Introduction to Point Set Topology, (Part I)
Prof. Anant R. Shastri
Department of Mathematics
Indian Institute of Technology, Bombay

Lecture - 58
Topological Groups-continued

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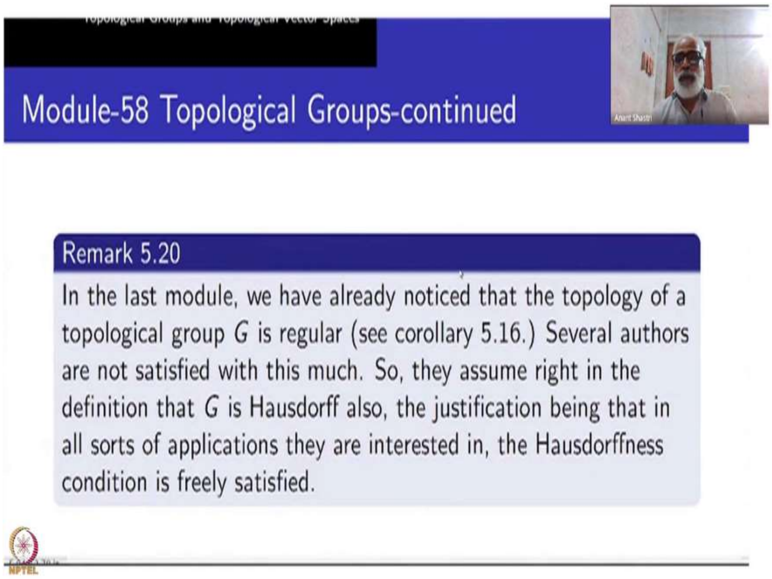

Welcome to module 58 of Point Set Topology part 1, we continue our study of topological groups. In the last module we have already noticed that the topology of a topological group is regular.

This we actually proved while proving even a stronger separability property for topological groups. Several authors are not satisfied with this much. In fact, there are two schools of mathematicians or we may say only topologist, one which sticks to Hausdorffness and other one sticking to regularity.

So, the Hausdorff-school people especially the Bourbaki oriented people, they would like to have every topological group Hausdorff. So, right in the beginning they have put this hypothesis, a Hausdorff space with a continuous multiplication blah blah blah. But we have not put that one. So, let us see how you can guarantee Hausdorffness with some minimal assumption ok.

So, we have already noticed that the moment it is T_1, it will be Hausdorff T_2 right. So, now we would like to show that the T_0 is just enough ok?

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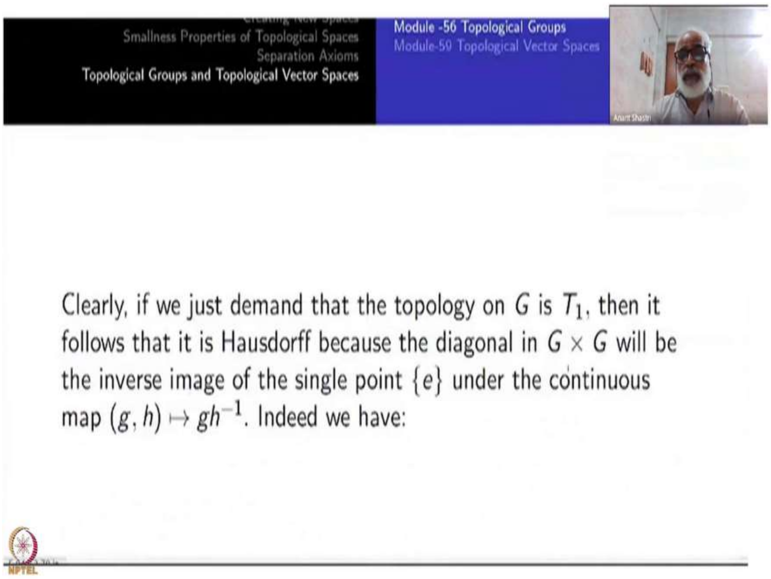

Since ν from $G \times G$ to given by (g, h) going gh^{-1} is continuous, and the inverse image of the $\{e\}$ is the diagonal Δ in $G \times G$. Hence if $\{e\}$ is closed, it follows that the diagonal will be closed, so G is Hausdorff. That is clear.

So, T_1 is enough, to conclude Hausdorffness. Even T_1 will be guaranteed by T_0, that is the first thing that we want to assure today ok.

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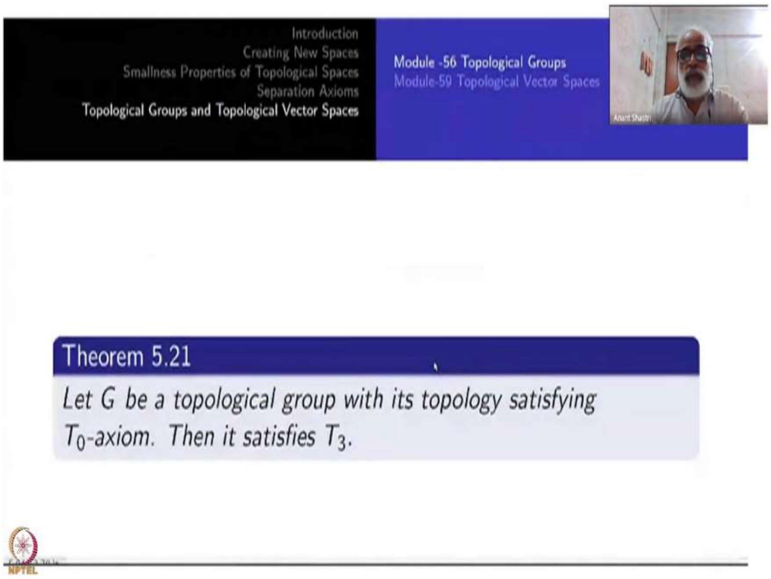

So, let G be a topological group with its topology satisfying T_0 axiom, automatically it will be T_3. All that we want to show is that it is T_1. Just now we have observed T_1 implies actually T_2, but even that is not necessary because you have already proved its regular. Regular plus T_1 is T_3, T_3 implies T_2 right? So, so let us just prove that T_0 imples T_1 ok.

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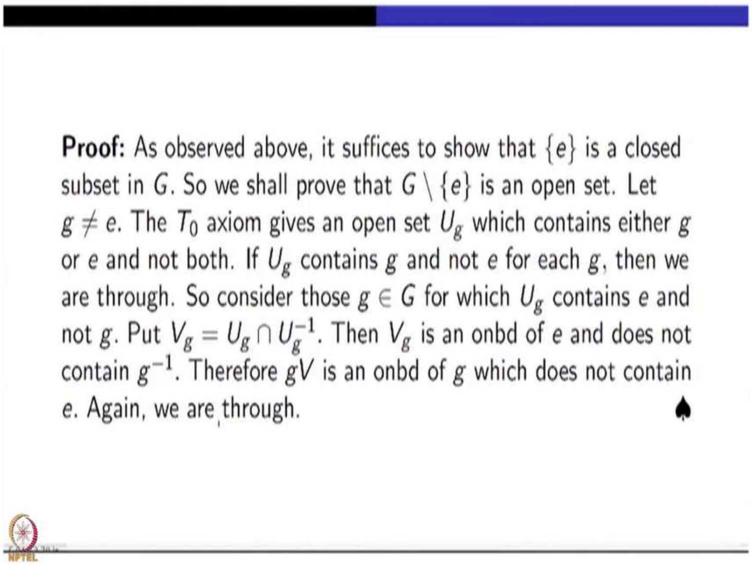

So, what is the meaning of T_0? T_0 means given two distinct points you may be able to find an open set around one of them not containing the other, but which one which one of the two points it will contain you do not know, That is a point. Whereas, if you can be sure that you can do it for both the points that is T_1, but in the case of a topological group you do not have to worry about all the points, you should just show that $\{e\}$ is closed. Then because the translation homeomorphisms are there all other points will be also closed ok? This we have observed earlier.

So, all that we want to show is that $G \setminus \{e\}$ is open ok. What does that mean? For each point $g \neq e$, we must find a neighbourhood U_g of g which does not contain e. So, that is all we have to do. The only problem is for some g we may find U_g which contains g, but not e, but for some others it may be it may contain e, but not g, ok? If for all g, we can find a U_g, all g means for all g inside $G \setminus \{e\}$, not equal to e, if we can find an onbd U_g of g, which does not contain e, then we are done.

So, consider those g such that U_g contains e and not g, alright. For such a g what do you do? Put V_g equal to U_g intersection $U_{g^{-1}}$. See now, just now I assumed that U_g contains e the identity element. So, $U_{g^{-1}}$ that will also contain e right? Inverse is just the image under i, right inverting all the elements. So, e will be there in both of them, so e will be in the intersection. So, V_g will be a neighbourhood of e, right. It will not contain g^{-1} right? If g^{-1} is here then g will be here and vice versa. So, so g is not here, is this g is smaller than U_g and this one. So, it will not contain g^{-1} either, first it did not contain g, but now it does not contain g^{-1} because we have taken the intersection of both of them.

Therefore, gV is a neighbourhood of g, which does not contain e. Because if this gV contains e would mean that there is g^{-1} inside V, gg^{-1} is the only way you can get e right. So, that is not possible.

So, V_g does not contain e. So, we have got a neighbourhood of the first kind namely, this V_g is an onbd of g and e will be inside V_g ok. So, that is the trick here, just T_0 implies T_1 and therefore, T_3.

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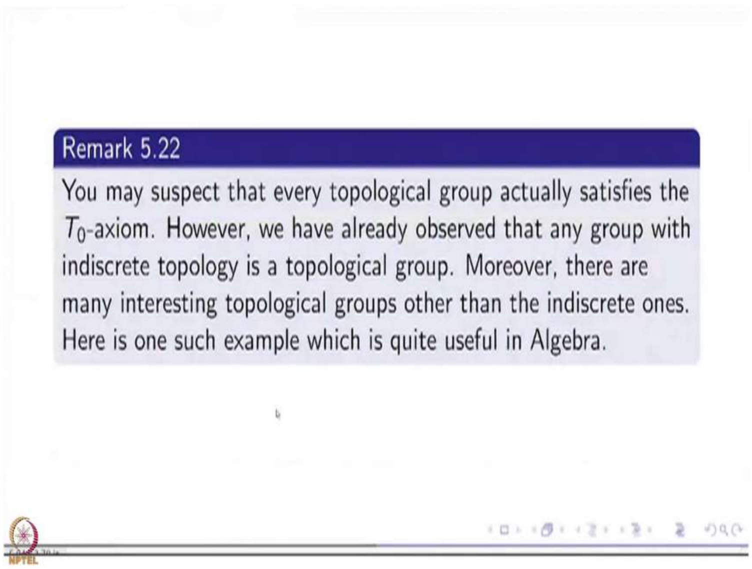

Now, you may suspect that every topological group actually satisfies T_0 axiom also, in which case, we should be happy. After all, we wanted to put minimal conditions. But that is not true. That may be the reason why so many good authors assume right in definition, Hausdorffness also. There is a big area of mathematics wherein, you know topological groups are used, those topological groups are not T_0.

In particular I am giving you one example here, I cannot deal with all those examples. So, especially useful in algebra ok? My point is that there many general topological groups, interesting ones which do not satisfy T_0 and that is why we should keep this general definition.

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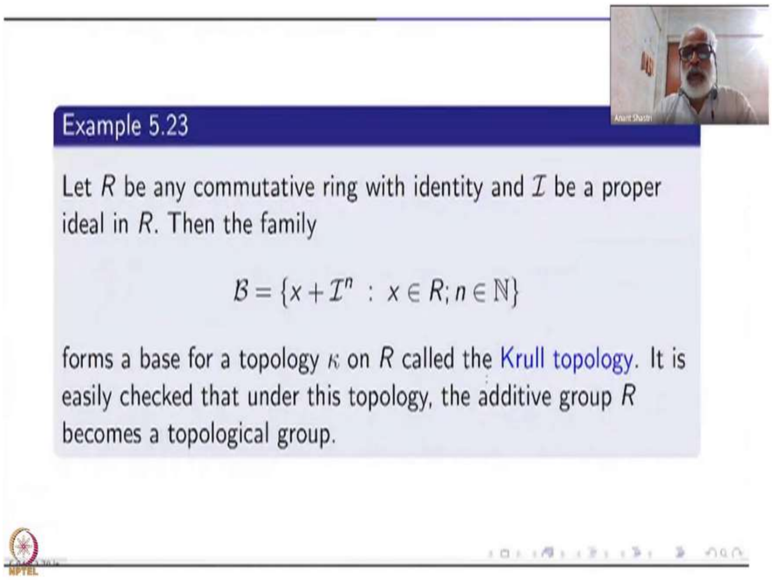

So, here is the example: If you do not know any ring theory, it may be a little difficult. However, here we use just some very elementary definitions of rings and ideals. You can just assume the results and go ahead ok. I have no time to explain what is a ring, what is a commutative ring with identity, what is a proper ideal and so on.

If you know these things then what I am going to tell is very elementary. So, you can just remember that some such thing was there and then go deeper into it when you come across into it ok. So, now, just concentrate on what I am saying. If you do not understand some terms here I have no time to introduce them.

Let R be any commutative ring with identity, just like integers ok, just like rational numbers. Rational numbers for a field so let us leave it. Just like integers you can say. And let I be a proper ideal in R, ideals are like $n\mathbb{Z}$, that is all.

Consider the family $\mathcal{B}$, of sets of the form obtained by shifting I^n by x, that is $x + I^n$, where x ranges over all of R and n is natural number. So, I am taking I, I^2, I^3 and so on ok, there is a multiplication in the commutative ring. So, I am writing II, III and so on, I^n is the standard notation for it. When I is an ideal, these powers will be also ideals ok?

Look at all these $x + I^n$. They will form a cover for the whole of R, because I am taking $x + I^n$. Indeed, they will form a base for a topology on R. Let us call this κ, in honour of Krull, it is called Krull topology ok. So, you have to check that $\mathcal{B}$ is a base for a topology on R. Addition in the ring, is a commutative group right.

So, addition in the ring becomes a topological group under this topology. That is both addition and subtraction are continuous. That is the meaning of that this is a topological group. And we know that if you take x equal to 0, here they will form a neighbourhood system for 0. Intersection of all these neighbourhoods namely just I^n, where n ranges over this one we know that it equal to $\overline{\{0\}}$, ok. It is a general fact about any topological groups.

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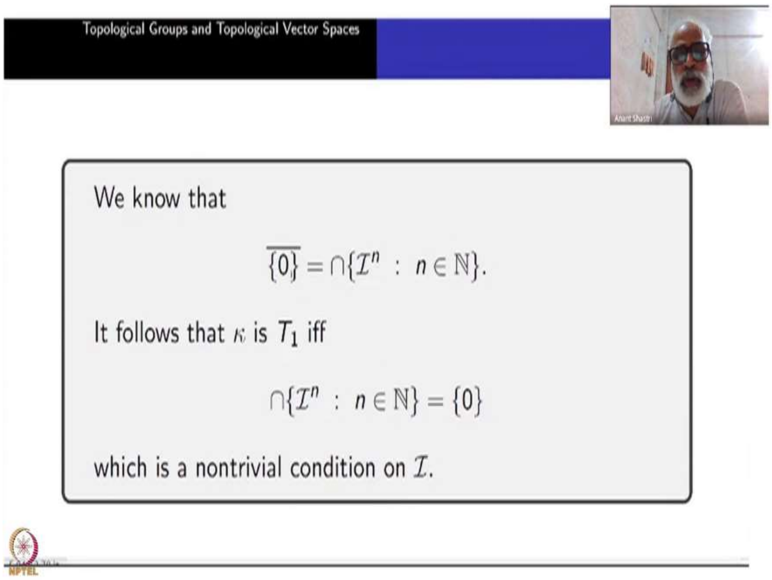

It follows that the Krull topology is T_1 iff the $\{0\}$ is closed if and only if this RHS is equal to $\{0\}$. T_1 is what? $\{0\}$ must be closed, $\{0\}$ is closed means $\{0\}$ is equal to $\overline{\{0\}}$; the $\overline{\{0\}}$ is equal to this one. This is the condition on I, the ideal I should have the property that intersections of all its powers. The powers of I, you know, are one includes the next one and so on. So, that should become $\{0\}$ and that is a nontrivial condition this does not happen always.

For integers you can see that it happens. Take a prime p, if every power of p divides a number that number must be 0, ok? So, in particular, if you take the ring to be integers, you can see that the Krull topology is T_1. That is all I want to tell you about the Krull topology.

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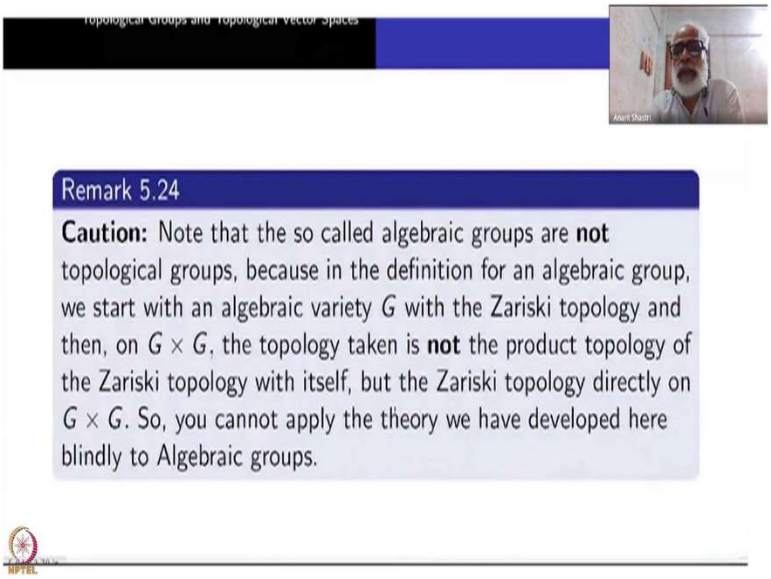

There is another aspect in algebraic geometry ok. What are called algebraic groups, they are not actually topological groups ok? You have to be careful. Because in the definition of algebraic group, we start with an algebraic variety G, with the Zariski topology, namely closed subsets are those which are given by vanishing of polynomial functions. And then on $G \times G$ you are not taking the product topology.

It is not the product of the Zariski topology with itself, but it is the Zariski topology directly on $G \times G$, which just means that the number of variables is doubled and all polynomials in those variables have to be taken and so on. So, you have to be a bit careful there. You know topological group theory that we are developing in this sequence of lecture, cannot be applied to Algebraic Groups directly.

There may be several parallel statements ok, parallel definitions etcetera. You have to check each of them correctly, properly and then only you can use them. Some of them are even wrong ok. So, anyway none of them you would have proved, because all our proof depends upon the topology on $G \times G$ being the product topology ok.

So, having said that, let us do something in our own definition, not Zariski topology now.

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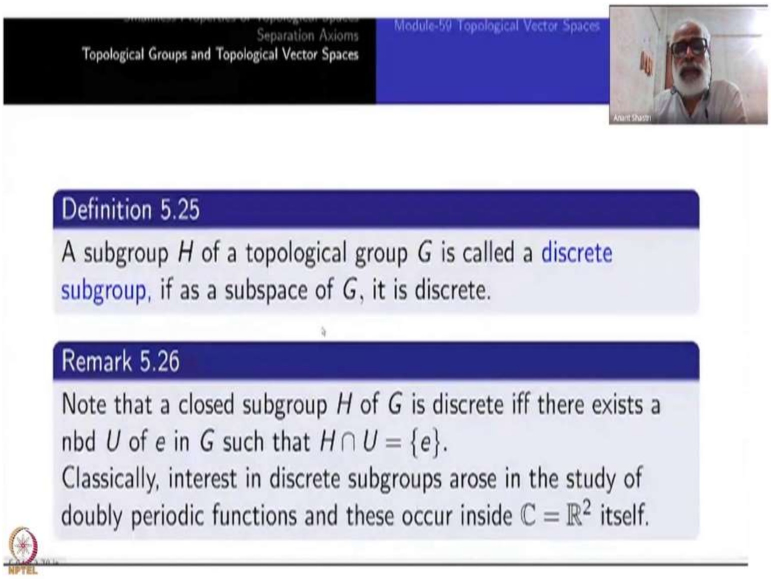

A subgroup H of a topological group G is called discrete ok, discrete subgroup, if as a subspace it is discrete, it is already subgroup there is no condition on group level. The subspace topology must be discrete. Discrete means what? Isolated and closed ok, it must be a closed subgroup, it must be a closed subset and each point must be isolated.

Note that a closed subgroup H of G is discrete, if and only if there exists a neighbourhood U of e such that $H \cap U$ is $\{e\}$. As soon as $\{e\}$ is isolated, again by using translations you can show that all elements of H are isolated, take any $h \in H$ ok. You take the same neighbourhood U, h of U intersection with H will be just $\{h\}$.

So, so this is easy to see that once singleton e is isolated in H; H will be a discrete subgroup. Only you do not know whether H is closed so you have to put that closedness condition also.

So, why one is interested in discrete subgroups? It is a very old notion, you know classically, interest in discrete subgroups arose in the study of doubly periodic functions ok. You can see that the exponential function is already periodic, but that did not really give rise to this study of discrete groups and so on.

But the same thing when you take doubly periodic functions inside complex plane ok, there you have to start worrying about more general things. And right in $\mathbb{C}$ itself, $\mathbb{R}^2$ itself it happens. So, why many many properties of this periodic function ok, they can be introduced easily if you understand the discreteness of the periods, the set of periods becomes a discrete subgroup that is how this is interesting.

Since you may not know what a periodic function is and so on, I will not elaborate on that one. So, this much motivation is enough ok? So, here is an easy lemma first of all and then we will improve upon this lemma later.

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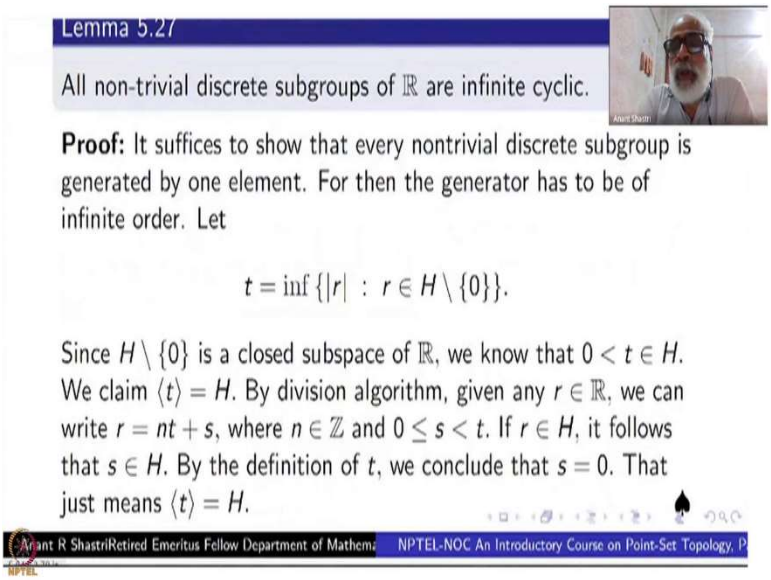

All non-trivial discrete subgroups of $\mathbb{R}$ are infinite cyclic. Typical example is integers sitting inside $\mathbb{R}$, ok? here I am looking at $\mathbb{R}$ as an additive group. So, it is a topological group additive topological group right?

So, $\mathbb{Z}$ is a subgroup which is discrete. So, what this theorem says is everything is infinite cycle, as soon as you take any non-trivial discrete subgroup it must be infinite cycle, the proof is very easy ok. It suffices to show that every non-trivial discrete subgroup is generated by one element ok; obviously, in the additive group of real number every element other than 0 is of infinite order. Therefore, this is all enough, the group generated by that one will be infinite cycle automatic.

So, so what I start, I am looking for that generator. So, put t equal to infimum of $|r|$, where r belongs to $H \setminus \{0\}$, non-zero elements of H. Look at the one which has least modulus ok, what I want to say there are exactly two of them ok. If one element is there, minus of that will be also there because H is an additive subgroup right.

So, look at the infimum of $|r|$; obviously, this is bounded below. So, infimum is well defined, but this is a discrete group, ok. So, it is a discrete set of points inside r right. $H \setminus \{0\}$ is a closed subset of $\mathbb{R}$. So, this infimum will belong to $H \setminus \{0\}$, therefore, this t is positive, it is attained, means it is actually minimum.

That means what? See if I take $0 < t, t$ itself is equal to $|t|$ and hence t is inside H ok? So, having found an element like this, we claim that t generates H ok.

So, for this all that you have to do is to use division algorithm ok? $t, 2t, 3t, 4t$ etc, and $-t, -2t, -3t, -4t$ etc are also there in H but nothing else is there, that is what you want to show right? So, given any r can be written uniquely as $nt + s, n$ is some integer plus or minus, but $|s|$ will be strictly less than t.

If you start with r inside H, t is already inside H. So, nt is inside H, $r - nt$ which is s that will be also inside H. But $|s|$ is less than t, how can that be. So, the only way it can happen is this s must be 0, s is not in $H \setminus 0$. So, s is 0 means $r - nt$, where n is in integer therefore, $H$is generated by t ok.

Now, without much effort, this same idea can be generalized to any $\mathbb{R}^n$. Only thing is we have to use now more linear algebra not just real numbers, but linear algebra we will have to use ok. Not very deep, very elementary linear algebra only.

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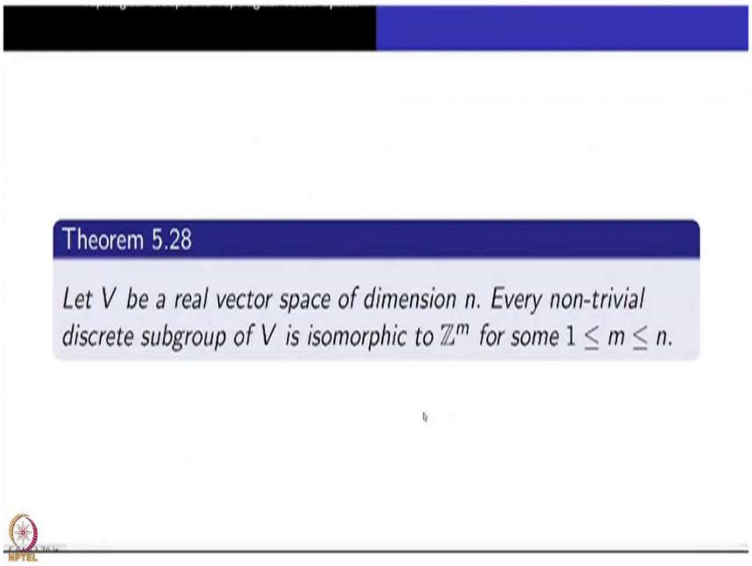

So, let us see how. So, statement is: let V be a real vector space of dimension n. Every non trivial discrete subgroup of V is isomorphic to $\mathbb{Z}^m$ for some $m \leq n$ ok, nontrivial I have assumed. So, $1 \leq m \leq n$. If n is 1, we have already proved it in the previous lemma. So, the idea is to use induction ok, induction and some linear algebra.

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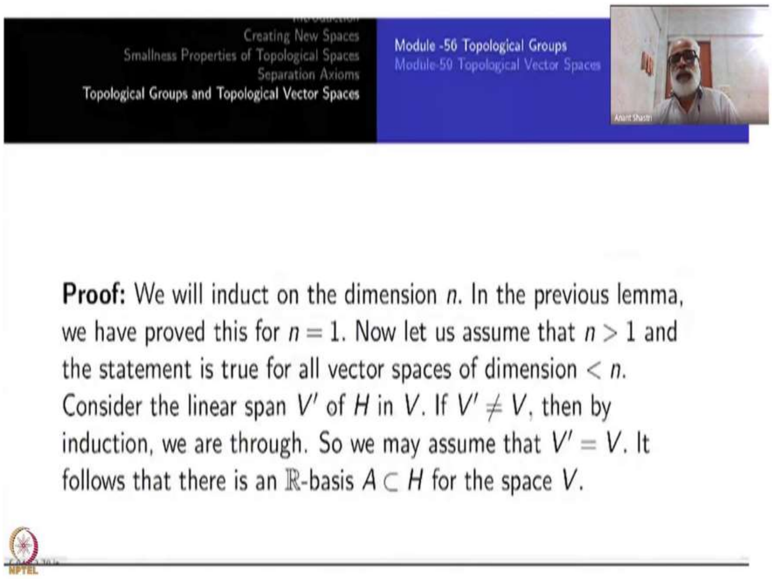

Let us assume this $n > 1$, and the statement is true for all vector spaces of dimension less than n. Take the linear span V' of H inside V, H is a subset of V. So, you can take the linear span that is some vector space, vector subspace of V.

Sorry, if V' is not the whole of V, a proper subspace, then in this subspace as dimension less than n, then by induction we are through. So, H sitting there right, it is a discrete subset inside V itself. So, it will be discrete in V' also ok. So, you can apply the induction.

So, without loss of generality we can assume that we are inside V' equal to V that is the case. What does that mean? That the set of vectors $h \in H$, they form a generating set for V.

Any generating set will admit a subset which is linearly independent and maximal, a basis. At least this is true very easily for a finite dimension vector spaces right. So, that is the only linear algebra I am using just now ok? It follows that there is an $\mathbb{R}$ basis A contained inside H ok for the space V. What is V? V is V', it is the span of H, alright?

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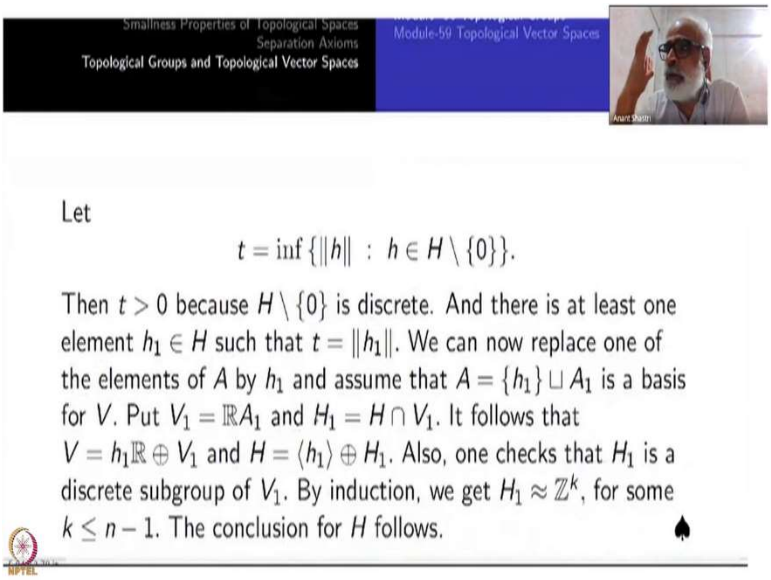

So, I could have made an elaborate statement in the in the theorem itself, but even this step will be use useful to you. So, you should remember the proof here, rather than just the final statement. Now, let us go ahead. Now, the idea is similar to the dimension 1 case. What you do? Put t equal to infimum of $\|h\|$ now, not just $|h|$ ok. Where h runs over $H \setminus \{0\}$. Once again H is discrete so $H \setminus \{0\}$ is also discrete closed and all that. Therefore, this t will be positive ok and will be attained.

That is there is at least one element h_1 belonging to H, such that this $t = \|h_1\|$ ok. You can now replace one of the elements of A by h_1 ok and assume that A is h_1 union some other finite set, which is a basis for V. You can trade, how do you do that? Remember write this h_1 as something $\alpha_1 v_1 + \alpha_2 v_2 + \ldots \alpha_n v_n$, one of the α_i's must be non-zero, say let us say α_1.

Then instead of v_1, you can put this h_1 and keep other $v_2, v_3, \ldots, v_n$ as it is that will be also basis. So, this is trading this thing is also part of linear algebra which is used to prove that any two basis have same number of elements right. I am just recalling some linear algebra that is all.

So, remember all elements of A are all in H itself. But now the first element is a special one, viz., norm of h_1 is minimal, there may be many elements with this property. I have taken one of them and assuming that this is one of the basis elements.

Now, I just split up the whole thing, write A equal to $\{h_1\} \cup A_1$. Put V_1 equal to the linear span of A_1 and H_1 equal to $H \cap V_1$. So, I am taking a subspace which is 1 dimension lower ok, and then I am taking H_1 equal to $H \cap V_1$. H is discrete inside V, H_1 will be discrete inside V_1, ok.

So, what you have is V is written as some copy of $\mathbb{R}$ spanned by h_1 direct sum with V_1 and H will be the infinite cyclic subgroup generated by this h_1 inside $\mathbb{R}$. So, that is the lemma 1, direct sum with H_1, this is a discrete group and this is the copy of $\mathbb{Z}$, that is also discrete of course.

So, you get the direct sum decomposition like this, because there is a direct sum decomposition for the whole vector space. Also, one checks that H_1 is a discrete subgroup of V_1, because its intersection of H with V_1. So, by induction because the dimension has dropped down here H_1 must be isomorphic to some $\mathbb{Z}^k$ for some $k \leq n-1$.

Add this one more component H_1, what does it give you? H is isomorphic to $\mathbb{Z}^{k+1}$.

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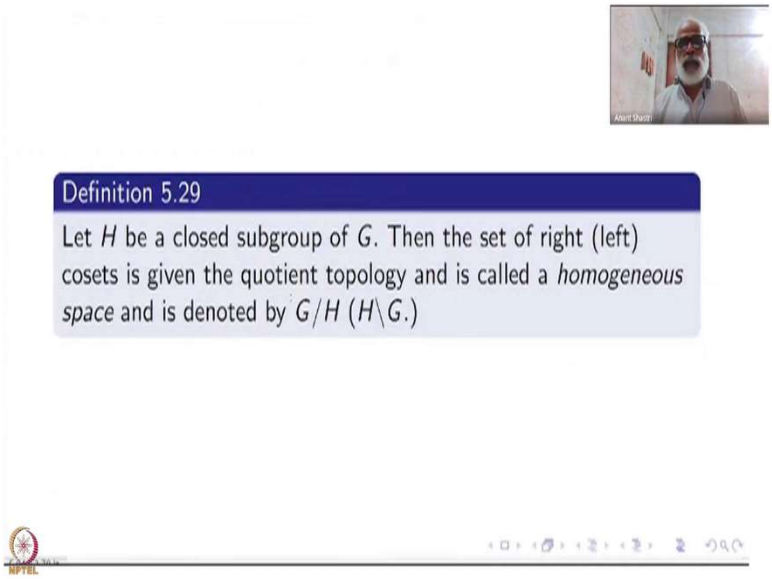

So, this is just a small beginning of the study of discrete subgroups, which is a vast subject. You know there are books written on with title Discrete Subgroups of Lie Groups ok.

Let us go ahead with the study of the subgroups. Take H to be a closed subgroup of G ok, then the set of right cosets, (similarly left cosets also you can take) is given the quotient topology because they are the orbits of the action of H on G right? H acting on the left of G.

Cosets are what? They are they are the orbits of the action. Left cosets or right coset depending upon which action you take, is given the quotient topology because it is a quotient set ok? This is called a homogeneous space ok.

What is the meaning of homogeneous space? You start with a topological group, take a closed subgroup then look at G/H. Assumption that H is closed is very important here because non closed subgroups are extremely badly behaved. You can always take G/H where H may not be closed, but you cannot do much topology on that one ok.

So, I am using this notation G/H for right cosets, I will read it as G by H only, but I do not know how to read it otherwise. This is very popular in Lie group theory and so on. So, they take left cosets, as well as right cosets both of them. That is why they have cooked up this notation.

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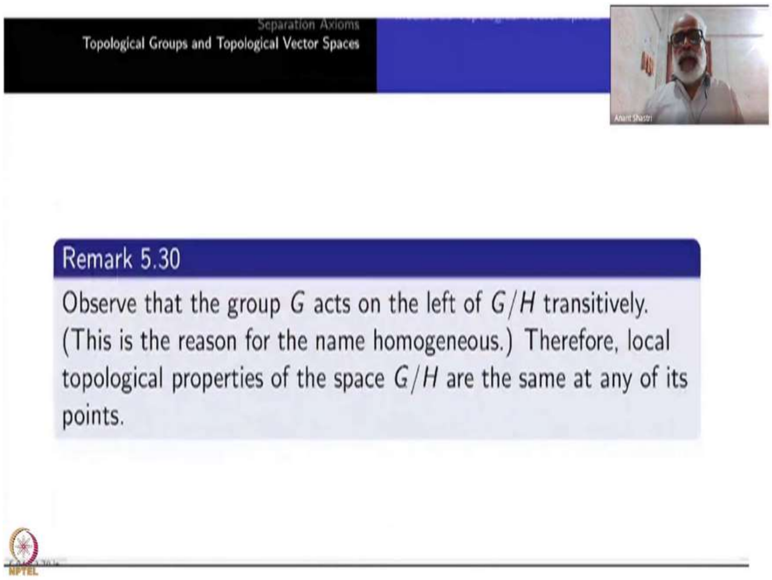

Observe that H acts on the left of G, then G acts on the right on H. Therefore, G acts on the right on the set of right cosets as well. This is the reason for the name homogeneous because action is transitive. What is the meaning of transitive? Given any two cosets here that is an element of g of G such that g of one will be other coset.

So, one point is taken to the other point, any two points are related by the action or the orbit space of this by this action will be just one single element. So, that is the meaning of transitivity ok. Such actions when you have such actions on a space that space is called homogeneous ok. that is the reason for homogeneity.

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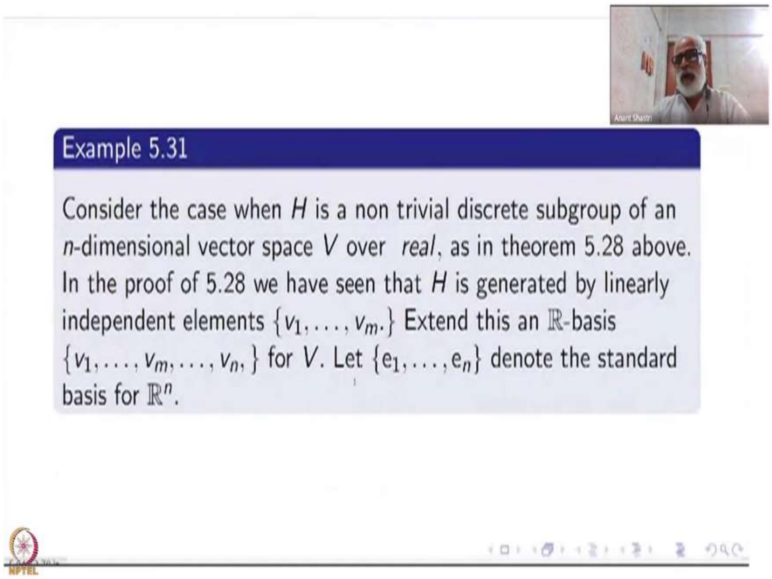

So, what is the, what is the good point of homogeneous spaces? All local properties, topological properties if they hold at one single point they will hold at all other points. Just like topological groups have that property, the homogeneous spaces will also have that property.

So, here is an example here, which we have just discussed already right. Consider the case when H is non trial discrete subgroup of an n-dimensional vector space over the reals. Just now we have discussed it ok.

In the proof of this that theorem as well as in the previous lemma, we have seen that the discrete subgroup H is generated by linearly independent elements $v_1, v_2, \ldots, v_m$ where $m \leq n$, right. Extend this $\mathbb{R}$-basis, extend this to an $\mathbb{R}$ basis $v_1, v_2, \ldots, v_m$, add some more elements to get a basis for the entire vector space V, ok.

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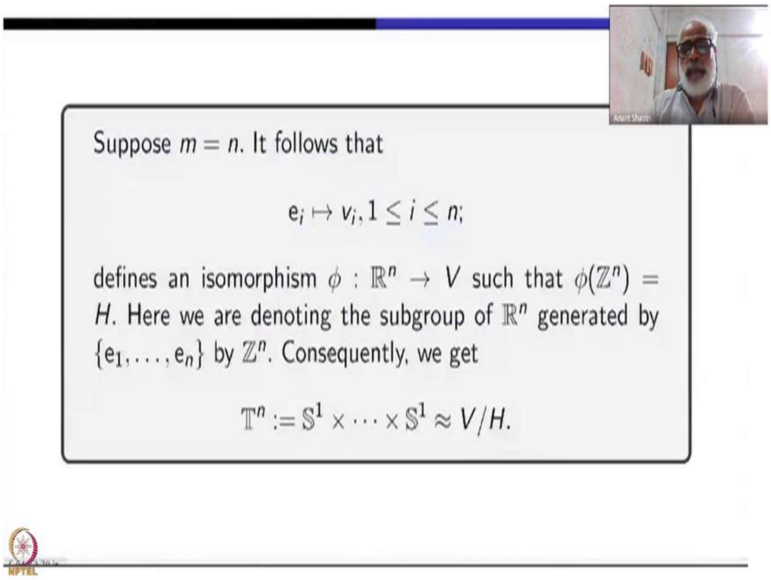

Take $e_1, e_2, \ldots, e_n$ as standard basis for $\mathbb{R}^n$. Now what I am doing? I am going to map e_1 to v_1 and so on e_n to v_n to get an isomorphism from $\mathbb{R}^n$ to V ok. Because they have the same number of basis elements after all. Map e_i to v_i, i let us say equal to 1 to n to get an isomorphism phi from $\mathbb{R}^n$ to V. This will have the property that if you take the subgroup generated by e_i's that is $\mathbb{Z}^n$.

So, they are they will generate $\mathbb{Z}^n$, the direct sum of $\mathbb{Z}$ with itself n times. Let us call $\phi(\mathbb{Z}^n)$ will be H automatically because they will be they will be mapped to v_i. So, H will be generated by v_i that is the assumption ok. So, here we are denoting the subgroup of $\mathbb{R}^n$ generated by e_1 to e_n by $\mathbb{Z}^n$, so that is H. Consequently, what you will get is $\mathbb{T}^n$ is a notation, $\mathbb{S}^1 \times \mathbb{S}^1 \cdots \times \mathbb{S}^1$ isomorphic to V/H ok.

I am sorry forgot to tell you that suppose $m = n$, then only this will be true ok. Suppose $m = n$, then $\phi(\mathbb{Z}^n)$ will be the lower of H because that is the subgroup generated by all the $v_1, v_2, \ldots, v_n$ because $m = n$. Then $\mathbb{T}^n$ is nothing but $\mathbb{S}^1 \times \mathbb{S}^1 \cdots \times \mathbb{S}^1$, which is V/H ok, V/H is this one, but how do you get this one, this is because this $\mathbb{R}^n/\mathbb{Z}^n$; $\mathbb{R}^n/\mathbb{Z}^n$ is isomorphic to the product $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \cdots \times \mathbb{R}/\mathbb{Z}$.

$\mathbb{R}/\mathbb{Z}$ you know is isomorphic to $\mathbb{S}^1, t$ going to $e^{2\pi it}$ giving you the isomorphism $\mathbb{R}/\mathbb{Z}$ to $\mathbb{S}^1$ ok. So, you know all this these are called tori, you know each of them is called a torus $\mathbb{S}^1 \times \mathbb{S}^1$ is the standard torus in dimension 2. The same name is used in higher dimension also, it is a tori. They are all what? Take any finite dimension vector space and take some largest kind of discrete subgroup sitting inside that and take the quotient.

If you change the subgroup, the topology does not change, but a lot of geometry will change here. So, that is why they are very very interesting objects. So, these groups are very interesting, even in the case of $\mathbb{S}^1 \times \mathbb{S}^1$. For $n = 2$, they are called elliptic curves.

Why? because each group here how it is sitting inside $\mathbb{R} \times \mathbb{R}$ will tell you a different story, different complex analysis is there. Complex analytically the quotients will be different. So, complex structure will be different. So, each of them is called, you know, an elliptic curve.

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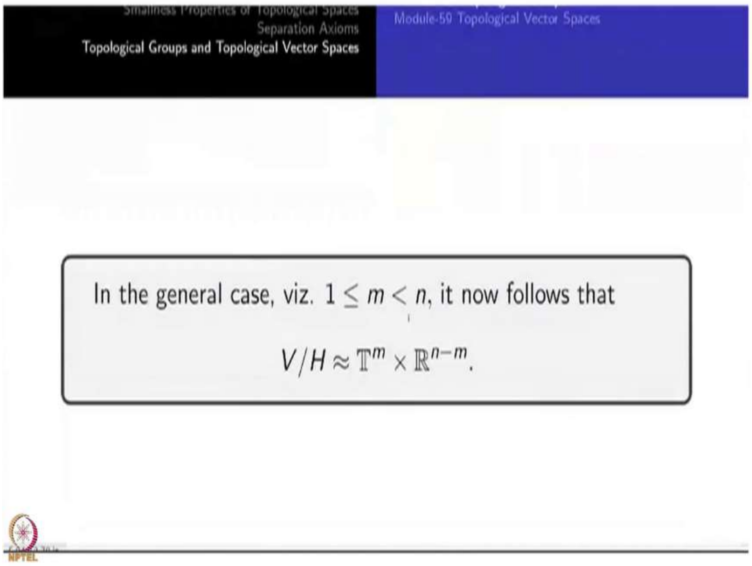

In the general case, when $m < n$ what happens? V/H will be $\mathbb{T}^m \times \mathbb{R}^{n-m}$. We obtain this the directly, all this way, we could have seen this one in the proof of our theorem itself. Up to m you have basis, you have extended it, extended part is $\mathbb{R}^{n-m}$. So, here it is already quotient. V_1/H is $\mathbb{T}^m$, where V_1 is corresponds to the linear span of the first m elements

$v_1, v_2, \ldots, v_m$ ok. So, this is a general picture. So I have told you the complete general picture of any discrete subgroup of a finite dimensional vector space ok.

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Now, here are some exercise which you can do easily. Arbitrary product of topological group is a topological groups alright.

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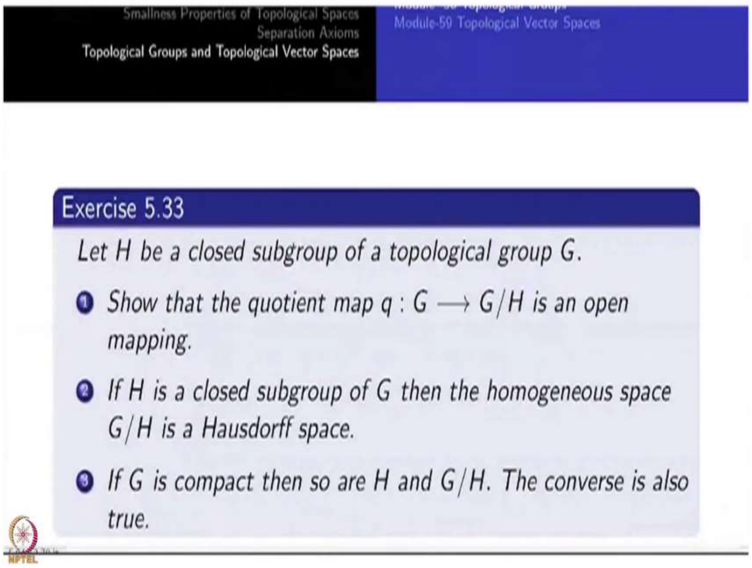

Then some more exercise are there which are not all that easy, but if you keep solving them one by one then it is ok. In fact, this one, the third one here, we have seen it in a different context though this will not be difficult for you at all ok.

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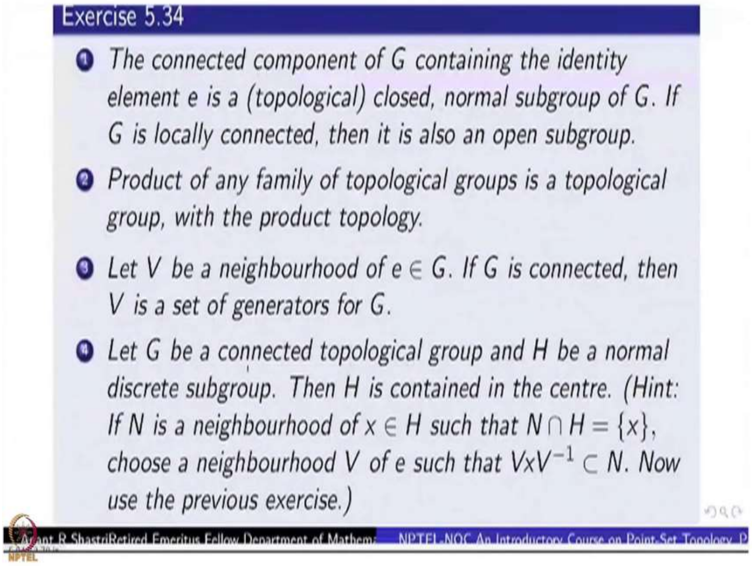

And then comes the connectivity reults. here you may have to spend a little more time, but you know, again try to solve them in that order, solve slowly then you will get all. You can solve all of them, that is the whole idea ok.

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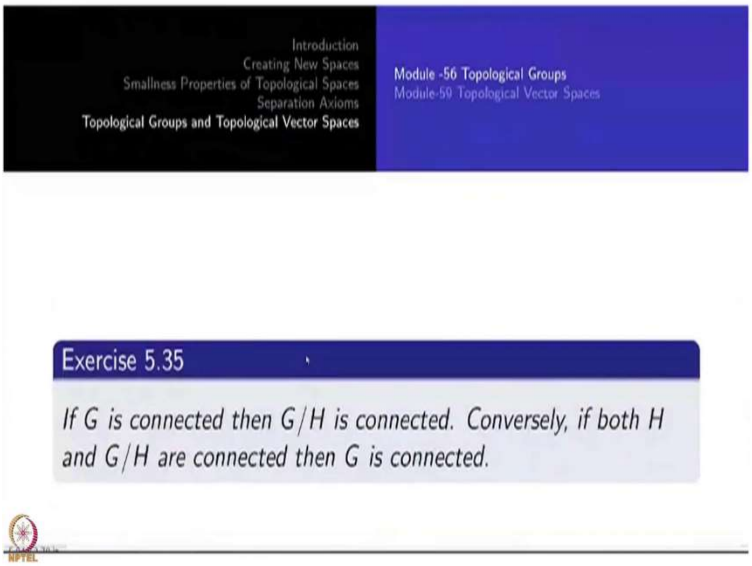

And then you can apply them to various different cases also.

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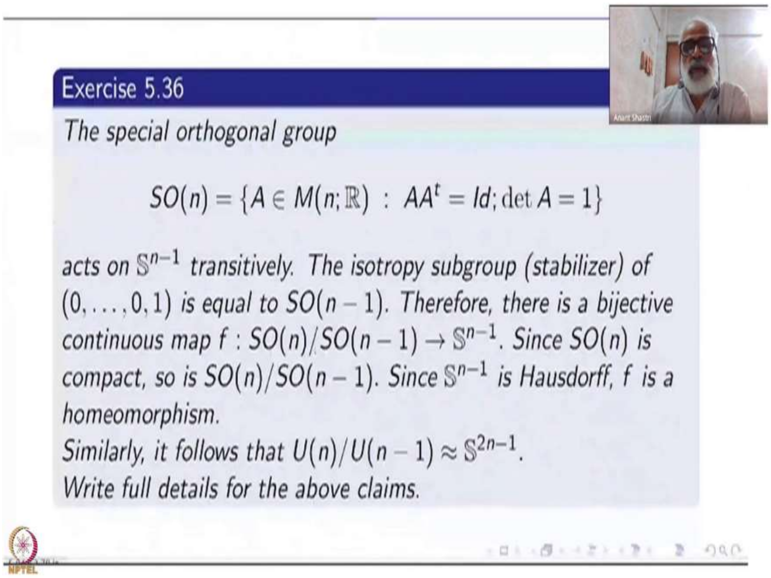

Namely, you can apply them to study these classical groups $SO(n), SO(n+1)$ and so on. So, there are some nice things happening her. So, I have tried to motivate these examples you know, by giving those elementary exercises so that you can solve these things easily. So, finally, these examples are the motivational examples for those exercises ok.

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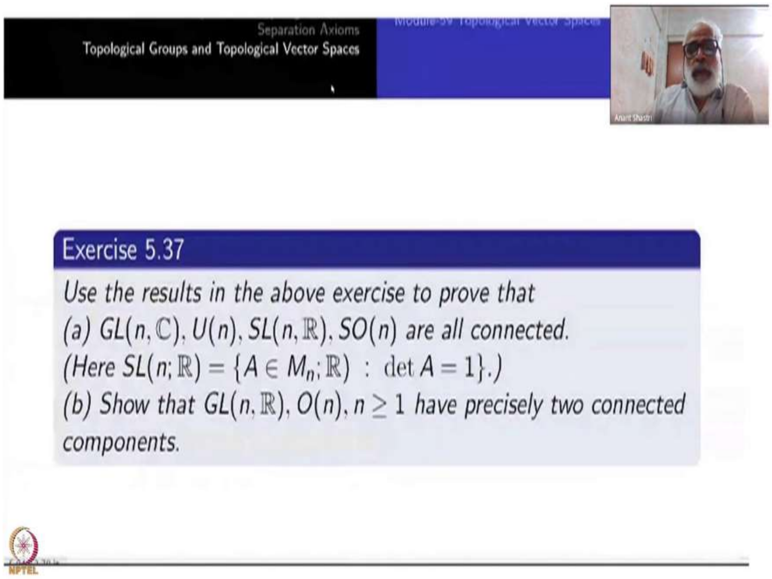

So, best of luck, you can try to do this one. Next time we will continue the study of topological vector spaces now ok. That will be the last topic for this course.

Thank you.