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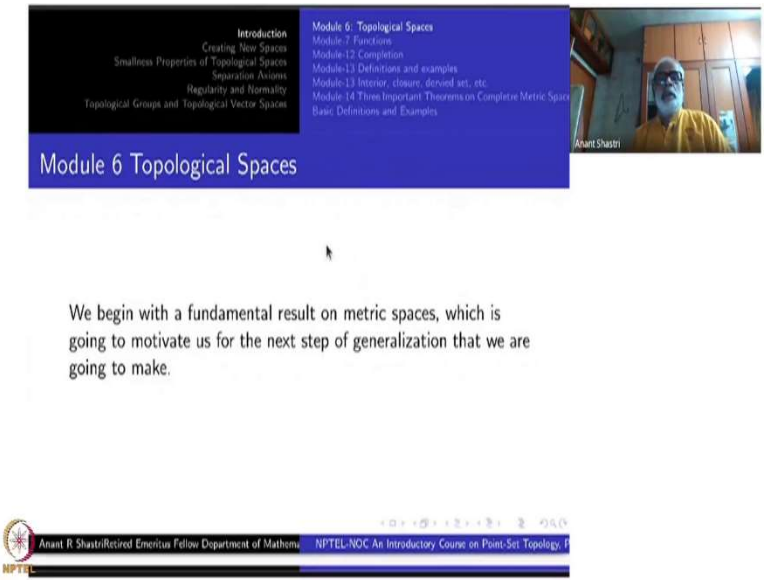


Welcome to module 6 of point set Topology course. Today at last we will start the definition of topological spaces and some few examples. Before that let us motivate this definition by studying one important aspect of same thing in metric spaces, ok.

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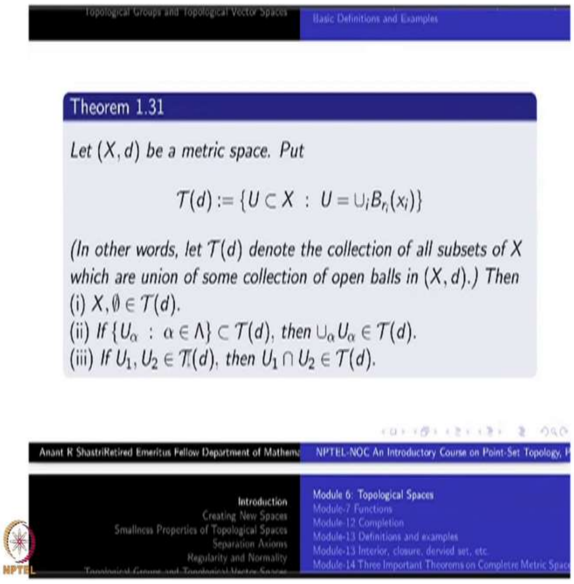


So, I have put it as a theorem, you can say it is just for surmising what is happening in metric spaces. Let $(X, d)$ be a metric space. Put $\mathcal{T}(d)$ equal to this collection of subsets of $X, U \subset X$ such that $U$ is the union of open balls $B_{r_i}(x_i)$, remember open balls centered at $x_i$ and radius $r_i$. So, I put union of that I do not know how many I have taken here that is why, $i$ ok, it could be finite, it could be infinite, it could be uncountable whatever it is it could be empty also.

When it is empty what is the union of an empty family, empty set is empty set which is also allowed, to take all subsets which can be written as union of balls. So, that is my $\mathcal{T}(d)$ ok, then this satisfies the following properties. First thing is just now I told you empty set is there because we can take empty union. $X$ is also there after all. Why? Because once any point $x$ belongs to $X$ ok it will belong some open ball and then that open ball will be there right. You can take union of all open balls around each point that will cover the whole of $X$, therefore $X$ can be written as union of balls like this. So, $X$ is there right.

If you take a sub family of $\mathcal{T}(d)$. any sub family ok then the union of that sub family will be also in $\mathcal{T}(d)$, ok.

The third thing which is very simple minded, but very important most important one: if two of them are there in $\mathcal{T}(d)$ then the intersection is there.

Once this is true it will follow that if finitely many of them are there then their intersection is also there. From 2 to 3, 3 to 4 any finite number you can we can get. So, these are the fundamental properties I have just collected them. I can collect a number of them lots of them, but these are the things which I am concentrating just like we started with modulus and took three important properties right?

We took them, made them into a norm, similarly here three important properties were made into a metric and so on. So, along with the same theme we have taken these three important properties. So, let us first test them. these are going to be axioms finally ok.

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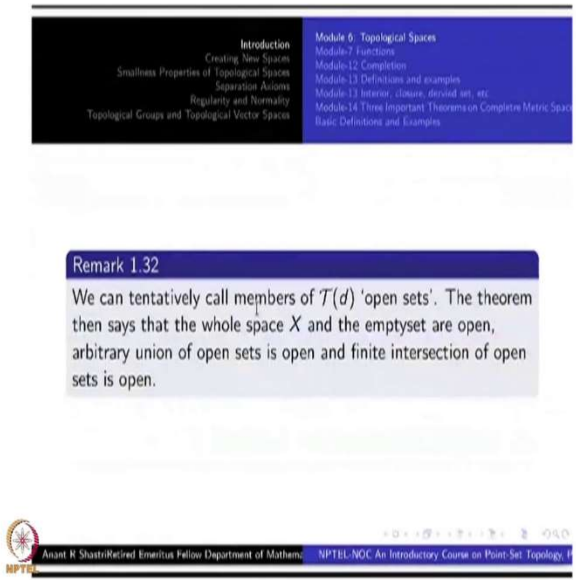


So, let us call the members of $\mathcal{T}(d)$ as open sets this is just a tentative definition soon we are going to change the definition of open set. So, right now let us call them open sets ok it is not the same thing as open balls; open balls are defined in a different way. Now an open set means a union of open balls, ok.

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Let us call them open sets. The theorem says that the whole space and empty set are open arbitrary union of open sets is open and finite intersection of open sets is open, ok.

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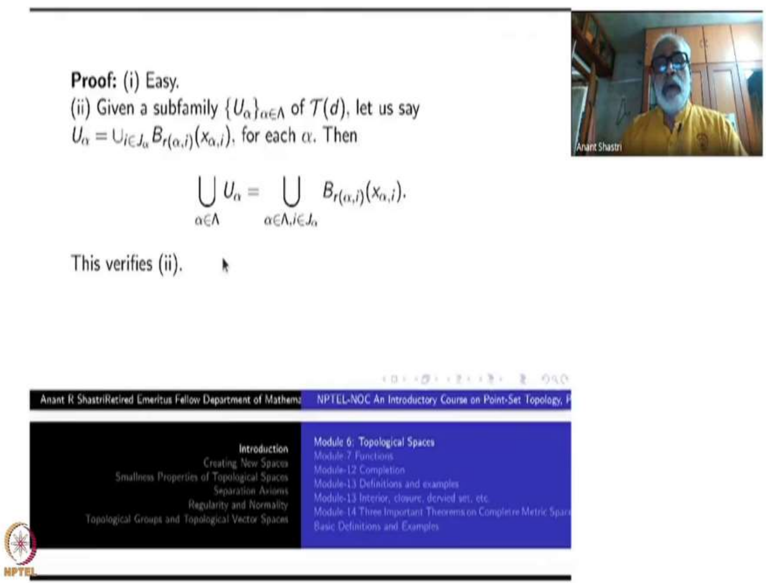


Given a sub family one part I have already explained you part 1. I have already explained to you why this happens. Given a sub family when you take the union over all of them right?

union of $U_\alpha$ each of them is a union. So, you can write it like this. Union over union is just a union with, its like a double summation right wherein in the in the case of numbers you will have to work bother about whether it is convergent and so on, here there is no question of convergence. Double summation whichever way you want to take you can sum it up it is like union of sets ok that verifies 2.

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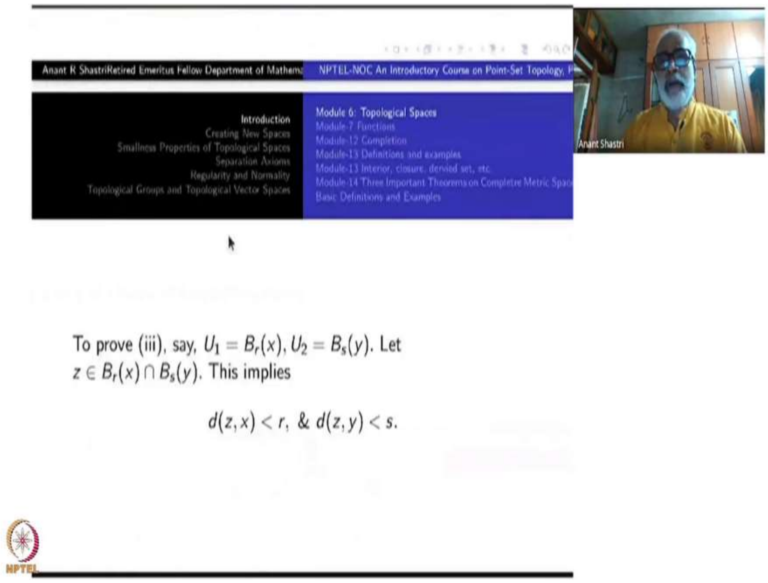


The 3rd one requires a little bit of analysis what is this namely you have to use the property of the distance function appropriately ok. So, take a member here ok, take a member here $U_1$ and $U_2$ they are themselves union of open balls when you intersect them what is that?

That is the U intersection of that is a union of intersection of one ball here and one ball there one ball here one ball there all pairs of intersections you have to take some of them may be empty some of them non empty.

But it is a union ok intersection of the union of something union something is union of the intersection that is what I am going to use. Therefore, it is enough to prove that intersection of just two of them ok two of the balls now is again a member of member like this is a union of balls.

See first I have to prove that intersection of $U_1$ and $U_2$ where $U_1$ and $U_2$ are arbitrary open sets like this are arbitrary unions like this intersection is so, but that will convert it into just proving the intersection of only two members, namely how they are themselves balls, $U_1$ is $B_r(x)$ and $U_2$ is $B_s(y)$. So, I show that this intersection is a member of $\mathcal{T}(d)$ in the union of all these will be the arbitrary intersection. So, that will be over there, ok.

So, $U_1$ is $B_r(x)$ and $U_2$ is $B_s(y)$. $z$ belongs to the intersection would imply that both $d(z,x)$ and $d(z,y)$ satisfy: this is less than $r$ and that is less than $s$. That is the meaning of this right. $z$ is in this ball means this one, $z$ is in this ball means this one, right.

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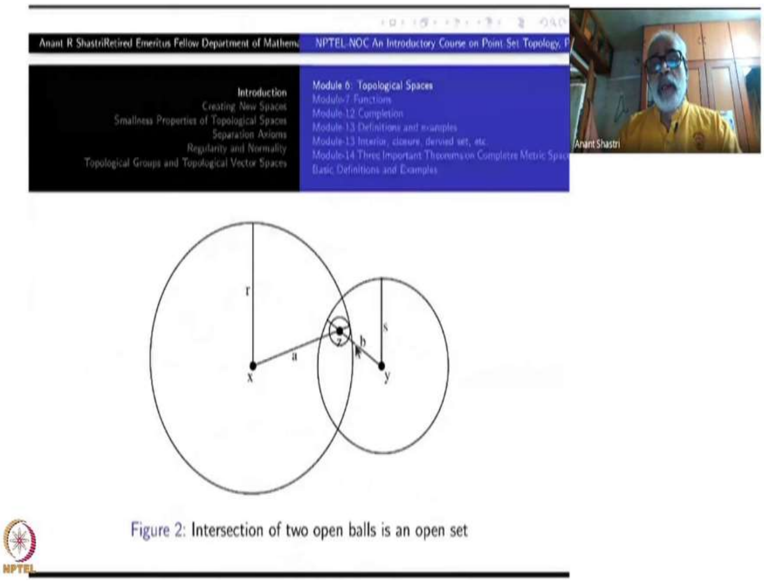


Figure 2: Intersection of two open balls is an open set

Now, now what is that I am going to look at here? Look at this one this is a ball of radius $r$, that is a ball of radius $s$ around $y$. Now $z$ is taken in the intersection you can see that intersection is not a ball ok, here it is a lens it is a lens like thing alright, but it will contain a small ball around $z$. That is true for every such $z$.

So, the intersection itself will be the union. So what is that ball? You have to choose this radius appropriately if the distance from $x$ to $z$ is $a$ and the radius is $r$ whatever is left out here is precisely $r - a$.

I can take as as big as $r-a$, but I have to worry about the another one also. So, here radius is $s$ and distance is $b$. So, I have take at the most $s-b$. Therefore, I have to take the minimum of the two. So, I take the minimum of the two then I am done. that is a positive number and that ball will be contained inside both of them, ok.

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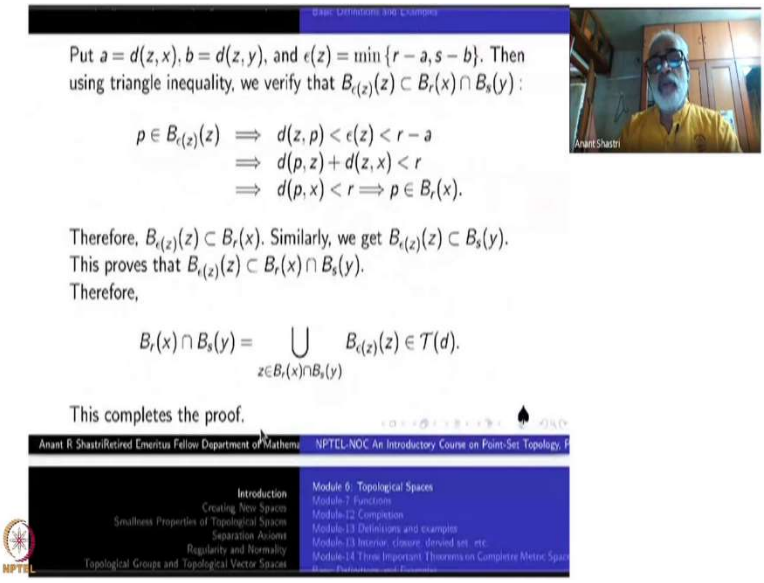


So, that is precisely what I have done here $a$ is $d(z,x)$ and $b$ is $d(z,y)$ and $\epsilon(z)$ is minimum of $r-a$ and $s-b$, alright. Then if $p$ is inside this ball you have to use triangle in equality distance between this and this ok is less than this number will imply the distance between here to there which is a plus distance here that will be less than $r$ similarly the other one. So, that is what I have to written.

$p$ belongs to $B_{\epsilon(z)}(z)$ implies $d(z,p) < \epsilon(z)$, but $\epsilon(z)$ is minimum of $r-a$ and $s-b$. So, it is less than $r-a$. So, I can write here if you take exactly minimum less than or equal to correctly to be sure does not matter less than equal to be good enough implies $d(p,z)$ is actually less than $\epsilon(z)$ because here itself I am taking less than.

So, this $\epsilon(z)$, but this may be equality here, but this is less than that is correct. $d(p, z) + d(z, x)$ this one I have replacing $r$ by this one taking $a$ on this side will less than $r$ you know take $a$ on this side that is less than $r$. So, $d(p, x)$ is less than $r$, that is $p$ is in the $B_r(x)$ similarly $p$ will be also in $B_s(x)$ therefore, it is in the intersection.

Therefore so, finally, this is what is intersection of these two balls is union of all these $B_{\epsilon(z)}$, this $\epsilon(z)$ will depend upon $z$ ok take them all inside in in the intersection. So, it will cover the whole thing ok. So, this union this one so, that will be inside $\mathcal{T}(d)$ ok.

So, now, the whole idea of doing this one is we have this three axioms here they become axioms for a topology, ok.

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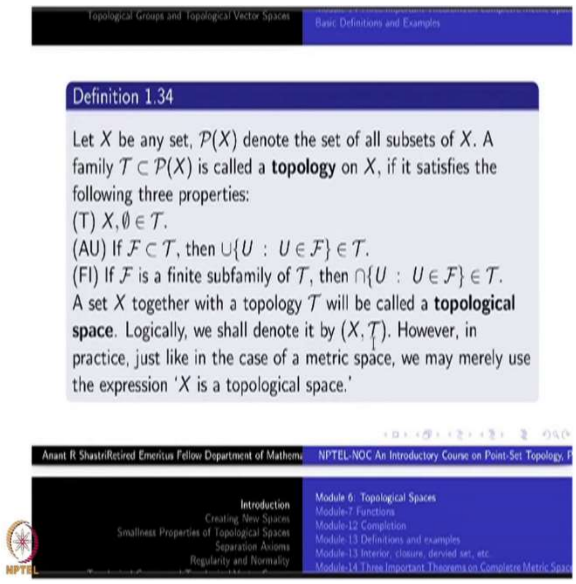


So, let us take any set $X$ take the power set $\mathcal{P}(X)$ of $X$ ok take a family $\mathcal{T}$ of subsets of $X$ that is sub a subset of $\mathcal{P}(X)$ itself. This should be called a topology on $X$ if it satisfies the three axioms which I have denoted now by (T), (AU) and (FI); $X$ and empty set are inside tau that corresponds to the first one there. The (AU) is arbitrary union, if $\mathcal{F}$ is a sub family of $\mathcal{T}$ take all members in $\mathcal{F}$ then take the union that will be also in $\mathcal{T}$.

If $\mathcal{F}$ is a finite sub family of $\mathcal{T}$, take the intersection over this finite family that will be also in tau instead of writing $U_1, U_2, \ldots, U_n$ I have just written $\mathcal{F}$ is a finite sub family, ok. So, these three axioms are there. Now a set $X$ together with a topology $\mathcal{T}$, it will be called a topological space. Logically we should write, just like we write $(X, d)$ for a metric space you should write $(X, \mathcal{T})$ for a topological space.

However, in practice we will cut down the notation. Just like in the case of metric spaces or normal spaces and so on. We may merely say that $X$ is a topological space when the $\mathcal{T}$ is understood. If we are discussing two different topologies on the same set $X$ ok then definitely we will write it sometimes there may be a different topology on a different set and so on that on those things you should write otherwise we need not write.

Writing is not a crime, only keep writing it. It takes more time and more space that is all. But when there are two of them you have to write which topology, ok?

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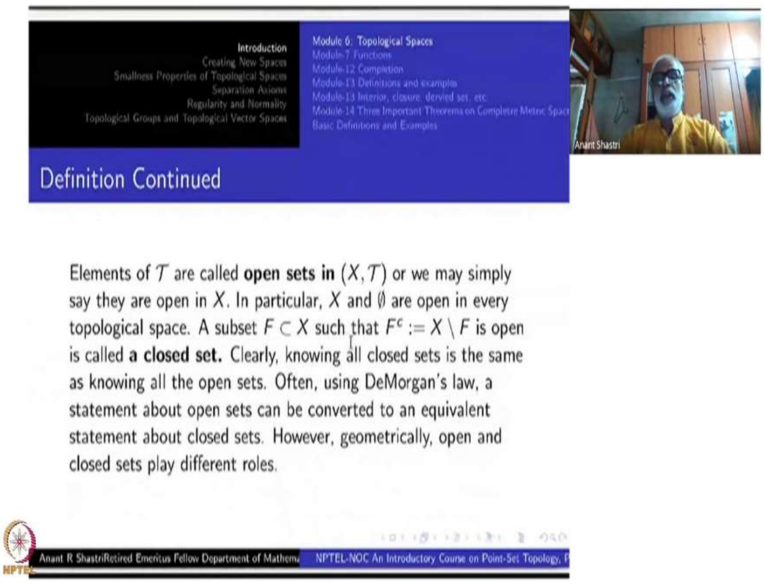


Let us continue with the definitions. Elements of $\mathcal{T}$ are called now open sets in $(X, \mathcal{T})$. They are also called merely open sets in $X$ when you say open set in $X$ we mean a topology is already understood it is in the background. ok.

In particular axiom 1 says $X$ and $\emptyset$ are open in every topological space alright. Next we will make a definition a subset $A$ is close if and only if its complement is open. Sometimes I write $F^c$ for the complement of $F \in X$. I prefer to use this longer notation $X \setminus F$. By the way, you can read it as minus F, but do not confuse it with the additivity of a real number of a complex number this is not the minus of that. So, you should not write the horizontal minus here. So, this slanted bar is the correct thing ok?

If the complement of $F$ is open then you call $F$ closed set ok. Compliment of an open set is a closed set and compliment of a closed set is an open by DeMorgan law that is all. Clearly knowing all closed sets is the same thing as knowing all open sets, to know all the closed set you must know their compliments. That is all.

Often using DeMorgan's law a statement about open sets can be converted into an equivalent statement about closed sets ok, you have to do that circus sometimes. However, geometrically open and closed sets play different roles ok. So, in our mind-set there is feeling that makes us believe as if open sets are more important, but logically there is no difference you must see that, ok.

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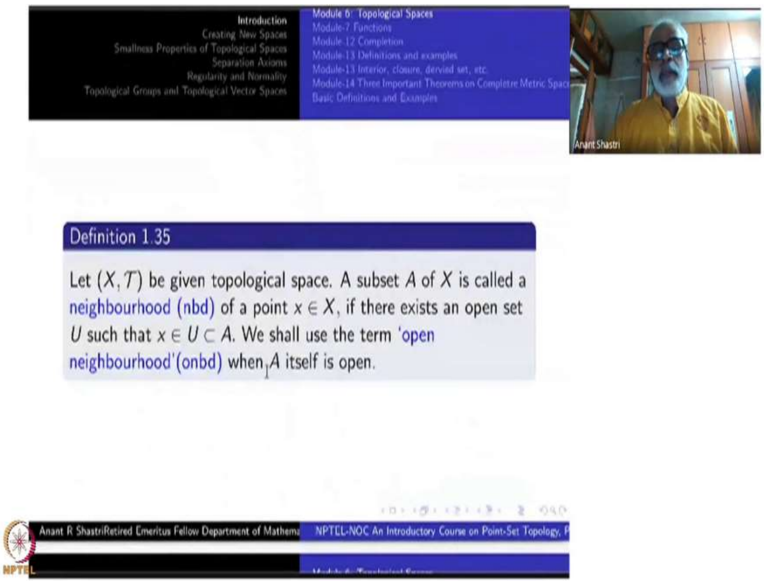

Let $(X, \mathcal{T})$ be any topological space. Now I can continue with the definitions. A subset is called a neighbourhood of a point the point is also inside $X$ of course, if there exist in an open set $U$ such that $X$ is inside $U$ and $U$ is inside $A$, ok this is a definition of a general neighbourhood ok? sometimes I am going to write nbd, this nomenclature, for neighbourhood.

We shall use the term open neighbourhood, and write onbd when $A$ itself is open. $A$ is open and $x$ belongs to $A$. Then $A$ will be an open neighbourhood of $x$. So, this is the terminology ok. It is fairly easy to see that a subset $A$ is open if and only if it is the neighbourhood of every point belonging to it.

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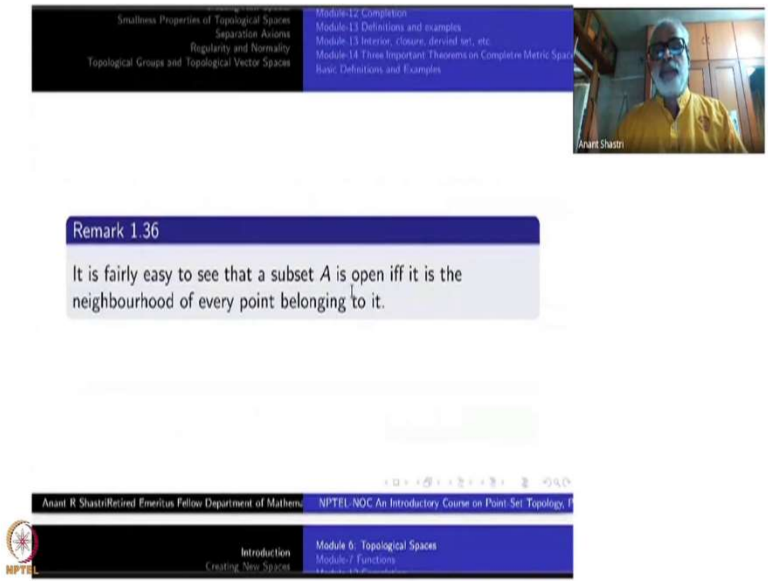


So, I will let you think about this one ok in the second part I am not assuming open neighbourhood just a neighbourhood yet if it happens for every point in $A$ if you know it is neighbourhood of every point inside $A$, then $A$ must be open and converse. The converse is obvious if $A$ is open it is a neighbourhood its actually open neighbourhood ok.

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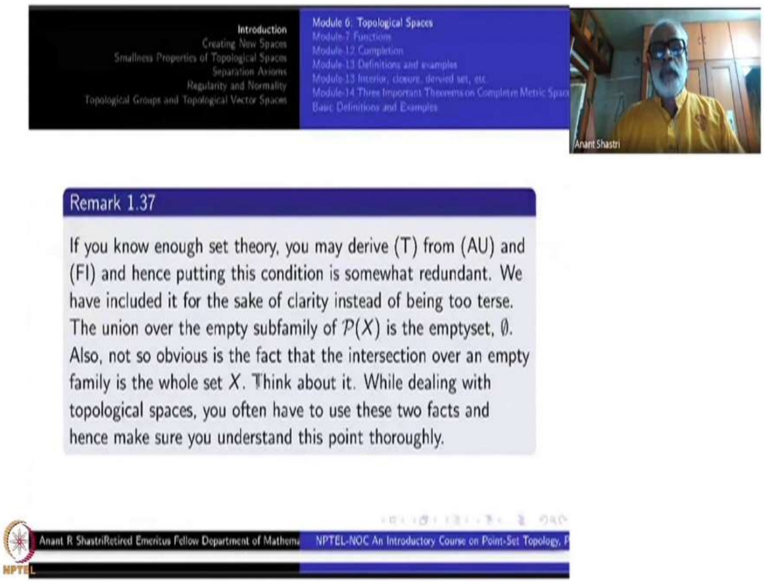


If you know enough set theory you may derive the first axiom (T) from (AU) and (FI). (AU) is arbitrary union and (FI) is finite intersection. Hence putting this condition is somewhat redundant and some books do that. They do not put it at all they just put only 2 axioms and others put it as 4 axioms by by cutting (T) itself into two, ok.

So, basically the important one are (AU) and (FI). So, we are included (T) for the sake of clarity instead of being too terse, though logically only these two are sufficient. By the way there is a general understanding that a definition should be as short as it can be ok? No extraneous explanation terms etc should be there. That is a guiding principle for making a definition ok.

If you follow that this (T) has no value (T) should be removed because that is a consequence of (AU) and (FI). Not so obvious is the fact that intersection over an empty family is the whole set. So, this gives at least the beginner, some problem ok. So, I want you to think about it. this a very important thing. If you proceed further and further in topology as well as in mathematics elsewhere, this becomes a necessity to understand this one ok?

So, make a beginning now. While dealing with topological spaces you often have to use these facts namely that an empty union is what? Empty union is empty set. empty intersection is the whole set.

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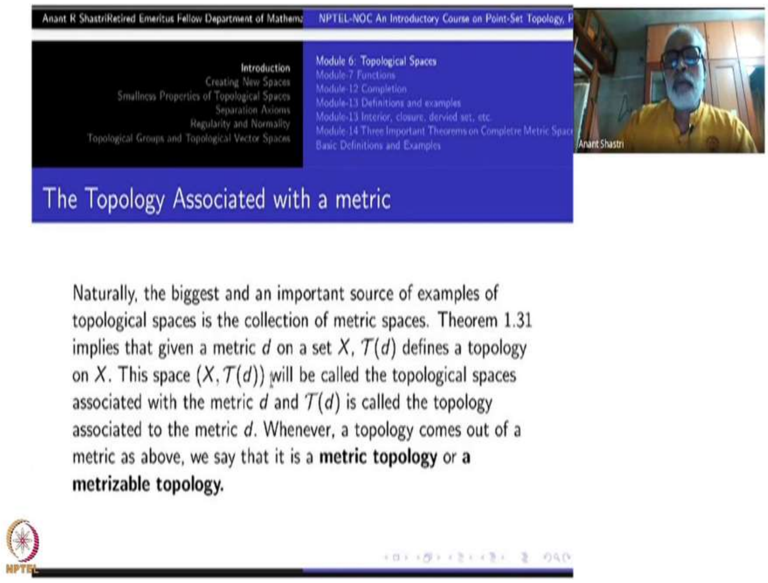


So, our topology was after all motivated by metric spaces. So, there is no wonder if metric spaces give you know a big source of topological spaces how start with any metric space and this take this tau d that is a topology.

So, our topology was after all motivated by metric spaces. So, there is no wonder if metric spaces give you, you know, a big source of topological spaces. How? Start with any metric space and take this $\mathcal{T}(d)$ that is a topology. That is what the previous theorem says right. 1, 2, 3 properties we approve and they are those properties have become axioms that is all. So, $(X, \mathcal{T}(d))$ is a topological space associated with the metric $d$ right.

So, such a topology we will call a metric topology that is, the one coming from a metric. Sometimes we say it is a metrizable instead of saying which metric actually gives it. There may be several metrics giving the same topology.

So,  it is just called a metrizable topology ok?. Obviously, there is going to be some stringent condition on a  topology to be a metrizable one, right? They are some small class among all topological spaces, the once you have `created' using this definition. It is like a monster ok it is no longer in our control. it will create lots and lots of topologies ok.

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So, that is what I am telling here if you have only metric topologies all the time then there was no need to make this general definition at all. The fact is that the great people who worked all these things out, the harbinger's of topology had foreseen the floodgates that this definition opens up. Our next step is to study a few examples of topological spaces which do not arise naturally from metric spaces, but this we will do next time.

Thank you.