Introduction to Point Set Topology, (Part I)
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Lecture - 56
Topological Groups

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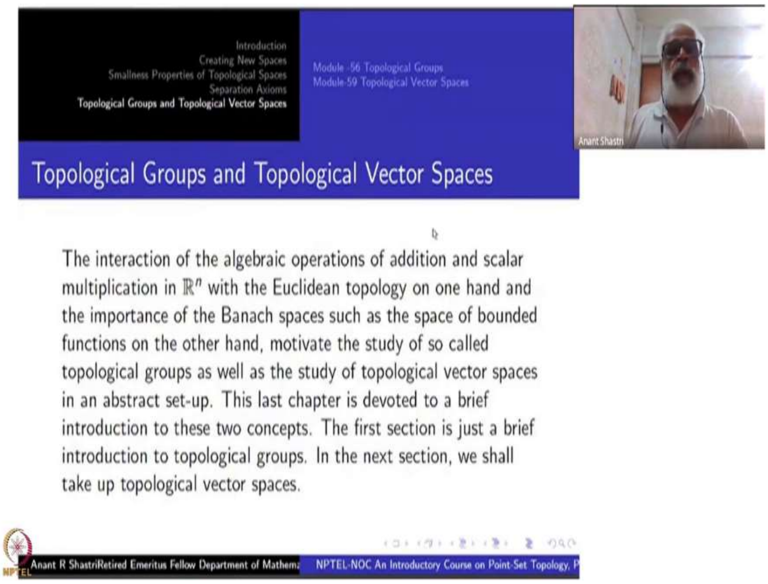

Welcome to the last chapter on Point Set Topology course Part 1. So, this will be topological groups and topological vector spaces. The interaction of algebraic operations of addition, scalar multiplication etc with the Euclidean topology on the one hand and the important example that we had Banach spaces right, they are very important ok. What is the Banach, one particular Banach space we have emphasized namely, the set of all bounded functions on a given set right.

So, all these things together motivate the study of the so called topological groups on the one hand and topological vector spaces on the other ok. So, this last chapter is devoted to a brief introduction to these two concepts. The first section is just a brief introduction to topological groups and the next section we will have take up topological vector spaces ok. It is notat all claimed that this is exhaustive or comprehensive. tThis is just a brief introduction ok?

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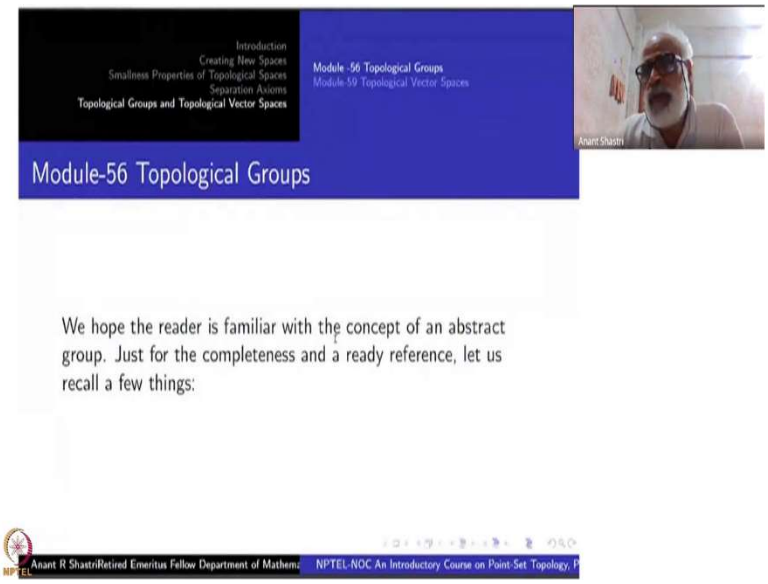

Let me start with recalling the definition of what is a group first, so that I can use those notations comfortably. I assume that you will know already groups because you have even studied group actions right? So, a little bit of group theory whatever groups, homomorphisms of group etc I suppose you know, but let me recall them first for ready reference also.

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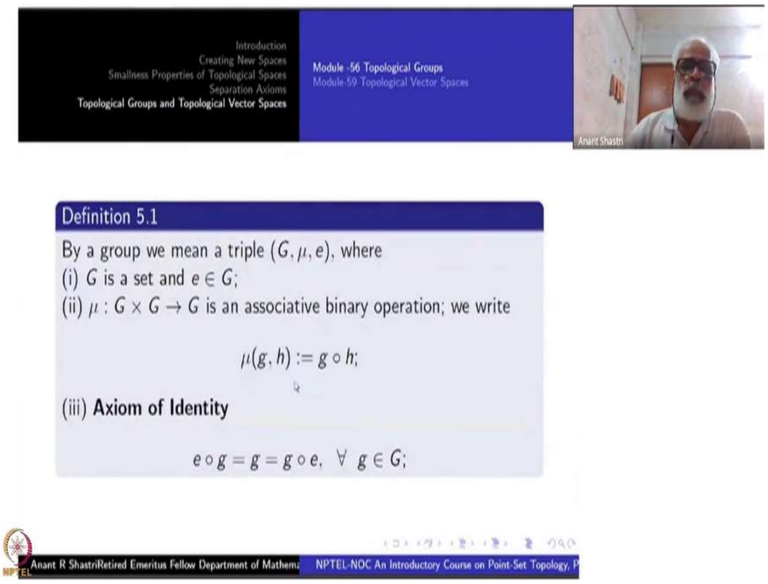

First of all, you may treat a group order triple of a set G and a binary operation μ and a distinguished element e; where G is a set, μ is a function from $G \times G$ to G that is the meaning of a binary operation, which is associative. A short notation we will use: $\mu(g, h)$ as $g \circ h$.

So, g and h are elements of G, $g \circ h$ is also inside G. You may read it as $g \circ h$.

Axiom of identity: there is this axiom which means just another condition here, that is, this distinguished element is called identity.Why? Because its action on other elements, $e \circ g$ is g and on the right side of the $g \circ e$ is also g for every g in G ok? So, it is acting identically on other elements. That is why the name identity element.

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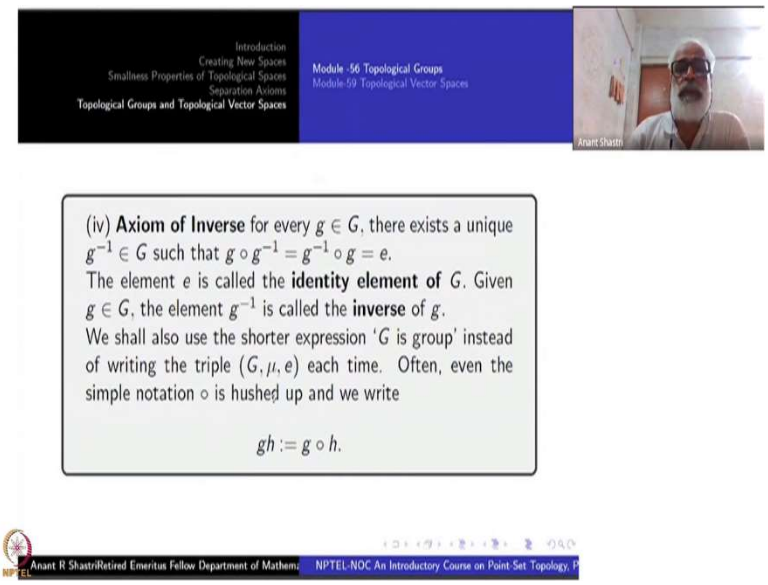

The third axiom is that for every g in G, there exist a unique element g^{-1} ok? We read it as g inverse such that, what is the property? $g \circ g^{-1}$ is equal to $g^{-1} \circ g$ is the identity element ok? For every g in G, the element g^{-1} is called actually inverse of g.

We will also use the short expression: G is a group just like we say X is a topological space. So, we can mention G is a group instead of writing the triple (G, μ, e) etc each time. Often even the simple notation this composite instead of $\mu(g, h)$ we are writing $g \circ h$, but most often when there are no other compositions we will just write it as gh.

If there are 2, 3 dfferent compositions, then you have distinguish them you cannot write all of as gh ok. Even when there are 2 different groups G, G' ok, say one is the domain and another co-domain of a function, we are using the same gh here and $g'h'$ there to mean represent the two corresponding multiplications inside G and G' respectively.

So, such short notation you know; you know short notations or abuse of notation which is there in practice by stalwarts all the way go back to, you know Euler and so on, we cannot change that. So, we better follow those rules.

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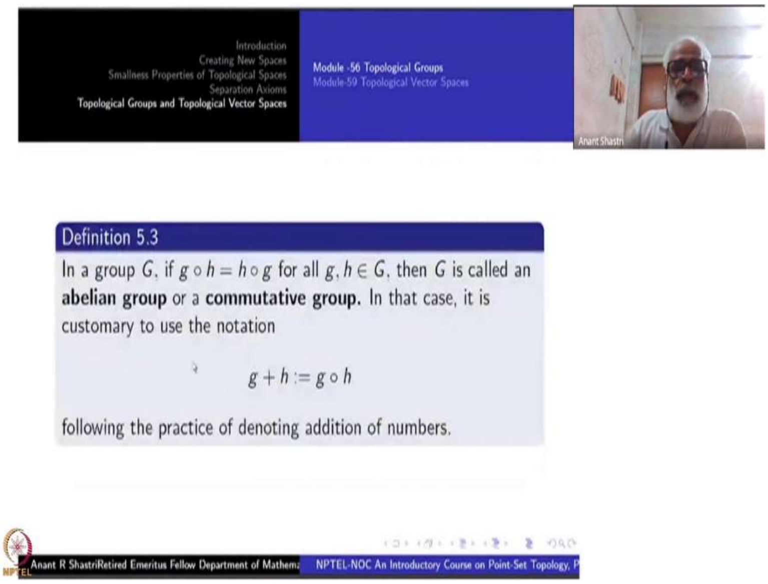

In a group G, if the composition happens to have this property namely $g \circ h$ is equal to $h \circ g$ for all g and h, then G is called an abelian group or a commutative group ok.

This is similar to the case of integers, rational numbers, real numbers and so on. So, there standard notation is plus: $g + h$ stands for $g \circ h$, but that is not forced on us. Because, there may be more than one composition both of them commutative ok. So, then you have to choose plus for one but for the second, you may choose $+'$ or some such thing.

So, that is also not customary, though you have to follow the customs here rather than rigid rules; rigid rules will be followed you know logically in our mind that much we have to do--

abuse or no abuse. This abuse of notations it just means that you are not going to get confused by this simplied notation. That is a whole idea ok.

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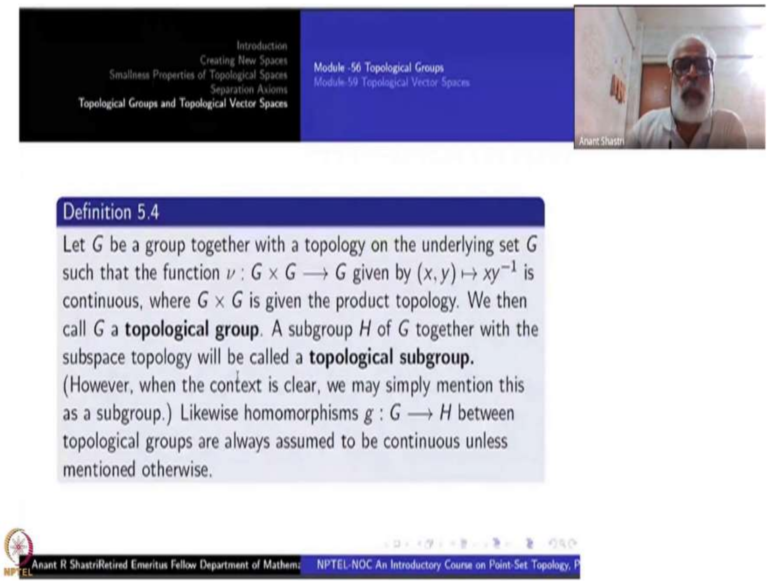

Let G be a group together with a topology on the underlying set G such that the product map namely, (x, y) going to xy^{-1}, that I am going to denote by ν, for a while that must be continuous ok.

What is the continuity here G is a topological group which means it has a topology ok? Also it is a multiplication. Then condition is that this multiplication is continuous. Means what? G has a topology, I have to take the product topology on $G \times G$. The product topology from given topology on both factors the same topology. That is what I have to take. Under that this ν must be continuous ok?

We then call G a topological group. A subgroup H of G together with the subspace topology will be called a topological subgroup provided what? What you have to do? The group operation is also should be taken as the restricted operation from that of G because it is a subgroup ok.

So however, when the context is clear we may simply mention this as a subgroup. Likewise homomorphisms from G to H between topological groups ok are always assumed to be continuous unless mentioned otherwise ok.

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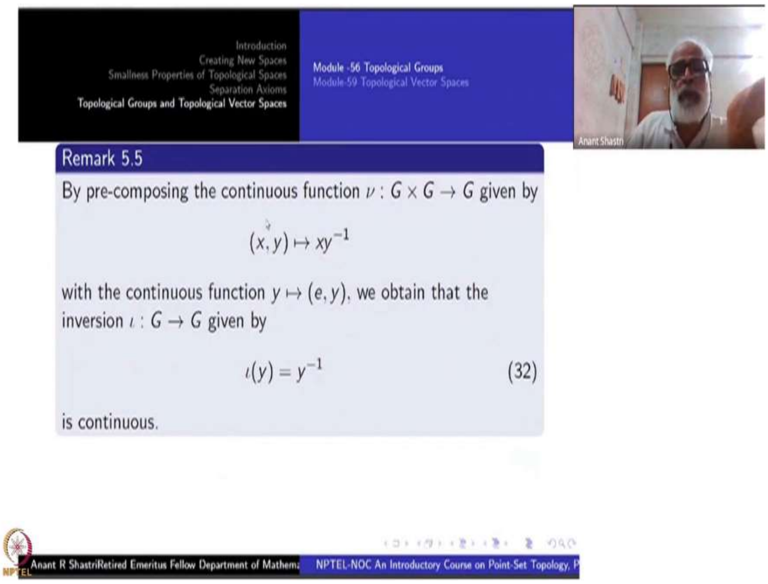

Pre-composing the continuous function μ from $G \times G$ to G by y going to (e, y). See y going to (e, y) is a continuous function from G to $G \times G$ right. It is like a coordinate inclusion; follow it by (x, y) going to y^{-1} is continuous. So, this x becomes e.

So, this composition is just y going to y^{-1}. So, that map I am denoting by η; $\eta(y) = y^{-1}$. It is called the inversion map $\eta(y)$ is y^{-1} because y^{-1} is uniquely defined ok. This map is called what inversion map and it is continuous by this observation. I am not making this as an extra hypothesis, extra condition.

It is a consequence because it is a composite of these two functions.

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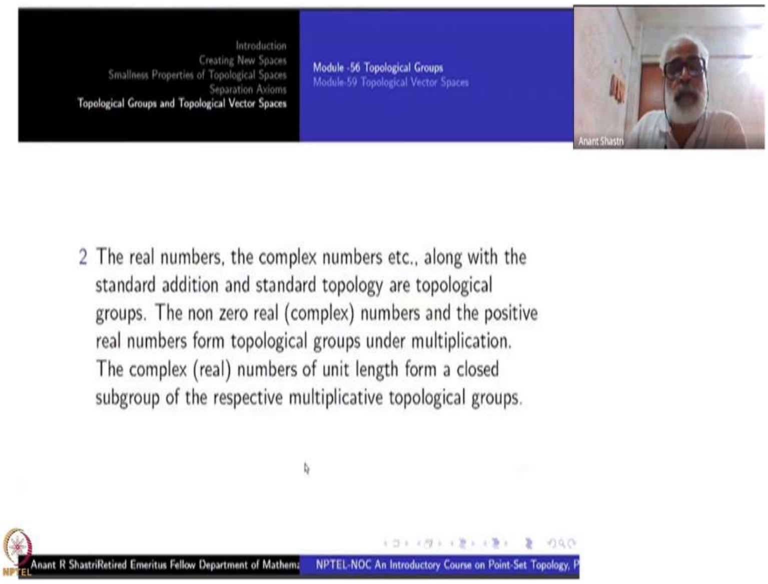

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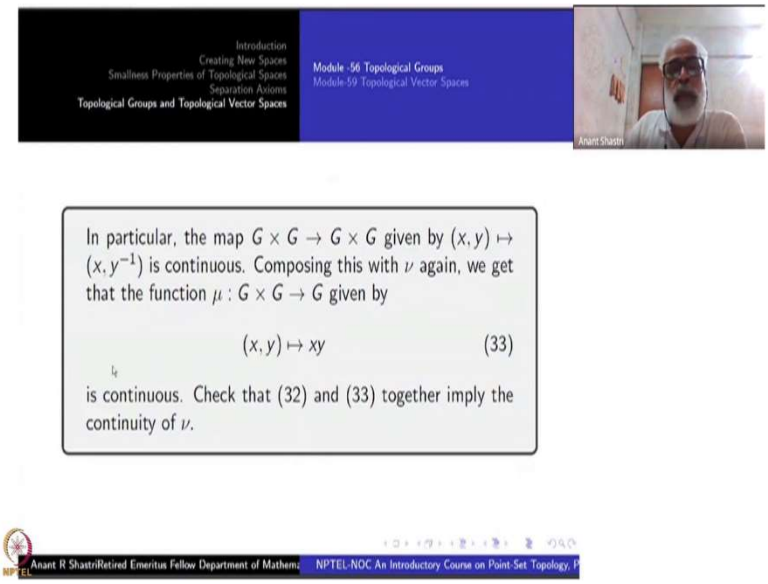

Likewise, if you take (x,y) going xy. The original composition μ. So, (x,y) going to (x,y^{-1}) is what? Identity comma η right, therefore this is continuous. Now compose this with $\nu:(x,y)$ going to xy^{-1}. That will become (x,y) going to xy. The $\nu(x,y^{-1})$ is equal to $\mu(x,y)$, right? Because inverse of the inverse is identity. Therefore, actually the multiplication map μ you started with is also continuous.

So, in one single go by taking this definition namely (x, y) going to xy^{-1} is continuous, we have made both operations of taking inverse of an element and taking (x, y) one to xy, continuous ok. Once you have both of them continuous you can recover the continuity of ν also by the formula $\nu(x, y)$ is equal to $\mu(x, y^{-1})$.

So, they are two different conditions combined equivalent to each other. So, 32 and 33 together imply continuity of ν. alright.

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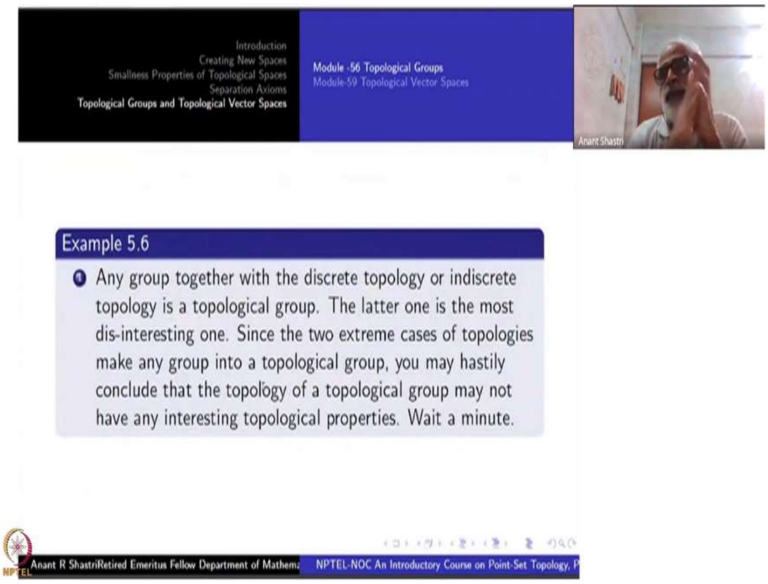

So, here are some examples now, any group together with the discrete topology or indiscrete topology. Instead of a topological group, see start with any group. So, group operation you do not have to change. You put a topology, we are very familiar with putting lots of different topologies on a given set. You put the topology, say. Discrete topology cross discrete topology is discrete the codomain is discrete, but any function from a discrete topology to any other space is always continuous.

Similarly, indiscrete to indiscrete any function into an indiscrete space is also continuous. So, out of this the first one, the discrete topology is not so disinteresting. We have already done one of them, in the context of discrete action of a group on set or on topological space.

But the second one, namely indiscrete topology is most disinteresting one. You will never have an occasion to use that one ok? But what is happening is that, you know, it triggers some thought process here. These two extremities are there right? on any given group. They are both topological groups without change in the group structure. So, you may think that topological groups after all have no special properties at all! Any group is a topological group. That is of course true.

However, can you conclude that the topology for a topological group may not have no extra properties at all? If it all there are additional properties it must be because of the group theory? You may see if you wait a minute that this is not the case. So, we are going to prove something out of this maybe you may think it is nothing ok. In other words, starting with an arbitrary topological space, you may be able to put group structure so as to get a topological group. So, we will see such things.

So, that being the abstract part, let us come back to some reality, namely, some genuine and useful examples. The real numbers, complex number. These were the motivating examples for us, for this abstract definition right, along with standard addition. And you take non-zero (real or) complex numbers, there is a multiplication also, they are all topological groups right?

The complex(or real) numbers of unit length, form a closed subgroup of the respective multiplicative topological groups complex or real numbers ok. If you take real numbers of unit length, it is just minus one plus one that is a subgroup. Similarly, the circle, the unit circle is a subgroup of the non-zero complex numbers under multiplication ok?

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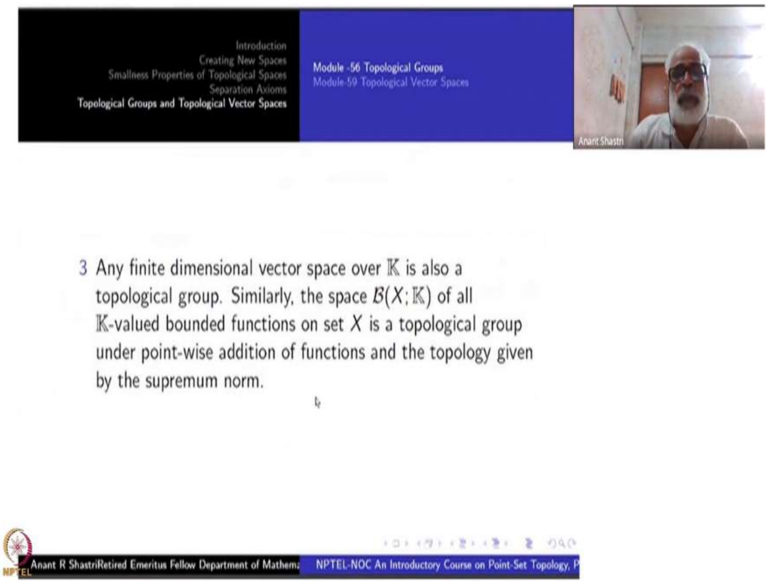

So, these are the easy examples. Any finite dimensional vector space over $\mathbb{K}$ is also a topological group. I am only looking at the addition. The scalar multiplication is there, we will study them little later. Similarly, our example namely I told you about Banach spaces-- the set of all bounded functions on X taking values in $\mathbb{R}$ or $\mathbb{C}$ ok? That is a Banach algebra right? So, there the addition, which will be automatically continuous with respect to the topology induced by the norm, the norm is supremum norm ok.

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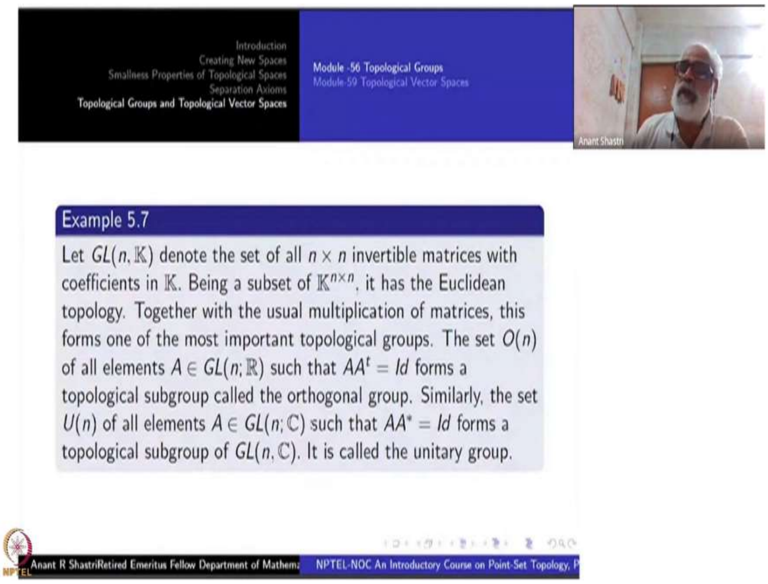

One of the most interesting case of all these topological groups occurs inside the space of all $n \times m$ matrices. Addition there already makes it a topological group ok. Even the multiplication for $m = n$, just like the in the case of complex numbers and real numbers of course, you have to throw away the 0 matrix of course, not only 0 matrix this time you have to throw away a lot more namely all matrices of determinant equal to 0, you have to throw away.

So, in other words you have to take only invertible $n \times n$ matrices. Invertibility with respect to multiplication. So, that will form a group. The group laws are continuous. how do you take inverse of an invertible $n \times n$ matrix. Each entry will be a polynomial namely the $(ji)^{th}$ cofactor divided by the determinant. So, this is polynomial divided by a polynomial, but the denominator is non zero therefore, they are continuous ok.

So, $GL(n, \mathbb{K})$ forms a group and the group laws are continuous with respect to what, with respect to the Euclidean topology. $GL(n, \mathbb{K})$ is an open subset of the space of all matrices $\mathbb{K}^{n \times n}$. I have told you earlier, rather than writing $\mathbb{K}^{n^2}$, you write $\mathbb{K}^{n \times n}$, which suggests that we are dealing with matrices as vectors. So, that way you get a Euclidean topology on the matrices.

Similarly, you can look at the orthogonal group, O_n, which is defined as all those real matrices A such that $AA^T = Id$. If you take complex matrices then take AA^* namely conjugate transpose right? AA^* equal to identity. That will be called the unitary group. So, verification that they are groups is easier, is just linear algebra, matrix theory.

The only missing thing is why the group multiplications are continuous. you can separately verify that (A, B) going to AB is continuous by looking at the matrix entries of this product. They are all polynomials. Then you have to look the function A goes to A^{-1}, whenever A is invertible. Of course, you have to know how to write the inverse.

So, for writing the inverse you have the Cramer's rule which says: take the adjoint matrix. which has again each entry a polynomial in the original entries. Then each entry you have to divide by the determinant of the given given matrix which is another polynomial. So, that is the whole idea of this $GL(n, \mathbb{C})$ ok and $GL(n, \mathbb{K})$ in general. They have various subgroups I have introduced only two of them here. O_n and U_n here then you can take groups with the determinant one also here. They will also form subgroups smaller subgroup and so on.

So, these groups and many of their groups are central to a lot of mathematics and there are theories here a small aspect of this you can call them matrix groups which will lead later on a deep theory, very beautiful theory call Lie groups which we will not be able to do in this course.

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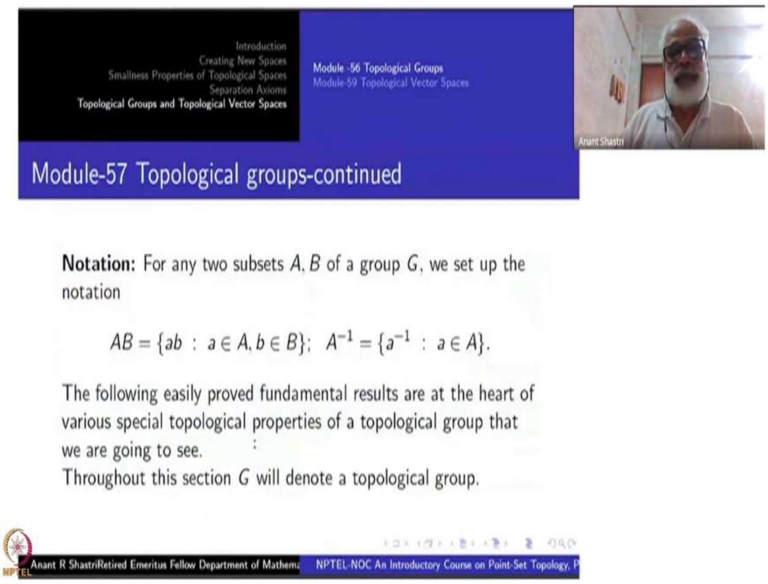

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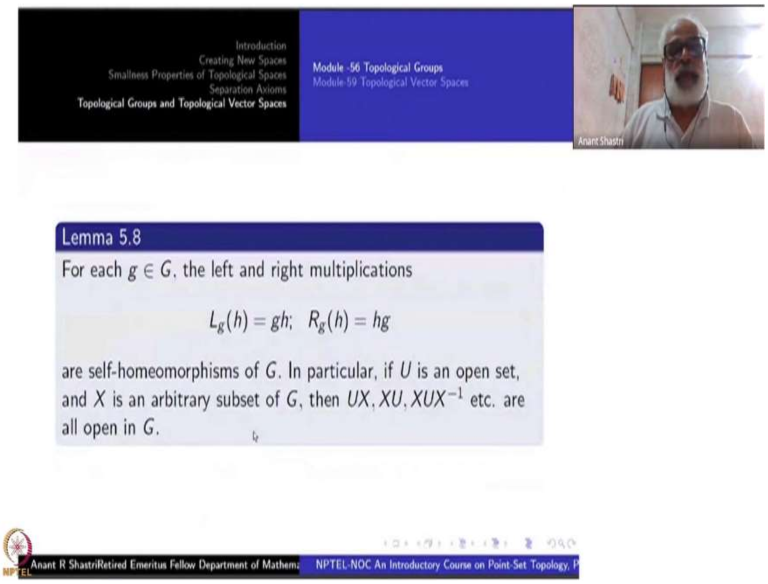

So, let us stop here and take up these discussions next time ok.

Thank you.