Introduction to Point Set Topology, (Part I)
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Lecture - 57
Topological groups-continued

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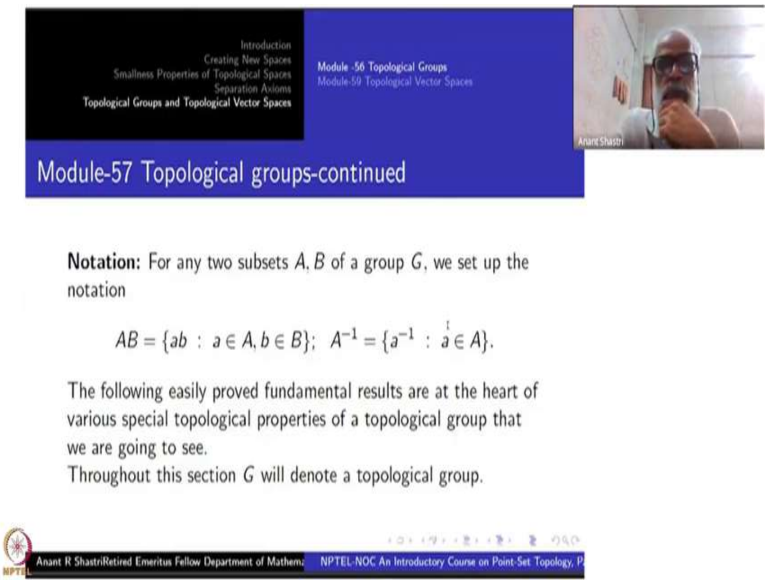

Welcome to module 57 of Point Set Topology Part 1 course. We will continue our study of topological groups today which we have started last time right. So, here are some notation which I am going to fix up at least for today maybe tomorrow for next time also. So, these notations will be for this section.

Take any topological group or just a group to begin with, for any group, you can have this notation A and B are subsets ok, A followed by B, AB you can writeis is equal to all ab, where $a \in A$ and $b \in B$ ok. Similarly, $A^{-1}B$ or A^{-1} etc are defined, alright.

I want to draw your attention. This is not a subgroup generated by something and so on. It is inside the group, but these are just sets and this is just (A, B) ok, but you have to combine them with the group law. Here, you have to take the inverse and take the corresponding set.

So, the following easily proved fundamental results are at the heart of various special topological properties of topological groups that we are going to obtain ok?

So, I will not keep saying that G is a group (G, μ, e) is a group and so on. So, this G will be always denote a topological group for some time.

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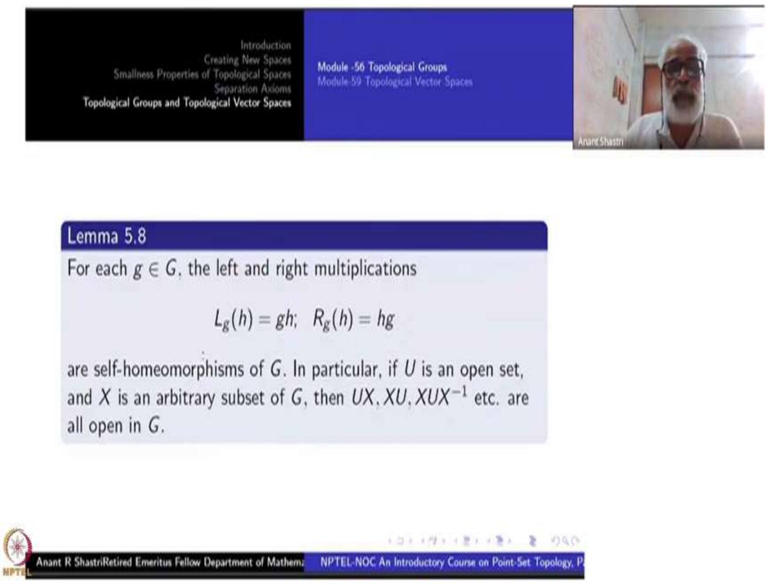

The first thing is this lemma. It says for each $g \in G$, look at the left multiplication map given by g, namely $L_g(h) = gh$. So, it is a map from G to G. Similarly, I can take the right multiplication $R_g(h) = hg$, ok.

So, one is multiplying on the left, the other multiplying on the right ok. So, both are maps from G to G. They are self-homeomorphisms of Gok? It is very clear that they are invertible, the inverse of L_g being equal to $L_{g^{-1}}$. Multiply by g^{-1}; g into g^{-1} is identity. So, it will be h. So, these are bijective function, you know L_g and R_g. Why they are homeomorphisms? Because they are also continuous ok.

Multiplication by g, you are freezing the element g, only h is variable. So, if it is like a two variable function you are taking and one variable is fixed. So, that is also continuous ok? since L_g and $L_{g^{-1}}$ both are continuous both of them are homeomorphisms.

Similarly, R_g and $R_{g^{-1}}$. So, an easy consequence of this is that you take any open subset and x be any arbitrary subset of G, then UX, What is UX? UX consists of ux, collect all of them together. So, that is UX and it is an open set.

Similarly, XU now multiplying on the other side. Or you can take XUX^{-1} or you can take many other combinations. Go on multiplying just like writing alphabets. One of them is open, that is necessary, then all these product subsets will be open.

Can you see why this is true? Look at U that is an open set. When you multiply on the right by x, what is that? It is just $R_x(U)$, R_x is a homeomorphism.

So, $R_x(U)$ one single x here ok, namely Ux that is an open set because it is the image of U, under R_x. Now UX is nothing but union of all these things where x varies over X. Therefore, this UX is open. similarly XU. So, once this is open you can apply this, you know to XU times X^{-1}, X^{-1} is some other set right and so on.

You can go on applying to finitely many products all the time. So, they will all be open subsets. I am repeatedly using that R_x's and L_x's which are homeomorphisms ok. So, this is the fantastic thing happening here inside a topological group.

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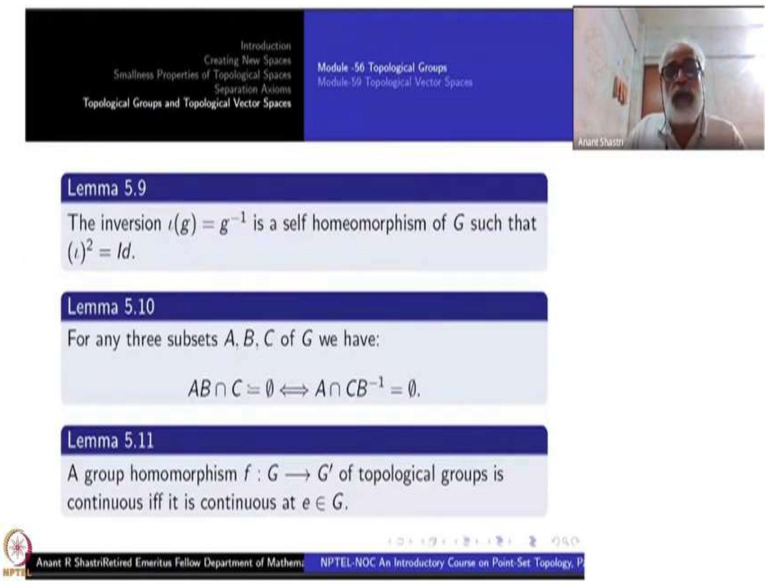

The second thing is that inversion $i(g)$ which I have denoted by $i(g)$ that $i(g)$ equal to g^{-1}, this is also a self-homeomorphism of G and if you take this again, that is $i \circ i$ namely i^2 that is the identity map, because inverse of g^{-1} is g itself ok. Since i continuous, that we have seen already, i^2 is identity. This is a homeomorphism. inverse of i is i itself. So, it is a homeomorphisms ok. It is a very special homeomorphism it is of order 2.

Next here is an easy thing which you will keep using without even mentioning it. So, here I have mentioned it for the first time maybe you know it will be used several times namely, take any 3 subsets ok. Suppose $(AB) \cap C$ is empty iff $A \cap CB^{-1})$ is empty. This is very easy to verify. Take a point here which mean which will look like ab, but it is an element of C. So, we have $c = ab$ right? But then $a = cb^{-1}$ So, there is an element a here and the same element is here cb^{-1}.

So, the RHS is nonempty. and conversely ok?

A group homomorphism f from G to G', ok. What is a group homomorphism? $f(xy)$ is equal to $f(x)f(y)$ for all $x, y \in G$, right. A group homomorphism of a topological group is continuous if and only if it is continuous at a single point, namely e belonging to G. Any other point will also do, but let us prove it for e belonging to G, ok

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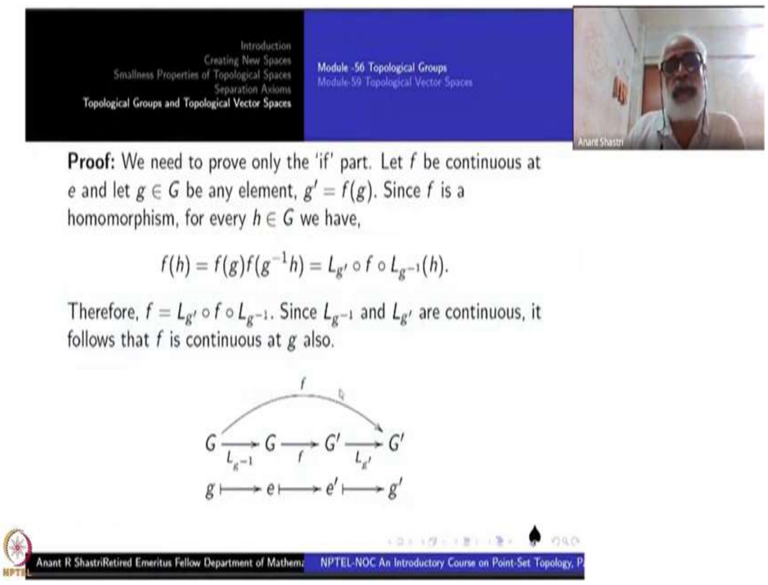

So, you must have learnt it elsewhere, but let me sum it up here, because of the importance of this little result here ok? We need to prove only the `if' part here because the `only if' part is obvious. If f is continuous, it will continuous at e also. Just continuity at a single point makes it continuous everywhere that is the part which you have to show ok.

So, let f be continuous at e and g be any other element some other element. Let us look at the image of that namely $g' = f(g)$. Since f is a homomorphism for every $h \in G$, we have $f(^{-1}gh)$ is equal to $f(g)^{-1}f(h)$. Therefore $f(g)f(g^{-1}h)$ is equal to $f(h)$. This is true for all $h \in G$.

So, what I have done? I multiplied by g^{-1} on the left inside the bracket. So, outside the bracket it will be $f(g^{-1})$ because of the f is a homomorphism. So, I have to compensate it. So, I am multiplying again by $f(g)$. So, it cancels out alright, but now what is this inside thing, it is a left multiplication by g^{-1}, right.

And $f(g)$ is a point of G'. So, it is a left multiplication by $L_{g'}$ inside G'. So, this whole thing will look like $L_{g'} \circ f \circ L_{g^{-1}}$ operating upon h. Therefore, f itself is equal to $L_{g'} \circ f \circ L_{g^{-1}}$ ok. So, I have written f itself like this in a complicated way.

But, this will help us now why because I know $L_{g^{-1}}$ is continuous, $L_{g'}$ is continuous to show that f is continuous at a point I have to only show that f is continuous at some other point. What is that? That is precisely what is happening here see, I want to show this map f is continuous at g ok.

So, I apply $L_{g^{-1}}$ to this; this g comes to e, now I apply f, but f is continuous at e. So, I can go all the way up till here when I come to f of that, it is e' and $L_{g'}$ will take it to g' that is nothing but $f(g)$ ok. So, middle thing is continuous at e, these two are continuous functions, the composite is continuous and starting point here is g.

So, f is continuous at g ok. There are many different ways of writing it down. So, I find it the elegant way of writing that is all. So, if you learn this kind of writing down it will make other concepts very clear. At elsewhere also.

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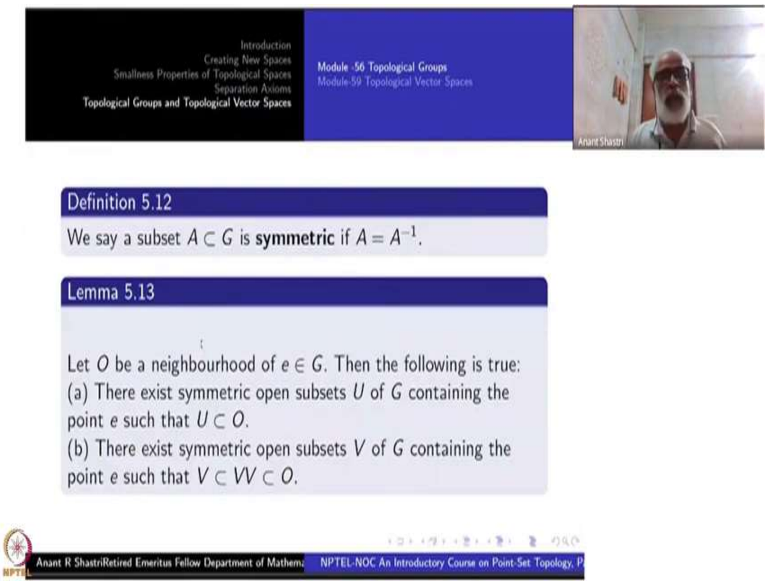

Now, I introduce a terminology here because this inversion keeps coming again and again right. You take a subset A contained inside G and call it symmetric if A equal to A^{-1}; that is if you take the i, the inversion. Under inversion, it is invariant $i(A) = A$ok. So, such a subset is called symmetric. Now this is a temporary notation this has been used by others. So, I am also using this terminology.

Now, here is a lemma for symmetric things. Let O be a neighbourhood of the identity element in G, ok. Then the following is true. There exist symmetric open subsets U of G such that the point e is in U and such that U is contained O. So, starting with any neighbourhood O, you can improve it to become a symmetric neighbourhood ok.

There are many ways of doing it. So, there are many of them. In fact, if this happens to nbd, this just means that there is a symmetric neighbourhood systems. So, this means just that instead of all neighbourhoods you can just take symmetric neighbourhoods to form a fundamental system at the origin.

So, that is the profoundest thing here ok. Similarly, even more profound is that there exist symmetric open sets V of G containing the point e and such that V is a subset of VV that is obvious anyway ok, but VV itself is contained inside O. Now, see V is symmetric. I can replace V by V^{-1} at my will.

So, this will imply a lot many things. That is what I told you. These are elementary observations to build up the topological theory. So, you have to come back here if you have made a mistakes or you have not understood these things correctly. Then you will have no problems you will see that ok.

So, starting with any neighbourhood of identity I can get such beautiful neighbourhoods; neighbourhoods which are symmetric, neighbourhoods which are contained inside their own product and the product is contained inside O and so on ok.

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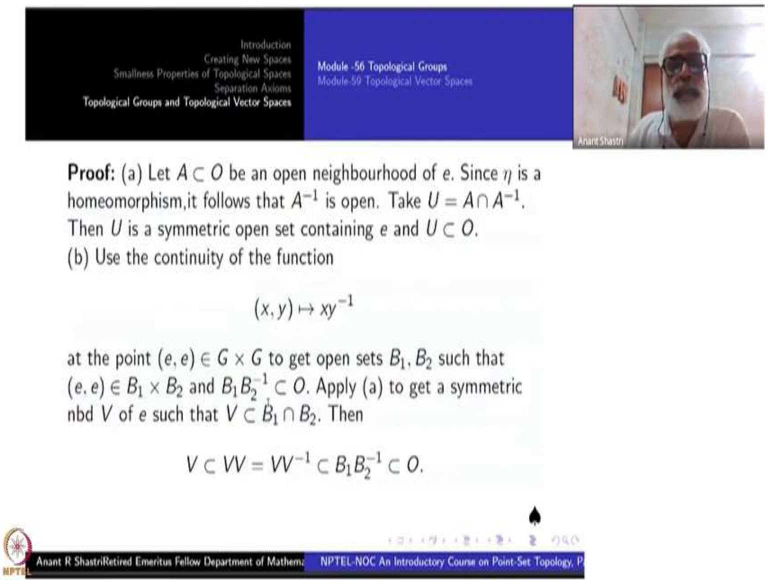

So, how to prove that? Proof is also very elegant in a sense. Here I am actually proving even stronger thing here namely, instead of single point you can do it for any neighbourhood. Let A contained inside O be an open subset of G ok. See I started with an arbitrary neighbourhood only now I am taking A to be open subset. A neighbourhood means after all there is an open subset right. So, e belongs to A, A is open and A contained inside O, let us start that way. Since the inversion is a homeomorphism, it follows that A^{-1} is also open ok?

because it is $\eta(A)$. Take $U = A \cap A^{-1}$.A is open A^{-1} is open, intersection is open. Inverse image of e is e itself.

So, e is in both of them, so that U is a neighbourhood of e open neighbourhood of e ok. What is U^{-1}? It will be $A^{-1} \cap A$ it is same thing as U. So, U is a symmetric open set containing e and U is contained inside O ok because U is $A^{-1} \cap A$ and A is also contained inside O ok.

So, that is what I am writing here namely (e, e) belong to $B_1 \times B_2, B_1$ and B_2 are open subsets ok. This $\nu(B_1, B_2)$ will be equal to $B_1 B_2^{-1}$ that will be contained inside O. So, this is by continuity of this map which I have denoted by nu earlier. Apply (a) to get a symmetric neighbourhood V of e such that this V is contained inside the intersection of B_1 and B_2.

B_1 and B_2 are neighbourhoods of e. So, intersection is also neighbourhood of e. So, now, I can improve it to become symmetric also. So, $V = V^{-1}$ is extra hypothesis I can put ok. So, once you have that V is always contained inside VV because identity is there right.

So, $V = eV$ that is contained VV so, but I can replace this V by V^{-1}. So, this will be VV^{-1} also because $V = V^{-1}$, but now this V is contained inside B_1 and it is also contained inside B_2. So, first one is B_1 and second one is B_2^{-1}, I have written that is all. And that is contained inside O ok.

So, proof is over. So, we will have opportunity to use this one, no problem let us see.

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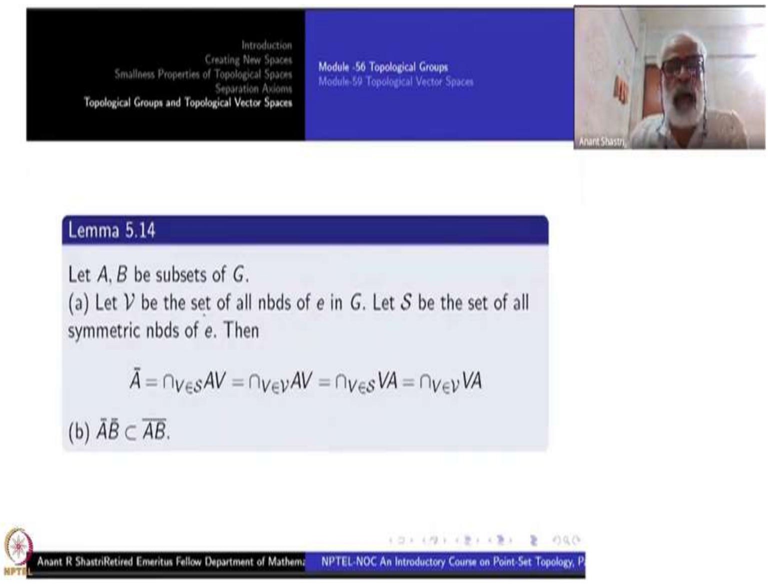

Now, a little about closures and so on. Let A and B be any subsets of G again ok. Let this $\mathcal{V}$ be the set of all neighbourhoods of $e \in G$. Let $\mathcal{S}$ be the set of all symmetric neighbourhoods of e, this is a smaller family ok.

They are all neighbourhoods of e. and these are only symmetric neighbourhoods. Then the claim is: A an arbitrary subset, I am making a statement about $\bar{A}$. $\bar{A}$ is intersection of all AV where V ranges over all symmetric neighbourhoods of e. It is also equal to the intersection of all AV where V ranges over all the neighbourhoods of e and then the other way round; instead of AV, I have VA here and VA here the other two things are similar here. So, either I can write it on the left side or write it on the right side. So, I take intersection of all of them and what I get is $\bar{A}$, ok. The second thing is much easier $\bar{A}$ into $\bar{B}$ is contained inside $\overline{AB}$. It is a neat statement ok. Let us see how this proofs work.

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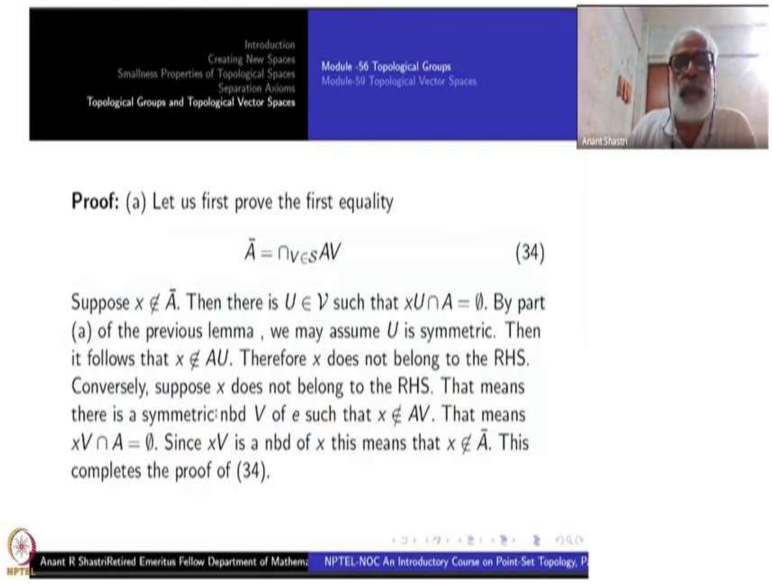


Let us first prove the first one. What is the first one? $\bar{A}$ equal to intersection of AV, V belonging to $\mathcal{S}$. So, where $\mathcal{S}$ is only symmetric ok. What is the meaning of $\bar{A}$? Take a point in $\bar{A}$, means every neighbourhood of that point intersects A that is the definition of closure point right.

We shall use that directly the definition of closure no other properties directly use the definition ok. Suppose x is not in $\bar{A}$, then I want to show that it is not in one of the sets here and therefore, it is cannot be in intersection. So, not in $\bar{A}$ means not on the right hand side and again not here means not here that is what I want to show ok.

So, suppose x is not in $\bar{A}$ then there is a U inside $\mathcal{V}$, What is curly $\mathcal{V}$? Memebrs are neighbourhoods of identity element such that $(xU) \cap A$ is empty. So, what is xU? xU will be a neighbourhood of x, x is not here. Every neighbourhood of x will look like xU where U is a neighbourhood of identity; this is what I am using here.

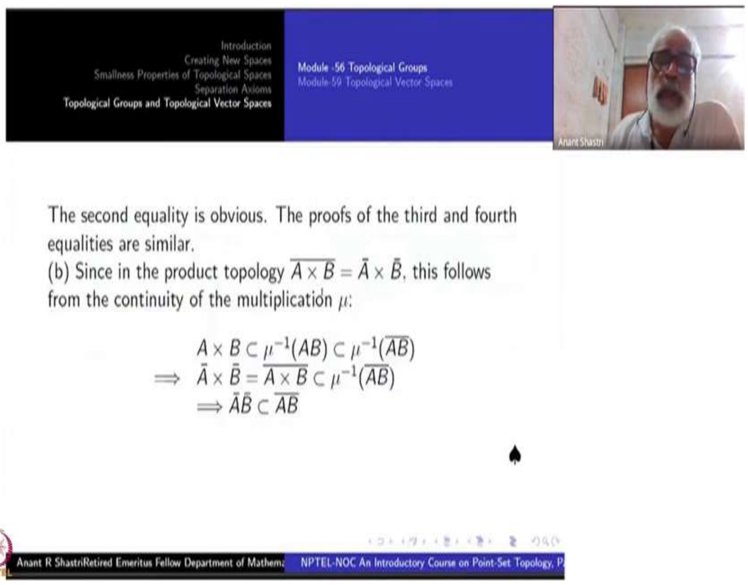
You see neighbourhoods of any other point you do not have to look anywhere you have to just translate them left translation or right translation does not matter. Starting with identity element and a neighbourhood multiply by x on the right or x on the right that will contain x and it will be neighbourhood because these are homeomorphisms multiplication on the right or left right.

That is what I am using here, xU is any neighbourhood of x. So, there is no specialty here. And one of them intersection A is empty because x is not in $\bar{A}$ that is all I have done. Now, by part (a) of the previous lemma, we can assume that U is symmetric. You can go back to a symmetric neighbourhood it will be smaller thing contained inside U.

So, that intersection with A will be still empty. So, I can assume U itself is symmetric ok. Now, it follows that x is not in AU, see I have AB intersection C is empty implies A intersection CB^{-1} is empty that is what I am using here ok. So, if this is empty x cannot be in AU. Now, this is symmetric therefore, it is one of the elements here one of the members here so; that means, that x cannot be in the intersection ok.

So, one part we have done. Now, start with the point which is not here then you can show that it is not in the closure this is what I have to write. Conversely, suppose x does not belong to RHS; that means, that is a symmetric neighbourhood now directly because I have put S here. Say let us call denote it by V ok such that x is not in AV, see V is symmetric neighbourhood and I am taking A times that, right.

So, x is not in AV. So, that is the meaning of this; this is not in the intersection. But that just means that again going back here $(xV) \cap A$ is empty. xV is a neighbourhood of x, this means x is not in the closure. So, part (a) is done.(Refer Slide Time: 24:17)



The next one what we have to do? $\bar{A}\bar{B}$ is contained inside $\overline{AB}$, ok. So, let us see why, the first thing is we know in a product topology if you have a subset A and a subset B of two spaces X and Y, then $\overline{A \times B}$ is same thing as $\bar{A} \times \bar{B}$ right; that is what I am using here $\overline{A \times B}$ is equal to $\bar{A} \times \bar{B}$ ok. Now, all that I have to use is the continuity of the multiplication that is all. The μ from $G \times G$ to G, (x, y) going to xy.

So, how? Let us see $A \times B$ is obviously contained inside $\mu^{-1}(AB)$ because $\mu(A \times B)$ is nothing, but AB ok. If you take inverse $A \times B$ will be contained in $\mu^{-1}(AB)$, alright. But AB is a smaller subset than $\overline{AB}$. So, this is contained in $\mu^{-1}(\overline{AB})$. But this is a closed subset.

Why it is closed subset? It is bar of something, then I am taking μ^{-1}, μ is continuous. So, μ^{-1} of this one is a closed subset containing a set. So, its closure will be contained inside that closed subset, since the closure is the smallest closed set containing the set right. So, I am using something which we have done a long long back perhaps.

But, $\overline{A \times B}$ is $\bar{A} \times \bar{B}$. So, $\bar{A} \times \bar{B}$ is $\overline{A \times B}$ that is contained inside $\mu^{-1}(\overline{AB})$ because this is a closed subset contained in $A \times B$ ok. Now, apply mu on both side this will be $\bar{A}\bar{B}$ and that is contained inside $\overline{AB}$.

So, do not make the mistake that $\overline{AB}$ is contained inside $\bar{A}\bar{B}$, ok. First thing you should observe is that $\bar{A}$ is closed, $\bar{B}$ is closed, but $\bar{A}\bar{B}$ may not be closed ok. So, these are the cautions.

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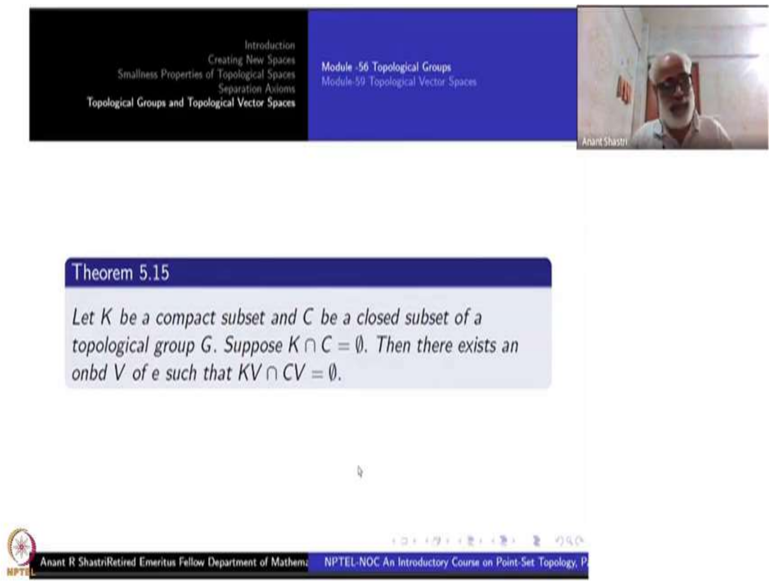

Now, we are going to do something quite deep suddenly ok. We have everything every machinery ready for it. So, start with a compact subset and C be a closed subset of a topological group G ok, no other assumptions K is compact and C is closed ok. Suppose they are disjoint. Then they can be separated by open sets that is just, but we can do better, namely there exist one single open neighbourhood V of e such that KV that would be neighbourhood of K that we know right $KV \cap CV$ is empty.

KV is a neighbourhood of K, CV will be a neighbourhood of C because V is a neighbourhood of identity, but these are quite large open subsets for that matter, but they contain K and C the intersection is empty. So, such a neighbourhood of identity I can find. So, this is the claim. In particular it will follow that K and C can be separated by open sets.

It is similar to normality, but far away from normality because I have assumed K is compact not a closed set ok; not just a closed set. If it is closed set then this would have given you normality. So, quite near normality it comes ok. So, that is why I am taking here suddenly proving this such strong results.

So, now do you understand that in topological groups the topology on a topological group has to be quite special alright though it can range from discrete to indiscrete, though not anything in between, it has to be special ok.

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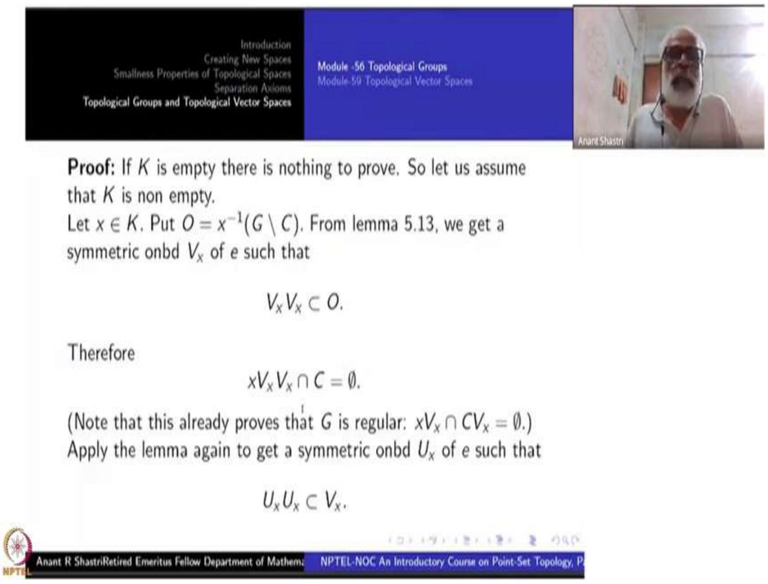

Let us prove this one. If K is empty there is nothing to prove because I can always take, you know, KV when K is empty is just empty set. Empty set intersection anything is empty that is no problem. So, let us assume that K is nonempty. Similarly, we can assume that C is nonempty alright. Since I start with x belonging to K that is why I am justifying that let K be non empty that is all.

Take a point x inside K put O equal to $x^{-1}G \setminus C$, the C is closed. So, $G \setminus C$ is open translated by x^{-1} that is also open.

See look at this one x is a point of point of K ok and hence x in in $G \setminus C$. So, I am I taking x^{-1} of this one. So, that is an open set that is all that I have ok. Now, from lemma 5.13 whatever we get a symmetric neighbourhood V_x of e such that $V_x V_x$ is contained inside O. See why this is true? Because O is a neighbourhood of identity, why? Because x is inside K therefore, x is inside $G \setminus C$.

Because x is not in C, K and C are disjoint that is all ok. If x is here x^{-1} of that will contain identity element. So, this O is open and it is a neighbourhood of identity you can improve it to get a symmetric neighbourhood V_x such that $V_x V_x$ is inside O. So, this was part (a) and (b) both combined here of this lemma which we have proved just now.

This implies that $(xV_xV_x)\cap C$ is empty this contained inside O and $x^{-1}G$ right. So, this x, I am bringing it on the left here. So, what I have what is $G\setminus C$ on that side ok. So, it is contained inside $G\setminus C$, but intersection with C is empty there ok.

So, for each x we have found V_x such that $(xV_xV_x)\cap C$ is empty. So, this already tells you that G is regular. The topology is regular why? bBecause suppose instead of K compact and so on K is singleton x, ok. Then if I prove whatever the statement that would be regularity That is all that is all I have proved xV_xV_x is a neighbourhood contained inside the complement of C.

So, here itself the proof is over of regularity ok. So, we have already proved that every topological group is regular on the way to proving this lemma ok.

Let us continue. Now, applying the lemma 5.13 again we get another symmetric neighbourhood which I have denote by U_x of V such that this U_xU_x is contained inside V_x ok. Because V_x is a neighbourhood of identity alright.

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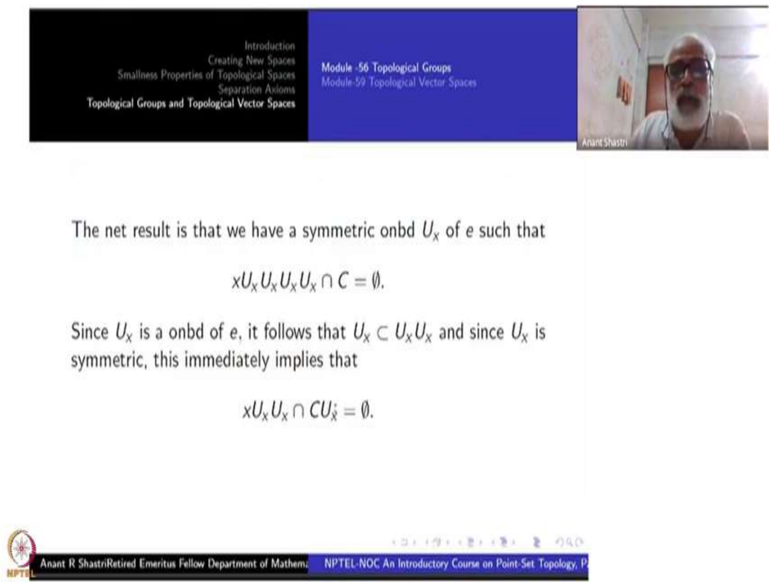

If you combine this one $(xV_xV_x) \cap C$ is empty, now I can replace V_x by U_xU_x which is a smaller neighbourhood than V_x. So, $(xU_xU_xU_xU_x) \cap C$ is empty ok. So, do not go on doing this again and again. We have already arrived at what we wanted.

U_x is a neighbourhood of e right. So, U x is contained inside U_xU_x. Therefore, $(xU_xU_xU_xU_x) \cap C$ is empty. Since U_x is symmetric, one U_x on this side I am translating it to the other side ok. When you translate you have to write U_x^{-1}, but you do not have to write U_x^{-1} because U_x is symmetric.

Out of the three you can translate to the one other side and out of three you can combine two of them and that just makes it one of them U_x is contained inside U_xU_x. So, $xU_xU_x \cap CU_x$ is empty ok.

We have not yet completed. So, one single thing we have done this much from regularity. We have improved it to this much ok the same neighbourhood here repeated twice comes on both sides.

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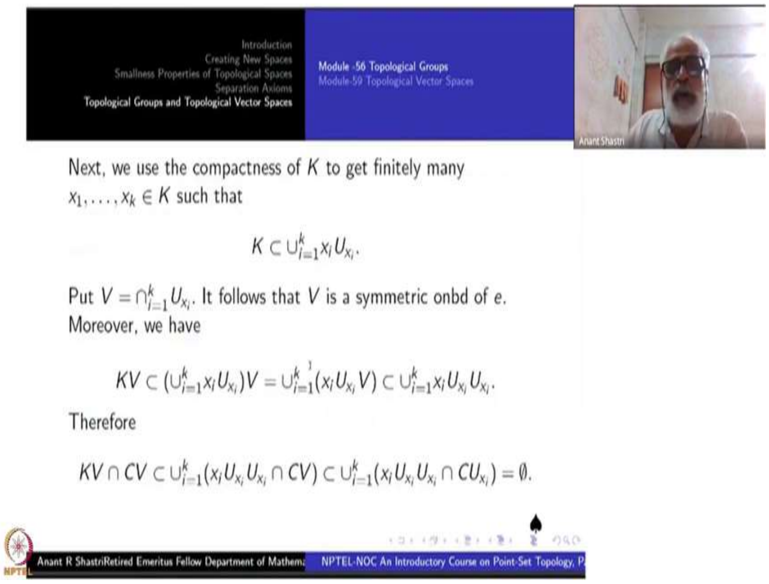

To prove regularity as I have indicated, just the $(xV_x) \cap C$ is empty that was enough ok. Now, we use the fact that K is compact. So, if K is compact there will be finitely many

$x_1, x_2, \ldots, x_k$ belonging to K such that K is contained inside union of $x_i U_{x_i}$; i going to 1 to k ok? Why? Because these xU_x's as x varies over K will form an open cover for K. So, out of that you extract a finite sub cover.

Now, you put V equal to intersection of U_{x_i}'s. Because this is finite intersection it is open. Because each U_{x_i} is symmetric, symmetric means what $\eta(U_{x_i})$ is U_{x_i}, ok. So, intersection is also. It follows that V is symmetric neighbourhood of e, open neighbourhood of e. Moreover we have KV is contained inside union of $x_i U_{x_i} V$, I am replacing K by this union times V.

Then I push this V inside the bracket. So, it is a union of $x_i U_{x_i} V$ ok, but then each $U_{x_i} V$ is what? V is the intersection of all these right. So, it is contained inside each U_{x_i}, I can replace V by U_{x_i} depending upon what this x_i is, so that I can this as $x_i U_{x_i} U_{x_i}$; I am writing ok.

So, KV is contained inside this finite union therefore, if I take the intersection of KV with CV, I want to show that this is empty right that was my final goal. Look at $KV \cap CV$, this KV is contained inside this one large thing ok. So, you take this thing intersect with CV, push it inside the bracket that is all, ok? So, it is the union of all these intersections finitely many of them, but that is contained inside union of $x_i U_{x_i} U_{x_i}$ as it is.

But, CV what is V, V is intersection of these things is contained inside CU_{x_i}. These are larger open sets ok. But, now what are these things each of them is empty by the choice. Remember that each of them is empty. I have taken only a finite many x_i's, coming from K ok. So, each of them is empty. So, union is empty.

So, we have proved that a topological group is strongly regular in the sense that a compact and a disjoint closed set can be separated by open sets in a neat way by one single neighbourhood V of e and its translates, $KV \cap CV$ is empty ok?

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So, I am just summing it up for posterity: every topological group is regular that much is fine alright.

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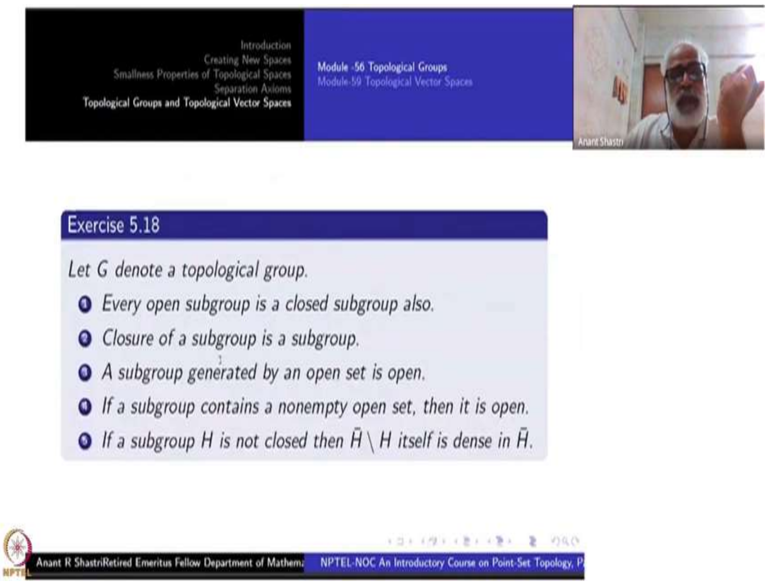

So, here are some elementary exercises again we can work out them, but these are serious examples serious exercises. If you work them in that order you will be able to do all of them. So, they are built up in that fashion ok.

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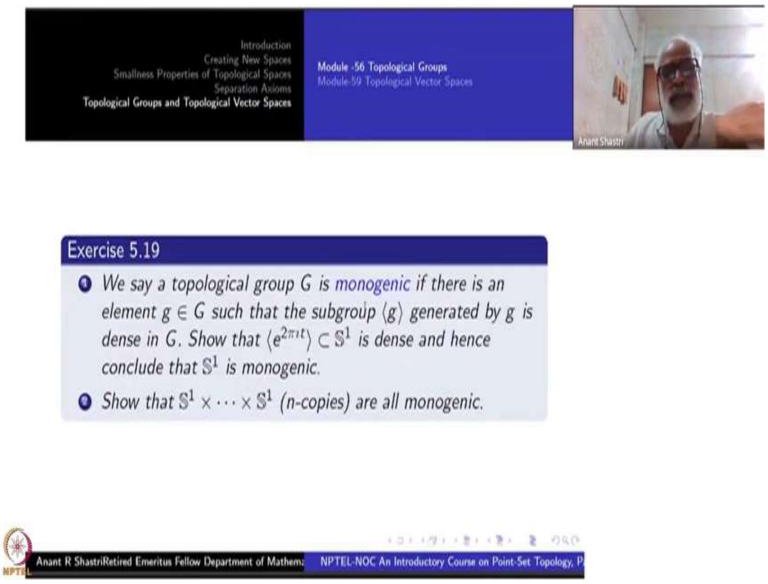

And then there are some other very nice things happening. I have no time to discuss them fully, but we will discuss them provided you show equal interest and come up with some solution, maybe wrong maybe right whatever it is. So, these are left to you as exercises and maybe specifically mentioned assignments ok.

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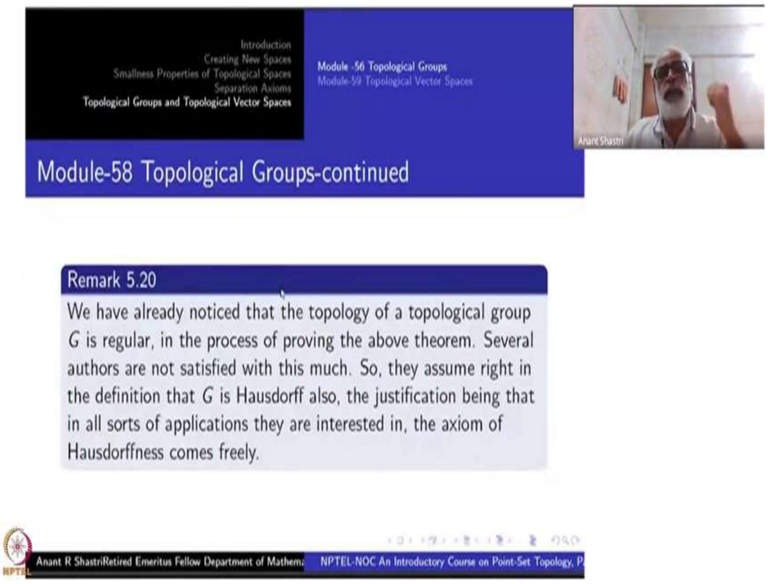

So, let us stop here. Next time we will continue the study of topological groups.

Thank you.