Introduction to Point Set Topology, (Part I)
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Lecture - 59
Topological Vector Spaces

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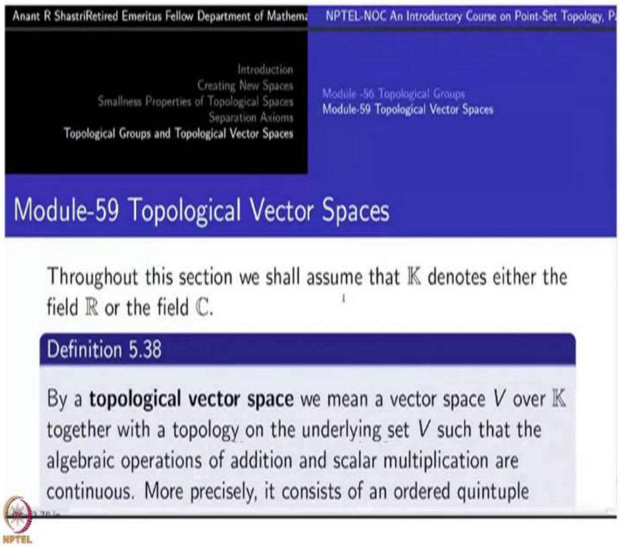

Welcome to module 59 Topological Vector Spaces. Throughout this section we shall assume that $\mathbb{K}$ denotes either the real field or the complex numbers field $\mathbb{C}$. By a topological vector space we mean a vector space V over $\mathbb{K}$ together with the topology on the underlying set V such that the algebraic operations of addition and scalar multiplication are continuous. So, this is not a formal a definition, I will give you precise definition in a moment ok? Just similar to a topological group.

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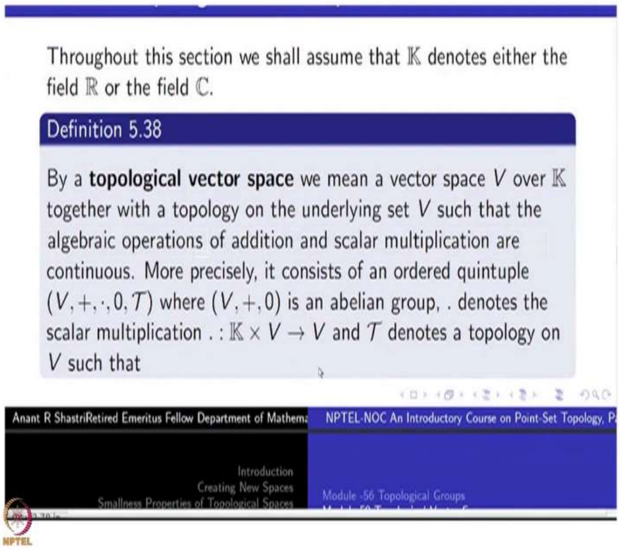

More precisely a topological vector space consists of an ordered 5-tuple, quintuple consisting of a set V, a $+$ operation, a . operation and a special element 0 and a topology $\mathcal{T}$. The first three $(V, +, 0)$ constitute an abelian group, wherein 0 is the 0 element of the abelian group. The . denotes a scalar multiplication, $\mathbb{K} \times V$ to V it is not from $V \times V$ to V ok. It is not a binary operation on, it is a scalar multiplication and tau is a topology on V, ok.

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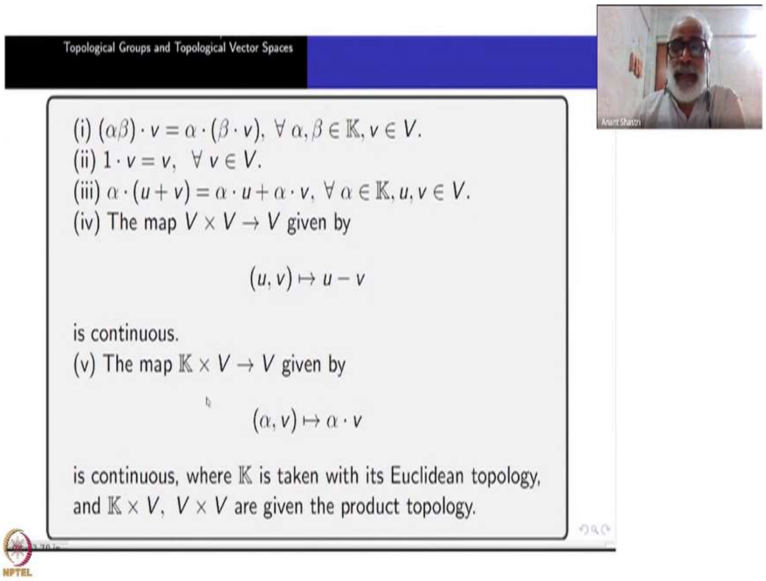

So, they have some interrelations here. What are they? First three are pure algebraic one, the scalar multiple you know, (α, v) goes to αv is associative that is the first condition. And if you multiply by 1, you know it is identity $1v$ is v for every $v \in V$. So, this is the scalar multiplication property of a vector space you may say ok. $\alpha(u+v)$ is $\alpha u + \alpha v$, distributivity of the scalar multiplication.

So, 1, 2, 3 actually constitute what is the meaning of a vector space, I am redefining it here that is all. The map $V \times V$ to V given by (u, v) going to $u - v$ is continuous. This is similar to what we have done in the topological group (x, y) going to xy^{-1} is continuous.

So, instead of multiplicative notation we are using now additive notation because we started with this one is an abelian group alright. So, additive notation is preferred here so it is $u - v$, it is not $u + v$; $u + v$ as well as v going to $-v$ both are continuous is equivalent to one single condition (u, v) going to $u - v$ is continuous. The map, the scalar multiplication $\mathbb{K} \times V$ to V given by (α, v) equal to this αv, this is the function . that is continuous.

What is the topology on $\mathbb{K} \times V$? V has a topology, $\mathbb{K}$ has the standard topology, Euclidean topology, $\mathbb{K} \times V$ is the product topology. Under that this map must be continuous ok. So, $\mathbb{K}$ is taken with Euclidean topology and $\mathbb{K} \times V$ and $V \times V$ etc are here given the product topology. So, for those who know what is a vector space already which you have used several times scalar multiplication and addition they are continuous is the easy way to remember that it is a topological vector space.

Vector space in which there is a topology with respect to which algebraic operations are continuous. This is the way to remember instead of all this list of conditions here ok?

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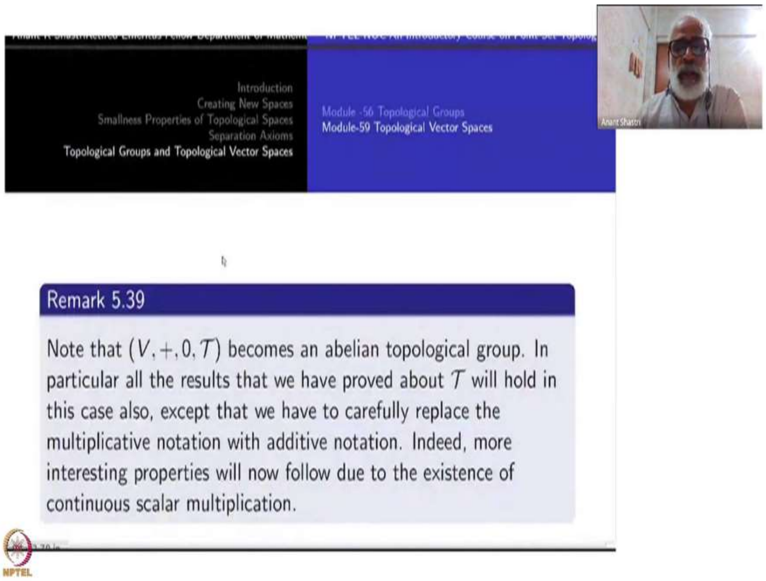
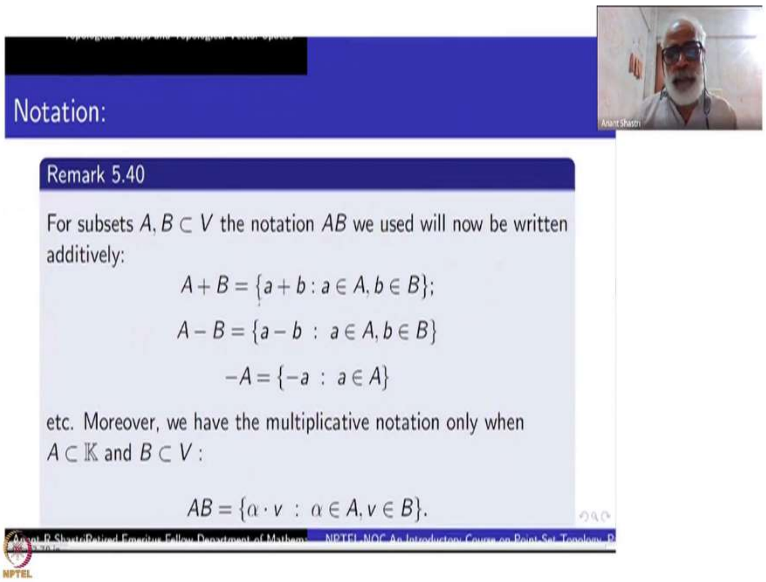


Notice that if we just ignore the scalar multiplication what we have is a topological group which is abelian. So, in particular all the result that we have proved about $\mathcal{T}$ in the previous section, will hold in this case also, except that we have to carefully replace the multiplicative notation with additive notation everywhere ok.

Now, what we expect is something more should happen, stronger result should happen, because of the existence of this continuous scalar multiplication. So, that is what we anticipate and well soon we will see that; that is what is true anyway. (Refer Slide Time: 05:58)



So, let us now reorient our notation ok according to these vector space notations here alright. So, A and B are any two subsets of V the notation AB, we used earlier will now be written additively; that AB for topological group was alright now it does not make sense for us.

So, you should write $A+B$ which just constitutes all points $a+b$, where a and b are respectively in A and B. Similarly, $A-B$ which in the earlier notation corresponds to AB^{-1}, right? So that inverse is now here minus, so $A-B$ is the collection of all $a-b$, where $a \in A$ and $b \in B$. Also, $-A$ is just the set of all $-a$ where $a \in A$, ok?

So, these are the prototype of whatever we used earlier in the case of a topological group and wrote in the multiplicity notation there. But, now we have another multiplicity notation here only scalar multiplication namely A must be a subset of $\mathbb{K}$ and B must be a subset of V. Then AB makes sense and is the set of all points αv where α is in A and v is in B. So, as far as possible, we will use this Greek symbols for the scalars or t, s etc, but a, b, c, d or v will denote the vectors.

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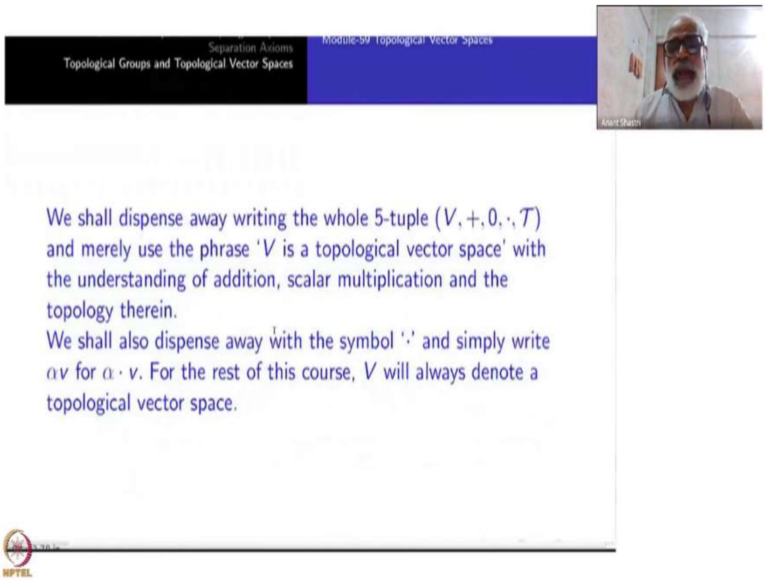

We shall also dispense away with the symbol . you know even . we will not write just simply write it is αv. Normally you know, when you have a vector space you write the scalars on the left and the vector on the right. Even that rule can be relaxed, you can use v times α also if there is no confusion ok? That is also allowed. But, standard notation is to write the scalars on the left.

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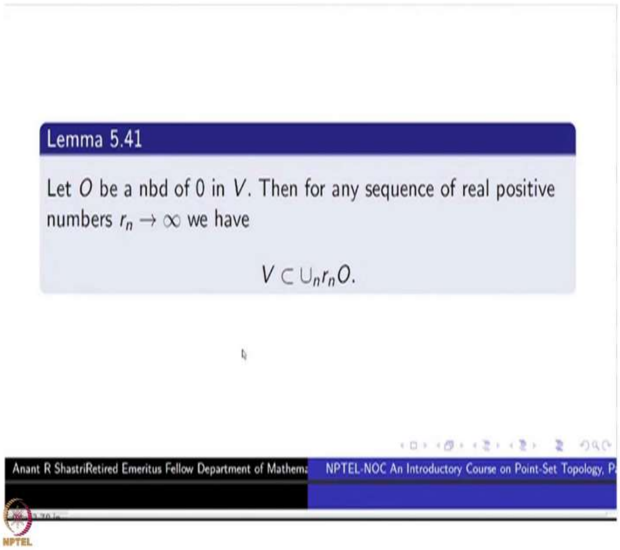

Now, you start doing some topology. Let O be a neighborhood of 0 in V. Then for any sequence of positive real numbers $r_n \to \infty$, we have the entire vector space V is covered, is contained inside the union of $r_n O$; where n ranges from 1 to infinity. Such a thing is obvious in the case of $\mathbb{R}$ or $\mathbb{R}^2, \mathbb{R}^3$ and so on ok.

So, why this is obvious for an arbitrary topological vector space? I will give you a minute think about it. So, you have an answer, some of you must have have an answer. So, congratulations if you have done it correctly. Here is the answer.

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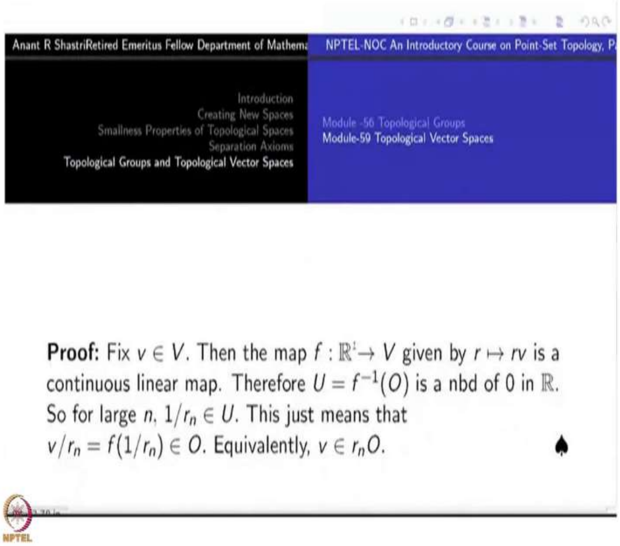

Fix a $v \in V$. I must show that it is inside r_n of O for some O. What is O? O is fixed neighborhood of 0 inside V, that is all ok. Look at the map from f from $\mathbb{R}$ to V given by r going to rv. This v is fixed ok, r goes to rv, this is a continuous function.

Why? Because scalar multiplication is continuous right. Therefore, the inverse image of an open set is open. O is an open subset of V right? The inverse image U of this open subset under f should be an open subset of $\mathbb{R}$. What happens if I put $r = 0$; 0 goes to 0 and 0 is inside O.

Therefore U is a neighborhood of 0 in $\mathbb{R}$. Then we know that if n is large since r_n goes to infinity, $1/r_n$ will be inside U right? Because U is a neighborhood of now 0. What is the meaning of $1/r_n$ is inside U? f of that is inside O. f of that is nothing, but $v/r_n; v/r_n$ is inside O means v is inside $r_n O$. ok? That is all you see.

So, that is the game we are going to play. Just that is why I stopped to give you a little more time to think about what is happening. The scalar multiplication is going to play a big role here as compared to just the additive structure which is there of course. We have seen something about topological groups, but this is the extra thing right? Ok. So, such a thing you could not say in a topological group.

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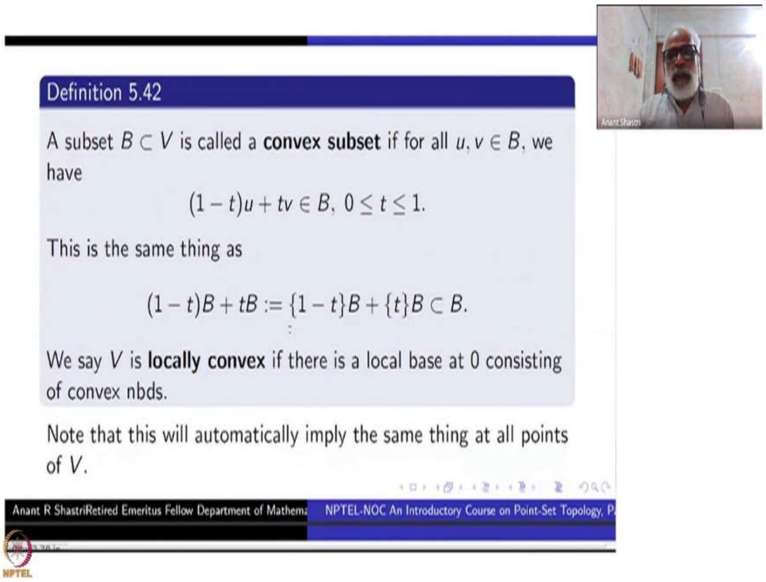

Now, I will introduce a number of definitions which will involve the scalar multiplication ok. $A \subset B$ of V is called convex subset you see this was always possible in a vector space. I am not doing anything else. The same definition, no change of definition here ok. If u and v belong to B should imply the line segment $(1-t)u+tv$, where $0 \le t \le 1$,this entire line segment is inside B. That is the definition of convex subset ok?

But, in the notation that we have introduced it is same thing as saying that $(1-t)B + tB$ is a subset of B. So, I have put an elaborate notation here because you may confuse it for something else. For me, there is no confusion. The singleton tB is the same thing as just tB this we have been using right.

So, I have used that one also tB is nothing, but singleton tB ok. So, that is all the shorter notation here these brackets are not written here, but this is by definition $(1-t)B + tB$. So, that is contained inside B same thing this is you know for each point here and each point ok for each point here and each point here (u, v). So, that line segment is inside V same thing as this one.

Once you have convexity you can talk about local convexity we say V is locally convex if there is a local base at 0 consisting of convex neighborhoods. Take any neighborhood of 0, inside that you must have a convex open subset around 0 contained inside the given nbd. That is the meaning of there is a base the local base consisting of locally convex open subsets convex open subsets.

So, it is true for topological vector spaces also ok. Thus this will automatically imply the same thing; same thing means what? V is locally convex at all the points of V . So, one point is enough for the definition. For example $\mathbb{R}^n$ is locally convex. There are many topological groups which are not locally convex. So, that is why we want to define this notion.

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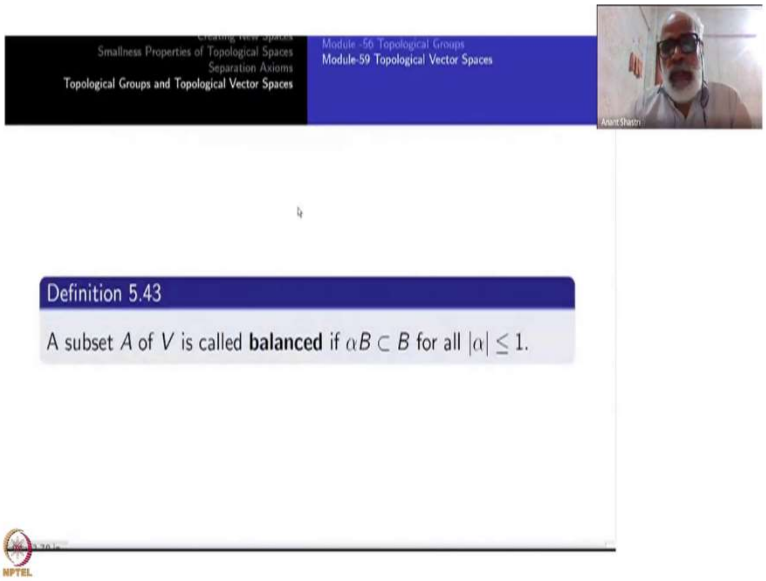

A subset B of V is called balanced ok? (this is something new you might not have come across, you might not have met with this one, in the standard study of vector spaces ok?) it is called balanced, if αB is contained inside B for all $|\alpha| \leq 1$, this $|\alpha| \leq 1$ becomes crucial here. (There is a typo in the slide.)

When you take complex numbers that will have different meaning you see if real numbers is just a line segment right complex numbers it is the whole disk. So, geometrically they will have different meanings ok. So, balanced means that. Let me illustrate this one a little more because this is a new concept ok. You should not confuse it with something which you may think is the right one.

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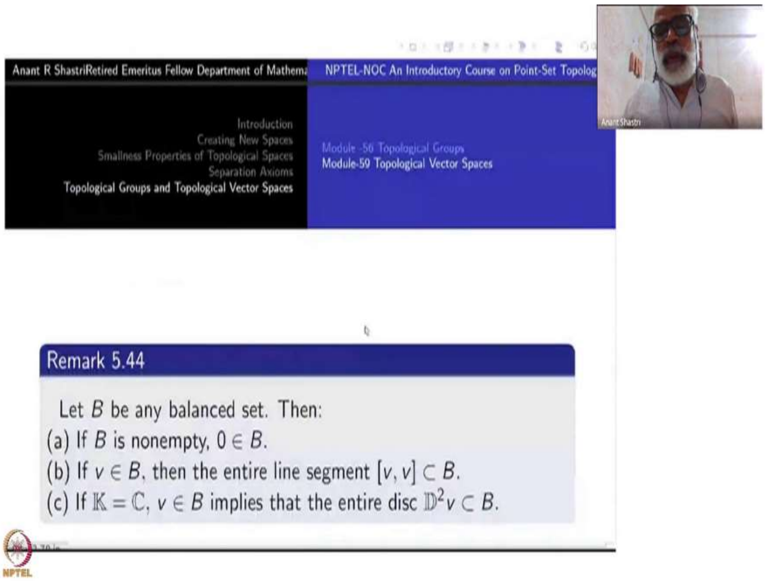

Suppose B is a balanced set ok? If B is nonempty then 0 must be inside B. Do you see why? Because look at this condition. Condition says that $|\alpha| \leq 1$, then αB must be inside B. So, you can put $\alpha = 0$, that is valid right? So, 0 must be inside B ok.

Only thing is I need to have some vector here if B is empty this is automatically satisfied empty set is balanced defined, but then you cannot say 0 is there right. So, only nonempty I have to assume that. If it is non-empty then 0 is B.

More generally take any vector $v \in B$ and B is balanced set ok. The entire line segment $[-v, v]$ will be inside B ok. See once, v is there I can take $\alpha = -1$.

So, $-v$ will be also there and $-v$ and plus v is there you know and you can take any other number also here between t between 1 and -1. So, all those multiples also there; that means, the entire line segment is there ok. One element is there then the line segment $[-v, v]$ is there. So far I have used only real numbers.

Next suppose suppose V is a vector space over complex numbers. I can take the entire disc $\mathbb{D}^2$, multiplied it with v, to get a subspace of V, homeomorphic $\mathbb{D}^2$, because multiplying by v

is an isomorphism, only assumption that v is non zeor. Unless v is 0 ok nonzero vector you have to take. If you take 0 vector this will be 0. That is also fine no problem with that ok.

So, if B is balanced and $v \in B$, then the entire disc $\mathbb{D}^2 v$ is contained inside B ok. So, this balancedness is an indirect way of bringing the open balls inside a metric space. You will see that balanced subsets balanced neighborhoods and so on they are going to play the role of the open balls in a metric space ok, that is the whole idea of this one. So, I have used B for balance, but they are also to remind you that they are prototypes of balls in metric space ok.

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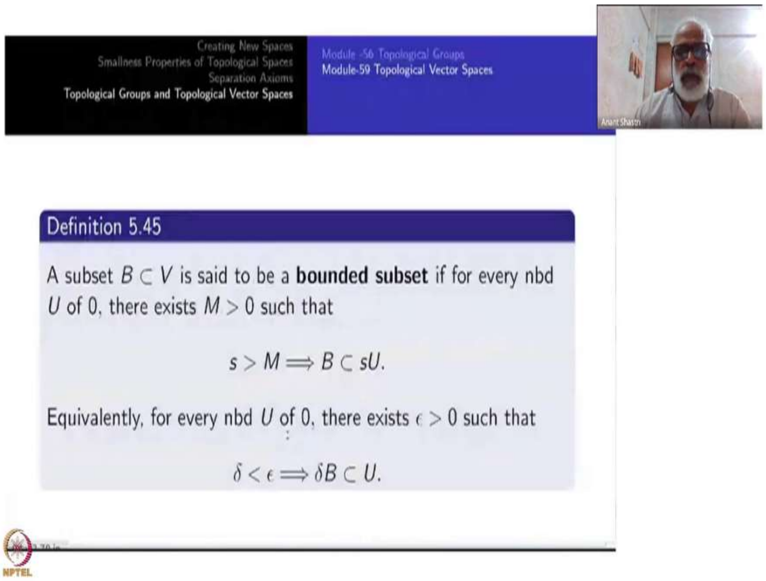

Now, let us go ahead with some more definitions. Now, I am going to introduce another term here though there are no metric spaces we are going to define some notion of boundedness. So, it is again another backdoor way of bringing boundedness though we do not have a metric ok.

So, pay attention to this definition. A subset B of V is said to be bounded if for every neighborhood U of 0, there exist $M > 0$, such that this $s > M$ implies B is contained inside sU; s is a real number, sU is like an expansion or contraction of U ok, s times that open set that is it. I started with U as a neighborhood of 0.

So, 0 will be always there all the vectors will be stretched by a factor of s. If $s > 1$, then sU is probably larger that U and if $0 < s < 1$, then it is probably smaller. But, whatever it is, s is bigger than M is important here. This is not a typo, this is not $|s|$, this is s ok. Here real numbers only. $s > M$ should imply B is contained inside sU.

There must be some M, how small how big I do not care. So, that is the meaning of boundedness.

This is a very strange definition. You will see slowly you will realize that this bondedness is actually stronger than the metric boundedness for a non bounded space ok. Equivalently let us look at the other way round here. For every neighborhood U of 0, there exists an ϵ positive such that $\delta < \epsilon$ should imply δB is contained inside U.

So, starting with a bouded set B by contracting B smaller and smaller you can bring it inside U. So, that is the meaning of this boundedness. Why this is equivalent to this one? You have to just invert it, you see, B is contained inside sU same thing as $1/sB$ is contained inside U. So, put $\delta = 1/s$.

So, here also you can invert all of them. The condition $s > M$ will become you know $1/s < 1/M$, right. So, this $1/s$ is your δ and $1/M$ is your ϵ. So, that is that is the way you can interchange these two. So, this is the meaning of something is bounded ok.

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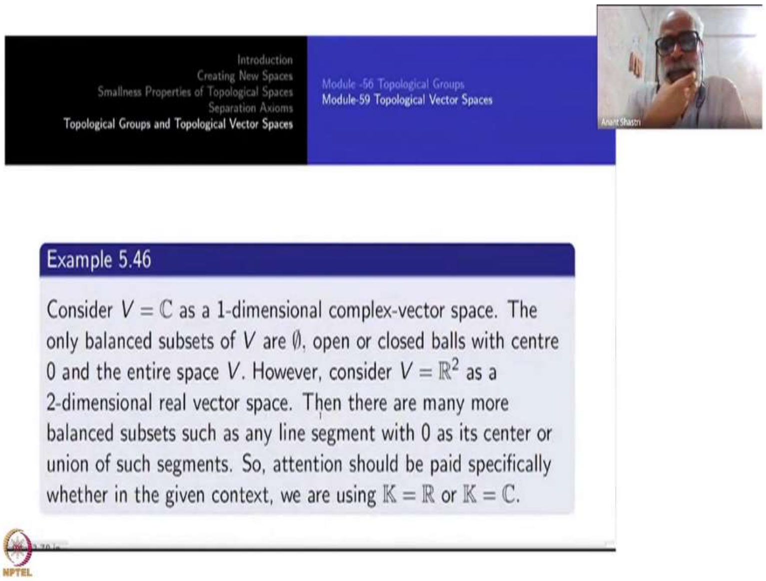

So, let me give you an example here. Consider V equal to the complex numbers as a 1-dimensional complex vector space. Do not go for you know, complicated example. You have to understand in the simple situations first. The only balanced subsets of V are empty set, open or closed balls with center 0 and the entire space V.

So, these sets are clearly balanced. They are open balls only thing is they are centered at 0. By translating them you will get all other balls you see. So, that is why this definition.

However, consider $V = \mathbb{R}^2$ as a vector space over $\mathbb{R}$ itself, not as complex vector space ok? As a 2-dimensional real vector space. Then there are many more balanced subsets here, such as any line segment with center 0, ok?

Any line segment with center 0 means, it must be a line segement of the form $[-v, v].v$ it could be any non zero vector. The line segment should passing through the origin that is all. All union of such segments are also balanced. For example, you can take x-axis and y-axis that will be balanced or $xy = 0, x + y = 0$ and $x - y = 0$ as well as x-axis, y-axis and so on.

Star shaped thing wherein all segments through they must be equally distanced from the center ok. So, that is why balanced the word balanced is used. So, attention should be paid specifically whether in the given context we are using $\mathbb{K}$ equal to $\mathbb{R}$ or $\mathbb{K}$ equal to $\mathbb{C}$ ok. Geometrically they have some different meanings here. So, when $\mathbb{K}$ equal to $\mathbb{R}$, it is not exactly balls. For $n = 1$, it is same thing intervals and balls. But in $\mathbb{R}^2, \mathbb{R}^3$ and so they are something different ok.

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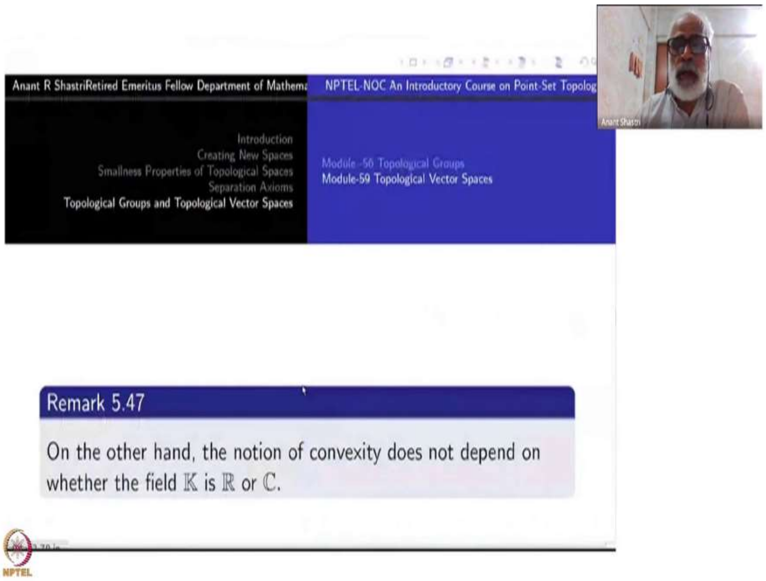

Whereas, convexity it is same thing no problem. Convexity does not depend upon whether you are using complex numbers or real numbers. Because in the definition of convexity it is only real, so $tV + (1-t)U$; where t is a real number between 0 and 1, because you do not use complex numbers at all. So, convexity is the same thing for whether you are treating a complex vector space or real vector space. So, I come back to an example of this boundedness also ok.

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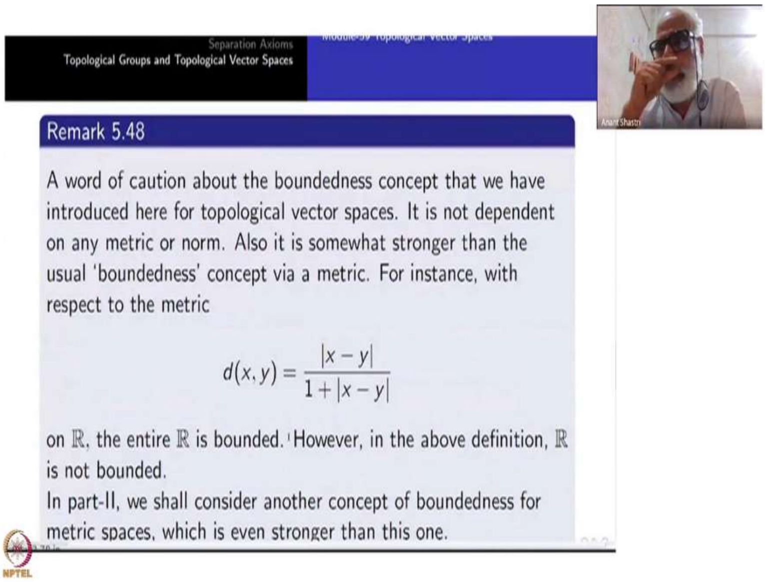

The word of caution about the boundedness concept that, we have introduced here for topological vector spaces. It is not dependent on any metric or norm. There is no metric or norm we have used. Everything is inside a topological vector space you see. Also it is somewhat stronger than the usual bondedness concept in a metric space.

So, how do you perceive that? Maybe you start with a normed linear space that is automaitically, a topological vector space right. Any normed linear space is a topological vector space with the norm you get a metric right? But, the metric you change it by using this rule. Here I have put $V = \mathbb{K}$, complex numbers or real numbers.

But you can do it for any norm linear space also by taking $d(x,y) = \|x-y\|/(1+\|x-y\|); d(x,y)/(1+d(x,y)$ that is the meaning of this right. Then this is always a bounded metric and gives you same topology. So, the original topological vector space will be a topological vector space with this topology, same topology right, only metric has changed it is not a linear metric it is not coming from norm ok?

But, if you use this metric the whole of V will be bounded because it is bounded by 1. Whereas, just look at $\mathbb{R}$ and the boundedness concept that we have introduced for topological vector spaces, $\mathbb{R}$ is not bounded in that sense neither in the standard meaning, it is bounded, $\mathbb{R}$ is not bounded. When $\mathbb{R}$ will be bounded?

Take any neighborhood then you must be able to collapse the whole of $\mathbb{R}$ by one single scalar inside that neighborhood that is not possible. Because no matter how small ϵ is $\epsilon\mathbb{R}$ is the whole of $\mathbb{R}$ itself right.

So, $\mathbb{R}$ or any other vector space are not bounded in that sense ok. So, I promise that in the second part we shall consider another concept of boundedness for metric spaces which is even stronger than this metric boundedness concept ok. So, that will be called totally boundedness and the name is quite justifiable. So, that will be done in only part II ok.

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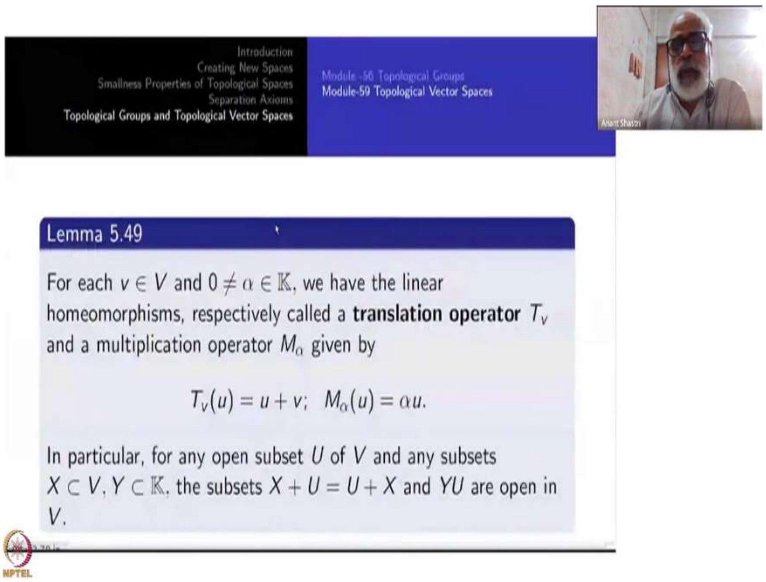

So, let me have one more simple lemma before giving a break here. For $v \in V$ and $\alpha \neq 0$, we have two linear homeomorphisms respectively called a translation operator T_v and a multiplication operator M_α, given by $T_v(u) = u + v$; and $M_\alpha(u) = \alpha u$. So, these are two homeomorphisms ok?

Of course if $v = 0$, then T_v will be identity, but for this one you have to assume that $\alpha \neq 0$, ok. Otherwise multiplication will collapse the whole thing, it will not be a homeomorphism.

In particular, for any subset open subset U of V and any subsets X and Y of V and $\mathbb{K}$ and so on. You will have similar results that we had earlier: namely $X + U = U + X$ here because this is commutative. YU I do not want to write UY because scalar should be written on the left only, they are both open subsets of V ok. So, this is something which we have done similar to just the case of topological groups.

Only thing this is new because then $\alpha \neq 0$, there is a multiplicative inverse also $M\alpha^{-1}$ will be inverse of $M\alpha$. So, $M\alpha$ is also a homeomorphism ok? this is the extra thing that is happening here ok.

So, lemma is obvious. Why? because you have to just use that these are homeomorphisms. So, $X + U$ you can write it as union of $X + U$, they are copies of U. Therefore, the each of them is open therefore, $X + U$ is open.

Similarly, YU is union of αU, where α runs over Y; αU is open because multiplication by α is a homeomorphism ok. So, we should stop here and reap the harvest of all this observations which we have made, next time. Remember the definition of convexity, balanced set, local convexity, boundedness. These are the new concepts I have introduced. Next time we will use these things and produce many interesting results here.

Thank you.